HENSON GRAPHS AND URYSOHN-HENSON GRAPHS AS CAYLEY GRAPHS

DEDICATED TO PROFESSOR ANATOLY VERSHIK ON THE OCCASION OF HIS 80TH BIRTHDAY

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Abstract. We discuss groups acting regularly on the Henson graphs Γ_n, answering a question posed by Peter Cameron, and we explore a number of related questions.

Introduction

Peter Cameron's paper [Ca00] provides a body of material relating to the following problem, and includes a survey of prior work (notably that of [CJ87]).

Problem. *For which pairs (G, Γ) consisting of a countably infinite group G and a homogeneous structure Γ is there an embedding of G into $\mathrm{Aut}(\Gamma)$ as a regular subgroup? Equivalently, when can we put a left G-invariant structure on G isomorphic to Γ?*

Many interesting open questions are raised as well, among them (on p. 751) the question whether the generic K_n-free graph (the Henson graph Γ_n) can be a Cayley graph for an infinite group when $n \geq 4$. An argument of Henson shows that the group acting cannot be abelian. We show here, using Cameron's analysis in a fairly direct way, that the free group of infinite rank has such an action. We then explore further the range of groups to which the construction applies, and generalizations to broader combinatorial settings.

The existence of a regular action by a free group will be shown in §1. In §2 we extend Henson's result from the abelian setting to the class of groups whose center has finite index; we put this in a more general setting suggested Cameron's treatment of Henson's argument. We tend to think that the same result should apply when there is an abelian subgroup of finite index, but we could not even handle the infinite dihedral case.

We show in §3 that the free nilpotent group of class 2 of infinite rank acts freely on the Henson graphs. Thus the gap between positive and negative results is not so very great. The result is given very generally, not just for Henson graphs but for homogeneous structures in a finite relational language for which the substructures are closed under free amalgamation.

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In §4 we look at abelian actions on digraphs of Henson type, as well regular normal actions in the sense of Cameron, as discussed in §2.

Our initial point of departure was actually [CV06], which deals with Urysohn's homogeneous metric space. In recent years we have been much occupied with an attempted classification of the metrically homogeneous graphs, that is the graphs which are homogeneous when viewed as metric spaces in the graph metric. Examples include the Henson graphs and integer-valued analogs of Urysohn space, as well as a mixture of the two which one may call the Urysohn-Henson graph (or space). In §5 we make some comments on the problem of finding regular actions of groups on metrically homogeneous graphs, a question also addressed in §9 of [Ca00] with a more restrictive notion of Urysohn-Henson graph.

It is a pleasure to acknowledge that my interest in this matter (and many other matters) was stimulated by conversations with Professor Vershik at HIM in Fall 2013 regarding [CV06] and more recent work.

1. Regular actions of free groups

The following result will be generalized in §3. One may omit the proof given here, but it presents the fundamental idea in its purest form.

Proposition 1. *There is a regular action of the free group F_ω of infinite rank on the generic K_n-free graph Γ_n (Henson graph).*

Definition. Let G be a group and S a symmetric subset of $G^\#$. The *Cayley graph* associated to (G, S) is the graph with vertex set G and edge set

$$\{(g, h) \mid h^{-1}g \in S\}$$

Note that this graph is left G-invariant, that is G acts as a group of automorphism by left translation. Loosely speaking, we may also called the induced graph on a set $X \subseteq G$ the Cayley graph associated to (X, S).

Proof. We need to construct a suitable Cayley graph on G, that is, find a suitable symmetric subset S of $G^\#$. We assume the reader is familiar with the extension properties that characterize the Henson graphs.

We view F_ω as coming together with a fixed sequence of free generators, and F_k then denotes the corresponding subgroup of F_ω of rank k.

The set S is built via finite approximations, by stages. One step of the construction goes as follows.

Construction. Given S a finite symmetric subset of $F_k^\# \subseteq F_\omega$, and U a finite subset of F_k, with U K_{n-1}-free in the induced graph defined by (F_k, S), take any $g \in F_{k+1} \setminus F_k$ and define

$$S' = S \cup (g^{-1}U) \cup (U^{-1}g)$$

Then we claim the following.

$$S' \cap F_k = S \tag{1}$$

$$gS' \cap F_k = U \tag{2}$$

$$\text{The graph associated to } (F_{k+1}, S') \text{ is } K_n\text{-free.} \tag{3}$$

The first two points are clear. The content of these two points is that the Cayley graph structures induced on F_k by S or by S' are the same, and that the Cayley graph structure on F_{k+1} associated with S' satisfies one instance of the required extension property, with g having only U as its set of neighbors in F_k. This last point can be delicate even for the random graph, when other groups are considered.

Once we have the third and final point, we can build the desired infinite symmetric subset of F_ω by stages so that the Cayley graph associated with (F_ω, S) is K_n-free and has the full extension property corresponding to the Henson graph Γ_n, and is therefore isomorphic to Γ_n.

So we take up the final point. Suppose there is an n-clique K in the Cayley graph associated with (F_{k+1}, S'). Shifting on the left, we may suppose this n-clique contains the vertex g. Therefore $K \subseteq \{g\} \cup gS \cup U \cup gU^{-1}g$.

What edges may occur between two of the sets gS, U, $gU^{-1}g$? Edges (gs, u), $(gs, gu^{-1}g)$, $(u_1, gu_2^{-1}g)$ would correspond to elements

$$u^{-1}gs,\ s^{-1}u^{-1}g, \text{ or } u_1^{-1}gu_2^{-1}g \in S'$$

As none of these can lie in S, we come down to the equations

(gS, U): $u^{-1}gs = (g^{-1}u')^{\pm 1}$

$(gS, gU^{-1}g)$: $s^{-1}u^{-1}g = (g^{-1}u')^{\pm 1}$

$(U, gU^{-1}g)$: $u_1^{-1}gu_2^{-1}g = (g^{-1}u')^{\pm 1}$

The last is impossible, and so is the first, since $s \neq 1$. The second reduces to

$$s^{-1}u^{-1} = u'^{-1}$$

or $u' = us$.

So if K meets U then K is contained in $U \cup \{g\}$ and we have a contradiction. Thus K is contained in $\{g\} \cup gS \cup gU^{-1}g$; shifting, we may suppose

$$K \subseteq S \cup \{1\} \cup U^{-1}g$$

As the induced graph on F_k is K_n-free, K must meet $U^{-1}g$.

An edge within $U^{-1}g$ would require a relation $(g^{-1}u_1u_2^{-1}g) \in S'$, with $u_1 \neq u_2$, and this is impossible. So K must have the form

$$\{1, u^{-1}g\} \cup S_0$$

with $S_0 \subseteq S$. Consideration of the edges $(u^{-1}g, s)$ for $s \in S_0$, as above, gives

$$uS_0 \subseteq U$$

As S_0 is a clique it follows that $(\{u\} \cup uS_0)$ is an $(n-1)$-clique in U, and this is a contradiction. $\square$

2. Constraints: abelian and nearly abelian groups

Henson has given an argument (for the group $\mathbb{Z}$) which shows that no abelian group can act regularly on a Henson graph Γ_n with $n \geq 4$. Cameron has shown that the same argument applies to all *regular normal actions* of arbitrary groups.

Here an action is called regular normal if both left and right translation preserve the graph structure; or equivalently, if left translation and conjugation preserve the graph structure.

We extend Henson's result to the case in which the center of the group has finite index. The appropriate generalization of that from Cameron's point of view is a regular action of a group G in which the action of a subgroup H of finite index is normal. (Note that we are considering the action of H on all of G.)

Lemma 2.1. *Let G be a group with a subgroup H of finite index. Then G has no regular action on Γ_n which is H-normal, for any $n \geq 4$.*

In particular if the center of G has finite index there is no regular action of G on Γ_n.

Our convention is that the full group G acts on the left side.

Proof. Supposing there is a regular G action on Γ which is H-normal, we identify Γ with G. By the indivisibility of Γ_n [EZS89] the graph induced on H contains a copy of Γ_n. In particular the graph induced on the neighbors of 1 in H contains a copy of Γ_{n-1}.

We follow through the Henson construction [Hen71].

We may take a clique $K \cup \{1\}$ of order $n-1$ in H.

Claim. There is an $h \in H$ adjacent to 1 such that $K \cup Kh$ is K_{n-1}-free.

Let $X = \{k_2^{-1}k_1 \mid k_1, k_2 \in K \text{ (distinct)}\}$. As K is a clique, X is contained in the set of neighbors of 1.

Let Δ_1 denote the set of neighbors of 1 in H and let Δ_1' be the union over $x \in X$ of the set of neighbors of x in Δ_1. The graph induced on Δ_1 contains a copy of Γ_{n-1}, while the graph induced on Δ_1' is a finite union of graphs omitting K_{n-2}. Therefore $\Delta_1 \setminus \Delta_1'$ is infinite.

An element $h \in \Delta_1 \setminus \Delta_1'$ will have the property that h is adjacent to 1 while the pairs $(k_1, k_2h) = k_2(k_2^{-1}k_1, h)$ are nonedges for $k_1 \neq k_2$. As $n \geq 4$, it follows that $K \cup Kh$ is K_{n-1}-free.

This proves the claim.

With h fixed as in our claim, there is then some $g \in \Gamma$ adjacent to $K \cup Kh$. We claim that

$$K \cup \{g, gh^{-1}\}$$

is an n-clique, which will give a contradiction.

We know that K is a clique, and by the choice of g and an application of right H-invariance, we have K adjacent to both g and gh^{-1}.

Finally $(g, gh^{-1}) = gh^{-1}(h, 1)$ is an edge, so we have arrived at a contradiction. □

It would be more satisfactory to eliminate abelian subgroups of finite index (or actions in which a subgroup of finite index acts normally on itself).

Problem 1. *Can an infinite abelian group operate regularly on a graph Γ_n with $n \geq 4$? In particular, is this possible in the case of the infinite dihedral group?*

This also suggests the following question.

Problem 2. *Can a K_n-free graph with transitive automorphism group contain Γ_n without being isomorphic to Γ_n?*

Unfortunately, this seems entirely possible. But if not, then it would follow easily that no group with an abelian subgroup of finite index can act regularly on Γ_n for $n \geq 4$.

In a more technical vein, it is useful to consider the FC-center in place of the center: this is the subgroup of elements whose centralizers have finite index, or equivalently the elements belonging to finite conjugacy classes (FC serves as a mnemonic for the latter).

Problem 3. *If the group G has an FC-center of finite index, does it follow that it cannot act regularly on a Henson graph Γ_n with $n \geq 4$? In particular, can this be proved when G is equal to its FC-center?*

In the next section we will see that under the assumption that the FC-center has infinite index we can in fact formulate a positive result, so this problem may be to the point in spite of its technical nature.

3. Nilpotent groups.

Now we aim to generalize Proposition 1 considerably. On the group theoretic side, we want a result which covers the free nilpotent group of class 2 on infinitely many generators. We also want to capture a broader range of homogeneous structures: those whose underlying language is a finite binary relational language, and which are closed under free amalgamation.

Here the free amalgam of two structures B, C over a substructure A is their relative disjoint union, both at the level of elements and at the level of relations. Some readers may think of free amalgamation in terms of the use of a "default" 2-type rather than "no relation." For such readers, we should specify that the default 2-type is symmetric.

We do not limit ourselves here to the case of *symmetric* binary relations, but the operation of free amalgamation is symmetric.

We know from Cameron's analysis in [Ca00] that the sets

$$S(a_1, \ldots, a_n) = \{x \in G \mid a_1^{-1} x a_2^{-1} x \cdot \cdots \cdot a_n^{-1} x = 1\}$$

play a special role in the theory of regular actions, at least for $n \leq 3$ (also the conjugacy sets $C(a, b) = \{x \mid a^x = b\}$ but these do not need to be made as visible).

Our generalization of Proposition 1 is the following.

Proposition 2. *Let G be a countable group, and Γ a homogeneous binary structure whose finite substructures are closed under free amalgamation. Suppose G satisfies the following conditions.*

- *The FC-center N of G has infinite index.*
- *G is not a finite union of sets of the following types.*
  - *Cosets of subgroups of infinite index;*
  - *Sets $S(a,b)$ (if all relations on Γ are symmetric we may omit the sets $S(a,a)$);*
  - *Sets $S(a,b,c)$.*

Then there is a regular action of G on Γ.

The reader who is skipping about will find a discussion of the FC-center at the end of the last section. We should also take note of Neumann's Lemma, much used in a variety of similar contexts: a group cannot be the union of finitely many cosets of subgroups of infinite index. In cases where the not very comprehensible sets $S(a,b)$ or $S(a,b,c)$ are each contained in a finite union of cosets of subgroups of infinite index, Neumann's Lemma ensures that our hypotheses are satisfied.

The proof will go much as in the original case, but with a good deal more detail required, and some care taken to avoid the FC-center. We first derive a corollary.

Corollary 3.1. *Let G be a countable group and Γ a homogeneous transitive binary structure whose finite substructures are closed under free amalgamation. Suppose that G satisfies the following conditions.*

- *The FC-center N of G has infinite index;*
- *In the quotient $\bar{G}/N$, the sets $S(\bar{a},\bar{b})$ and $S(\bar{a},\bar{b},\bar{c})$ are all finite.*

Then G acts regularly on Γ.

Proof. The sets $S(a,b)$ and $S(a,b,c)$ lie in the preimages of the corresponding sets $S(\bar{a},\bar{b})$ and $S(\bar{a},\bar{b},\bar{c})$ in $\bar{G}$, so all of the exceptional sets in the sense of the lemma are contained in finite unions of cosets of subgroups of infinite index. So as noted above, Neumann's Lemma applies here. □

Application. We have in particular—against our initial expectations—a class of nilpotent groups acting regularly on all Henson graphs Γ_n. Namely, let G be nilpotent of class 2 and suppose the following conditions are satisfied.

- The FC-center of G is the center $Z(G)$;
- $G/Z(G)$ is infinite and contains no element of order 2 or 3

Then G acts regularly on all Γ_n.

Proof of Proposition 2. We make the usual construction by finite approximations with some care taken to avoid elements of the FC-center N as codes for edges.

The term "Cayley graphs"' should now be replaced by "Cayley structures," after which we proceed in the same way as for graphs, with a heavier notation. But for our purposes, as we deal exclusively with finite approximations, a more appropriate place to begin is with the notion of "G-fragment," defined as follows, incorporating our concerns about the FC-center N.

A *G-fragment* consists of a finite structure

$$(\Delta, (\Delta_R)_{R\in L})$$

with the following properties.

(1) $1 \in \Delta = \Delta^{-1} \subseteq G$;
(2) Δ_R is a subset of Δ for $R \in L$;
(3) $\Delta_R \cap N = \emptyset$ for $R \in L$.

We write Δ for both the G-fragment and its underlying set. We associate to a G-fragment Δ the corresponding G-invariant binary structure on G as follows.

$$\begin{aligned} G_\Delta &= (G, (R_\Delta)_{R\in L}); \\ R_\Delta(x,y) &\iff x^{-1}y \in \Delta_R \end{aligned}$$

We also consider the induced structure on Δ, which we denote by $\bar{\Delta}$.

$$\bar{\Delta} = G_\Delta \restriction \Delta$$

We say that a G-fragment Δ is Γ-*admissible* if

(4) $\bar{\Delta}$ embeds into Γ.

We will choose a sequence of Γ-admissible G-fragments Δ whose union gives G a G-invariant structure isomorphic with Γ.

Claim 1. If Δ is a Γ-admissible G-fragment, then G_Δ embeds into Γ.

Here we make use of the notion of a clique, by which we mean a clique with respect to the edge relation defined as the union of the relations R in L. As the finite substructures of Γ are closed under free amalgamation, if G_Δ does not embed in Γ then some finite clique C embeds in G_Δ, but not in Γ.

Translating, we may suppose that the identity element 1 belongs to C. Then as C is a clique and all sets Δ_R are contained in Δ, it follows that C is contained in Δ, and hence C embeds into $\bar{\Delta}$, contradicting the Γ-admissibility.

This proves the claim. From the claim it follows that whenever Δ is an admissible G-fragment, we can extend Δ to a larger admissible fragment containing any specified element of G. This deals with one of the constraints on our construction, namely that the G-fragments involved should eventually exhaust G.

Next we will state our main claim, an extension property for admissible G-fragments. First we establish notation for the main construction.

Construction. Let Δ be a G-fragment, and $B = \bar{\Delta} \cup \{b\}$ an extension of the structure $\bar{\Delta}$ by one additional vertex, where B also carries an L-structure. Let $h \in G$ be given. We then make the following definitions.

$$\Delta' = \Delta \cup (h^{-1}\Delta) \cup (\Delta^{-1}h)$$
$$\Delta_R^+ = \{a \in \Delta \mid R(b,a) \text{ holds in } B\} \qquad \Delta_R^- = \{a \in \Delta \mid R(a,b) \text{ holds in } B\}$$
$$\Delta'_R = \Delta_R \cup (h^{-1}\Delta_R^+) \cup ((\Delta_R^-)^{-1}h)$$

The point of course will be to choose the element h properly.

Claim 2. Let Δ be a Γ-admissible G-fragment and $B = \bar{\Delta} \cup \{b\}$ an extension of $\bar{\Delta}$ by one vertex. Suppose that B embeds into Γ. Then there is an element h such that the extension $(\Delta', (\Delta'_R)_{R\in L})$ has the following properties.

(2.1) Δ' is a G-fragment;
(2.2) $\bar{\Delta}' \restriction (\Delta \cup \{h\}) \cong B$ over $\bar{\Delta}$;
(2.3) Δ' is Γ-admissible.

Claim 1, or the remark following, and Claim 2, taken together, are sufficient to build a G-invariant structure on G which is isomorphic to Γ, by successive finite approximations. So it suffices to establish the second claim.

What we must show in each case is that the elements h violating one of our conditions (2.1–2.3) lie in a finite number of exceptional sets, that is cosets of subgroups of infinite index and subsets $S(a,b)$ or $S(a,b,c)$.

We deal with our three conditions (2.1–2.3) in order.

(2.1) Δ' is a G-fragment

The content of this condition is that $\Delta'_R \cap N = \emptyset$ for all R. It will suffice to have

$$(h^{-1}\Delta \cup \Delta^{-1}h) \cap N = \emptyset$$

The exceptional h for this constraint lie in ΔN, a finite number of cosets of N, as required.

(2.2) $\bar{\Delta}' \restriction \Delta \cup \{h\} \cong B$ over $\bar{\Delta}$

All relations present in B have been encoded into $\bar{\Delta}'$, so we concern ourselves with the converse. We first consider the structure induced by $G_{\Delta'}$ on Δ. Suppose $a_1, a_2 \in \Delta$ and a relation $R(a_1, a_2)$ holds in $G_{\Delta'}$ but not in $\bar{\Delta}$. Then

$$a_1^{-1}a_2 \in \Delta'_R \setminus \Delta_R \subseteq h^{-1}\Delta \cup \Delta^{-1}h$$

The set of exceptional h for which this occurs is finite.

Now we must consider the case where a relation $R(a,h)$ or $R(h,a)$ holds in $\bar{\Delta}'$, but the corresponding relation $R(a,b)$ or $R(b,a)$ does not hold in B.

If $R(a,h)$ holds then $a^{-1}h \in \Delta'_R$ and thus there are three possibilities.

$$a^{-1}h \in \Delta_R$$
$$a^{-1}h \in (\Delta_R^-)^{-1}h$$
$$a^{-1}h \in h^{-1}\Delta_R^+$$

The first possibility corresponds to finitely many exceptional choices of h. The second possibility means that $a \in \Delta_R^-$, or in other words that $R(a,b)$ does hold in B. So we are left with the third possibility.

$$a^{-1}h = h^{-1}a_1$$

where $a_1 \in \Delta$. Thus $h \in S(a, a_1)$. So again h lies in one of a finite number of exceptional sets.

If $R(h,a)$ holds then similarly we come down to the case $h \in S(a_1, a)$.

Note: This analysis may be slightly refined when the relations on Γ are symmetric. Namely, we have $a^{-1}h = h^{-1}a_1$ with $a_1 \in \Delta_R^+$, so $R(b, a_1)$ holds in B and therefore $R(a_1, b)$ also holds. Thus we may suppose in this case that $a \neq a_1$; this significantly broadens the class of groups to which the analysis applies.

Our final condition is the following.

$$\bar{\Delta}' \text{ embeds into } \Gamma \tag{2.3}$$

By our assumptions on Γ, it suffices to check that every clique C in $\bar{\Delta}'$ embeds into Γ. It will be more convenient to prove directly that every clique C in $G_{\Delta'}$ embeds into Γ. Then we can translate the clique C so as to contain the element 1, and observe that as it is a clique, the elements of $C \setminus \{1\}$ all belong to the union

$$\bigcup_R \Delta'_R$$

and in particular lie in Δ'. Furthermore as Δ' is a G-fragment we conclude

$$C \cap N = \{1\}$$

Now we must extend our analysis of relations holding in $\Delta \cup \{h\}$ to relations holding in Δ'.

Relations holding between $h^{-1}\Delta$ and $\Delta^{-1}h$

Suppose the relation $R(h^{-1}a_1, a_2^{-1}h)$ holds, that is

$$a_1^{-1}ha_2^{-1}h \in \Delta'_R$$

We have the following possibilities, with $a \in \Delta$.

$$a_1^{-1}ha_2^{-1}h = a$$
$$a_1^{-1}ha_2^{-1}h = h^{-1}a$$
$$a_1^{-1}ha_2^{-1}h = a^{-1}h$$

In the first two cases h lies in one of the exceptional sets $S(a_1a, a_2)$ or $S(a_1, a_2, a)$. In the last case h lies in a finite set.

So (with h chosen appropriately) our clique C will not meet both $h^{-1}\Delta$ and $\Delta^{-1}h$.

This leaves us with two possibilities to analyze.

Case I. $C \subseteq \Delta \cup h^{-1}\Delta$.

We consider edges between Δ and $h^{-1}\Delta$. Suppose $(a_1, h^{-1}a_2)$ is such an edge, that is

$$a_1^{-1}h^{-1}a_2 \in \Delta'_R$$

Again we write out the possibilities, with $a \in \Delta$.

$$\begin{aligned} a_1^{-1}h^{-1}a_2 &= a \\ a_1^{-1}h^{-1}a_2 &= h^{-1}a \\ a_1^{-1}h^{-1}a_2 &= a^{-1}h \end{aligned}$$

The first and third possibilities involve finitely many exceptional sets. In the second case we have

$$(a_2a^{-1})^h = a_1$$

The elements h here lie in a coset of $C_G(a_1)$. Now if $a_1 \in C \cap \Delta$ and $a_1 \neq 1$, then $a_1 \notin N$, so $C_G(a_1)$ has infinite index in G and the relevant h lie in finitely many exceptional sets. So for appropriate h, since we cannot have $C \subseteq \Delta$ we must have

$$C \subseteq \{1\} \cup h^{-1}\Delta$$

Then translating by h we may take $C \subseteq \Delta \cup \{h\} \cong B$, and we have a contradiction.

Case II. $C \subseteq \Delta \cup \Delta^{-1}h$.

We consider the edges within $\Delta^{-1}h$. So suppose

$$(a_1^{-1}h)^{-1}(a_2^{-1}h) = (a_1a_2^{-1})^h \in \Delta'_R$$

This gives the following three possibilities with $a \in \Delta$.

$$\begin{aligned} (a_1a_2^{-1})^h &= a \\ (a_1a_2^{-1})^h &= h^{-1}a \\ (a_1a_2^{-1})^h &= a^{-1}h \end{aligned}$$

The second and third equations correspond to finitely many values of h, so we consider the first possibility. Since the left hand side is in Δ'_R, it follows that $a \notin N$, and as h is restricted to a coset of $C(a)$ this is again an exceptional set.

So we may suppose that the clique C contains a unique element $a^{-1}h$ with $a \in \Delta$. We now consider the edges $(a_1, a^{-1}h)$ in C, that is we suppose

$$a_1^{-1}a^{-1}h = (aa_1)^{-1}h \in \Delta'_R$$

The possibilities are as follows, with $a_2 \in \Delta$.

$$(aa_1)^{-1}h = a_2$$
$$(aa_1)^{-1}h = h^{-1}a_2$$
$$(aa_1)^{-1}h = a_2^{-1}h$$

This time the first two equations define exceptional sets, and the last possibility becomes

$$aa_1 = a_2$$

That is, $a(C \cap \Delta) \subseteq \Delta$. Thus translating by a, the clique aC has as its underlying set

$$a(C \cap \Delta) \cup \{h\}$$

which is included in $\Delta \cup \{h\}$ and thus embeds into B, a contradiction. □

4. Homogeneous Directed Graphs

We now consider constraints on regular actions of abelian groups on *digraphs* of Henson type. Our conclusion is that the situation is much as in the case of graphs of Henson type, but that a generalization to free amalgamation classes in general binary languages may be more subtle, in an interesting way.

As in §2, we place this discussion in the more general setting of *regular normal actions* introduced by Cameron.

If one restricts attention to Henson digraphs with triangle constraints (i.e., one or both of the tournaments of order 3 is forbidden, and nothing else) then it seems the analysis in [Ca00] still applies and one gets such actions, e.g. with $\mathbb{Z}$ acting. But our focus here is on the negative results, which in this context are applicable to uncountably many structures.

Proposition 3. *Let H be a countably infinite group and Γ the generic $\mathcal{T}$-free digraph, where $\mathcal{T}$ consists of an antichain of finite tournaments, at least one of which is of order at least 4. Then there is no regular normal action of H on Γ by isomorphisms.*

Proof. We suppose on the contrary we have an H-biinvariant structure isomorphic to Γ on H.

We take a forbidden tournament $T \in \mathcal{T}$ of order $n \geq 4$ and express it as

$$T = T_0 \cup \{a, b\}$$

with T_0 of order $n - 2$ and $a \to b$.

Take a copy of T_0 in Γ. We will look for elements $g, h \in H$ so that $T_0 \cup \{g, gh^{-1}\}$ is isomorphic to T over T_0 with g, gh^{-1} corresponding to a, b respectively.

The constraints on g, h are then the following.

- The type of g over T_0 is the type of a over T_0;
- The type of g over T_0h is the type of b over T_0; by right invariance we then have the type of gh^{-1} over T_0 equal to the type of b over T_0;
- We require $h \to 1$, so that by left invariance, $g \to gh^{-1}$.

So if we find such g, h then $T_0 \cup \{g, gh^{-1}\}$ will be a forbidden tournament and we arrive at a contradiction.

It is easy to rephrase all of this as a set of conditions on h, namely the first two sets of requirements on g should be consistent with the constraints of Γ, and the third constraint should apply.

This gives us the following requirements on h.

- T_0 and T_0h are disjoint;
- The required structure on $T_0 \cup T_0h \cup \{g\}$ does not contain a forbidden tournament;
- $h \to 1$.

We modify the second condition and aim at the following.

- T_0 and T_0h are disjoint;
- There are no arcs between T_0 and T_0h apart from the arcs (th, t) with $t \in T_0$;
- $h \to 1$.

We need to show that such an h exists, and that with such a choice of h, the corresponding configuration $T_0 \cup T_0h \cup \{g\}$ contains no forbidden tournament.

The condition that T_0 and T_0h be disjoint excludes only finitely many values of h. The other two conditions on h amount to the following, after applying appropriate left translations by elements of T_0.

- $(t_1^{-1}t_2, h)$ is never an arc;
- $(h, t_1^{-1}t_2)$ is an arc if and only if $t_1 = t_2$.

This is the description of a digraph structure on $T_0^{-1}T_0 \cup \{h\}$ for which any subtournament with more than two vertices is contained in $T_0^{-1}T_0 \subseteq H$, and hence embeds into Γ. Thus the conditions on the element h are consistent, and have infinitely many realizations in H. Therefore we can meet all three conditions.

Now we return to a consideration of the tournaments embedding into $T_0 \cup T_0h \cup \{g\}$. Those which lie in $T_0 \cup \{g\}$ or $T_0h \cup \{g\}$ embed into Γ since they are proper subtournaments of T, and $\mathcal{T}$ is an antichain.

Any others will be contained in a tournament of order 3 with vertices

$$\{t, th, g\}$$

with $t \in T_0$, $th \to t$, and the orientation of (t, g) and (th, g) determined by the orientations of (t, a) and (t, b) respectively.

As $th \to t$ and $a \to b$, one may see that an oriented 3-cycle (th, t, g) would correspond to an oriented 3-cycle (t, a, b) in T, and therefore a linearly ordered triple would also correspond to a linearly ordered triple. (Here we seem to be using some particular statement about tournaments of order 3.) So this is again a proper subtournament of T, and embeds in Γ. Thus we have all required conditions. $\square$

5. Homogeneous Metric Spaces

A number of results and problems concerning regular actions on Urysohn space are found in [Ca00]. One may consider Urysohn space, particularly the countable rational-valued version, as a close relative of the random graph; and the countable integer-valued version is an even closer relative. This last is in fact a *metrically homogeneous graph,* that is a graph which is homogeneous when viewed as a metric space in the graph metric.

One noteworthy problem raised in [Ca00] is the following.

Problem 4. *Which abelian groups act regularly on integer Urysohn space? In particular, which elementary abelian groups act regularly on integer Urysohn space?*

Cameron and Vershik showed [CV06] that an elementary abelian 3-group cannot act regularly on integer Urysohn space (or on rational Urysohn space, or any dense subset of Urysohn space), but that an elementary abelian 2-group can act regularly. The remaining cases are open.

From our present point of view it is natural to consider *Henson variations* on integer valued Urysohn space. In their simplest form, these are defined in essentially the same way as the Henson graphs, by forbidding K_n where now K_n is a metric space in which all distances equal 1. It is also natural to consider the case of bounded diameter. So we have the integer Urysohn space $\mathbb{U}_\delta$ of diameter δ and the Henson-Urysohn space $\mathbb{U}_{\delta,n}$, the generic K_n-free graph of diameter δ (viewed as an integer-valued metric space). It turns out that in the bounded diameter case there is a richer class of Henson variations $\mathbb{U}_{\delta,\mathcal{S}}$ where $\mathcal{S}$ is any set of $(1,d)$-spaces, that is spaces in which only the minimal and maximal distances 1 and δ occur. That is, $\mathbb{U}_{\delta,\mathcal{S}}$ is the generic $\mathcal{S}$-free integer metric space of diameter δ.

There is also a conjectured classification of all metrically homogeneous graphs [Ch11], in which the Henson variations play a considerable role. While that classification has not been proved complete, it can serve for the present as a catalog of all known examples.

These structures do not have the free amalgamation property required for Proposition 2, but one might expect their behavior to have some similarities with Henson graph case. So it is natural to ask the following.

Problem 5. *Which of the known metrically homogeneous graphs*

- *are Cayley graphs;*
- *admit regular $\mathbb{Z}$-actions (or abelian actions, or regular normal actions);*
- *admit regular nilpotent actions?*

And it is also natural to consider actions by elementary abelian p-groups, at least for $p = 2, 3$.

Cameron took up the question of regular normal actions on $\mathbb{U}_{\delta,n}$ in §9 of [Ca00], with positive results for $n \leq 3$ and negative results for $n \geq 4$. In favorable cases ($n \leq 3$) a precise characterization of the groups acting regularly seems still far away.

We will give a modest generalization of the result for $n \geq 4$ to the more general class of Henson-Urysohn graphs $\mathbb{U}_{\delta,\mathcal{S}}$ where the constraint set $\mathcal{S}$ contains some space of order at least 4. We expect similar things for the more general class of Henson-type graphs allowing side conditions on triangles which is delineated in [Ch11], but we have not looked into this.

Proposition 4 (Metrically Homogeneous Henson Graphs)**.** *Suppose $\delta < \infty$ and $\mathcal{S}$ is a set of finite $(1, d)$-metric spaces, none of which embeds isometrically into any other. Suppose in addition $\mathcal{S}$ contains some space of order at least 4. Then there is no regular normal action of a group G on the corresponding graph $\mathbb{U}_{\delta,\mathcal{S}}$, namely the generic $\mathcal{S}$-free metrically homogeneous graph of diameter δ.*

Proof. We suppose we have such a regular normal action of H.

We fix $S \in \mathcal{S}$ with $|S| \geq 4$, and we write $S = S_0 \cup \{a, b\}$, where we choose the pair a, b so that $d(a, b) = 1$ if possible; otherwise, S is a δ-clique.

Claim. There is an element h satisfying

(1) S_0, $S_0 h$ are disjoint;

(2) For $s_1, s_2 \in S_0$, the distance from s_1 to $s_2 h$ is $\begin{cases} d(a,b) & \text{if } s_1 = s_2; \\ 2 & \text{if } d(s_1, s_2) = 1; \\ \delta - 1 & \text{if } d(s_1, s_2) = \delta. \end{cases}$

We express condition (2) in terms of the type of h over $S_0^{-1} S_0$, as follows.

$$\begin{aligned} d(1, h) &= d(a, b); \\ d(s_2^{-1} s_1, h) &= 2 \text{ if } d(s_1, s_2) = 1; \\ d(s_2^{-1} s_1, h) &= \delta - 1 \text{ if } d(s_1, s_2) = \delta. \end{aligned}$$

Note that if $s_2^{-1} s_1 = s_3^{-1} s_4$ then $d(s_1, s_2) = d(s_3, s_4)$, so these specifications are at least meaningful. We must check that these conditions respect the triangle inequality in $(S_0^{-1} S_0) \cup \{h\}$, for triangles containing h.

If $\mathcal{S}$ is a δ-clique then all distances $d(t, h)$ are δ or $\delta - 1$, for $t \in S_0^{-1} S_0$, and the claim is clear. So suppose $d(a, b) = 1$.

If the triangle in question contains 1 and h, that is, has the form $(1, h, s_2^{-1} s_1)$ with $s_1 \neq s_2$, then $d(s_2^{-1} s_1, h) = d(s_2^{-1} s_1, 1) \pm 1$, and the triangle inequality is clear.

Now suppose the triangle has the form $(h, s_2^{-1} s_1, s_4^{-1} s_3)$ with $s_1 \neq s_2$ and $s_3 \neq s_4$. Write $t = s_2^{-1} s_1$, $t' = s_4^{-1} s_3$. If $d(s_1, s_2) = d(s_3, s_4)$ then $d(h, t) = d(h, t')$ and the triangle inequality holds for (h, t, t').

So we may may suppose $d(s_1, s_2) = 1$ and $d(s_3, s_4) = \delta$. So $d(1, t) = 1$ and $d(1, t') = \delta$. Therefore $d(t, t') \geq \delta - 1$. As $d(h, t) = 2$ and $d(h, t') = \delta - 1$ we again have the triangle inequality.

Since these conditions are consistent, condition (2) has infinitely many solutions, while condition (1) excludes finitely many solutions. So the desired element h exists and the claim is proved.

Next we require an element g satisfying the following.

$$d(g,s) = d(a,s) \text{ for } s \in S_0;$$
$$d(g,sh) = d(b,s) \text{ for } s \in S_0h.$$

If this is achieved, then $S_0 \cup \{g, gh^{-1}\}$ is the desired copy of S, for a contradiction. So now we need to check the triangle inequality in $S_0 \cup (S_0h) \cup \{g\}$, and also the absence of forbidden $(1,\delta)$-subspaces. But there are no nontrivial $(1,\delta)$-subspaces except those embedding in $S_0 \cup \{g\}$ and $S_0h \cup \{g\}$, which are isometric to proper subspaces of S. So we may confine ourselves to the triangle inequality.

Again there are two cases to consider: either $\mathcal{S}$ is a δ-clique, or $d(a,b) = 1$.

If $\mathcal{S}$ is a δ-clique then all distances in $S_0 \cup (S_0h) \cup \{g\}$ are δ or $\delta - 1$, and the claim is clear. So we suppose $d(a,b) = 1$.

As $S_0 \cup \{g\}$ and $(S_0h) \cup \{g\}$ both embed properly into S, the only triangles of interest are those of the form

$$(g, s_1, s_2h)$$

with s_1, s_2 in S_0.

If $d(s_1,s_2) \leq 1$ then as $d(a,b) = 1$ we get $d(a,s_1) = d(b,s_2)$, hence $d(g,a_1) = d(g,s_2)$, and the triangle inequality holds.

If $d(s_1,s_2) = \delta$ then $d(s_1,s_2h) = \delta - 1$ while $d(g,s_1)$ and $d(g,s_2)$ are not both equal to 1. Thus we have a triangle of type $(\delta-1,\delta,\delta)$ or $(\delta-1,\delta,1)$, and the triangle inequality holds.

This completes the construction: the element g exists, and gives a contradiction. □

The typical known metrically homogeneous graph is obtained as follows.

- Specify the diameter δ
- Forbid some triangles (metric spaces with 3 points) in a fairly complicated way, depending on 4 auxiliary parameters, three of which control the triangles of odd perimeter, and one of which controls the triangles of even perimeter;
- Add some Henson constraints

Various numerical constraints must be met for the corresponding metrically homogeneous graph to exist [Ch11].

It is reasonable to expect that the results of [Hen71, Ca00], will pass to all such examples: namely everything should depend on whether there is a minimal forbidden $(1,d)$-space of order greater than 3, with negative results in that case, and positive results otherwise. But the constraints on triangles and the associated amalgamation process are quite complicated in general, so this remains unclear.

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