L^*-GROUPS OF ODD TYPE WITH RESTRICTED 2-TORAL ACTIONS I. HIGH Prüfer 2-RANK (September 2022)

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Abstract. This is the first of a series of papers devoted to the algebraicity conjecture for simple groups of finite Morley rank of odd type. One the hand, we aim to extend the existing theory for K^*-groups to a broader class of L^*-groups with a further assumption on definable automorphism groups. Some of the results aimed at in the series are new in the K^*-case as well. In particular we intend to give an identification theorem for PSp_4 in this broader context; this is new even in the K^*-case.

In the present paper the main result is an extension [Bur09] from the K^* setting to our version of the L^* setting. Namely, if G is an odd type simple group of finite Morley rank having Prüfer 2-rank at least 3 which satisfy both an L^* hypothesis suitable for odd type, and the condition that definable automorphism groups of definable simple sections of degenerate type are again of degenerate type, then the group G is either algebraic or has a definable strongly embedded subgroup.

We also develop the general theory, notably the signalizer functor theory and a theory of algebraic components whose unipotent subgroups are strongly unipotent in a model theoretic sense, in a form suitable for further applications, laying the groundwork for the identification of PSp_4, and for identification of G_2 in favorable cases (e.g., over fields not of characteristic 3).

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1. Introduction

1.1. **The Odd Type Algebraicity Conjecture.** The Algebraicity Conjecture for simple groups of finite Morley rank, also known as the Cherlin-Zilber conjecture, states that connected simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields.

The Sylow 2-subgroup in groups of finite Morley rank has a subgroup of finite index of the form

$$U * T$$

with U definable, connected, of finite exponent (2-unipotent), and T a divisible abelian 2-group (2-torus). The group is said to have *even, odd, mixed or* degenerate type according as T is trivial, or U is trivial, or neither is trivial, or both are, respectively. Odd type includes algebraic groups over algebraically closed fields of any characteristic other than 2, including characteristic 0. These notions are independent of the choice of Sylow 2-subgroup as they are conjugate in this context.

The Borovik program of transferring methods from finite group theory to this setting has led to considerable progress, notably when the subgroup U is non-trivial.

Even & Mixed Type Theorem ([ABC08]).

There are no connected simple groups of finite Morley and mixed type.

Those of even type are algebraic; more precisely, they are Chevalley groups over algebraically closed fields of characteristic 2.

It is noteworthy that this result is proved with no prior classification of groups of degenerate type, which may in principle appear as sections. From the point of view of finite group theory, this would correspond to classifying groups of characteristic 2 type without first proving the Odd Order Theorem!

For the algebraicity problem, degenerate type is much more problematic. However, a mix of methods from finite group theory (both the theory of finite simple groups, and black box group theory) with model theoretic ideas of a more geometric character suffices to eliminate the case of 2-Sylow subgroup which is finite, but non-trivial.

Fact 1.1 (Degenerate Type Theorem [BBC07])**.** *A connected group of finite Morley rank and degenerate type has trivial Sylow 2-subgroup.*

In other words, any connected group of finite Morley rank which contains an involution has an infinite Sylow 2-subgroup.[1]

[1] It also transpires that one can say something about useful about torsion in general, not just 2-torsion.

Our concern here is exclusively with *odd type,* or equivalently, the case of *finite and non-zero* 2*-rank* (bearing in mind Fact 1.1). Here there is also a large body of work, much of it inspired by techniques of finite simple group theory. So our subject is the following.

Odd Type Algebraicity Conjecture. A connected simple group of finite Morley rank of odd type is a Chevalley group over an algebraically closed field of characteristic other than 2.

If one combines this conjecture with the known results one can phrase the conjecture more generally, and more usefully for the sake of applications, as follows.

Non-Degenerate Type Algebraicity Conjecture. A connected simple group of finite Morley rank which contains an involution is a Chevalley group over an algebraically closed field.

More generally still, the conjecture can be stated as follows: for any connected group G of finite Morley rank, if $\hat{O}(G)$ denotes the largest connected definable normal subgroup of G whose simple sections are all of degenerate type, then $G/\hat{O}(G)$ is a central product of Chevalley groups over algebraically closed fields.

We explain this last point, using structural notions familiar in finite group theory whose analogs in our setting will be reviewed below, as well as the more geometrical notion of *connected component.* We also make use of the theory of central extensions of simple algebraic groups in the category of groups of finite Morley rank, and some consequences of the general theory of automorphisms of simple algebraic groups, specialized to the case of finite Morley rank and definable automorphism groups.

From odd type to non-degenerate type. After factoring out $\hat{O}(G)$ we have $F^{*\circ}(\bar{G}) = E(\bar{G})$. Assuming the Odd Type Conjecture, this is a product of Chevalley groups over algebraically closed fields; this point relies on the theory of central extensions. It then follows from the theory of definable automorphism groups of simple algebraic groups that $E(\bar{G})$ has finite index in $\bar{G}$, and hence equals $\bar{G}$. □

1.1.1. *Thin, quasi-thin, and generic cases: Prüfer* 2*-rank.* Just as the theory of finite simple groups involves three substantially distinct cases, known as thin, quasi-thin, and generic, an entirely parallel—but more straightforward—division occurs in the setting of groups of finite Morley rank of odd type. In terms of the Algebraicity Conjecture these three layers of theory should correspond to Lie ranks 1, 2, or higher, respectively.

To make this case division precise in our setting, we consider the connected component T of a Sylow 2-subgroup. This is a divisible abelian group of finite 2-rank, and the thin, quasi-thin, and generic cases correspond to this 2-rank being 1, 2, or higher, respectively. Indeed, T will be the product of a finite number of so-called "quasi-cyclic" or "Prüfer" 2-groups, and the number of factors is its 2-rank, which is also called the Prüfer 2-rank of G. In a simple algebraic group of odd type, this is the Lie rank.

In the present paper we will develop methods which are helpful even in Prüfer rank 2, as long as the 2-rank is at least 3. But the dividing line between Prüfer rank 2 and 3 remains very sharp and when one comes o apply these methods, this can be done in a uniform way in higher Prüfer rank and only in a very ad hoc way in Prüfer rank 2. So for the applications, we confine ourselves here to the generic case. We intend to return to the subject in succeeding articles.

Whether one is working with finite simple groups or with simple groups of finite Morley rank, there is another important case division which is largely independent of the thin/quasi-thin/generic division. Namely, there are special configurations called "uniqueness cases," the best known being the case of a "strongly embedded" subgroup. We will have a good deal to say about such uniqueness cases further on, as well.

1.1.2. *K^* and L^*.* Most results on finite simple groups of odd type assume a strong inductive hypothesis: the group in question is a K^*-group. This means that all proper connected definable simple sections are algebraic. Here K stands for "known," and unlike the finite case, there are no known sporadic, or even twisted, groups.[2] Thus one aims here to refine a catalog of "minimal configurations" which must be either eliminated in some fashion, or may possibly suggest real phenomena to be explored via Hrushovski constructions, as in the case of bad fields, or by some other approach.

However, it is not so helpful to list "degenerate type" as a possible obstruction to the proof of the Odd Type Conjecture. One would prefer to adapt the inductive hypothesis to the case at hand. Ideally, this would simply be the "odd type L^*-hypothesis:" any proper definable section of odd type is algebraic. A similar approach worked very well in the treatment of even type (and less directly, also mixed type). But as we will explain next, to get an odd type theory at all comparable to the even type L^* theory we will need to impose another fairly

[2]The closest known relatives of the latter would be certain simple stable but not superstable groups of an exceptional type associated with Moufang quadrangles in characteristic 2.

strong hypothesis. On the other hand, the effect of doing so is to refine and clarify very considerably our list of "minimal configurations" representing potential obstructions to the Odd Type Conjecture.

In this connection, we note that one nice result in the K^* setting is that the case of "strong embedding" falls under the thin case, and therefore can be set aside much of the time. This result becomes very difficult in a more general setting as it passes through the theory of minimal simple groups, which tends to rely heavily on the theory of solvable groups. If one allows degenerate type sections matters become considerably more complex.

1.2. **Altınel's lemma and** NTA_2**.** As shown in [ABC08], one can prove the Even and Mixed Type Theorems by an inductive strategy even without a great deal of information about potential non-algebraic simple groups of finite Morley rank of degenerate type.

In both the even and mixed type settings one simply replaces the notion of K^*-group by the more general notion of L^*-group: a group of finite Morley rank whose proper connected definable simple sections *of even type* are algebraic.

To make this work, it suffices to have *Altınel's Lemma,* which says that a group of finite Morley rank of degenerate type (or even, finite 2-rank) admits no faithful definable action by an infinite elementary abelian 2-subgroup. This has the effect of uncoupling the degenerate case from the inductive analysis in even type.

Then one first proves the even type classification by analyzing the structure of L^*-groups of even type inductively. After that one can eliminate a connected simple group of mixed type by considering a minimal example and using the structure of the even type sections.

We will consider straightforward analogs of these two ingredients—the L^*-notion and Altınel's Lemma—which are suitable for the context of groups of odd type.

Definition 1.2. A group G of finite Morley rank and odd type is an L^*-group (in the odd type sense) if every proper definable simple section of odd type is a Chevalley group over an algebraically closed field of characteristic other than 2.

This definition clashes with the traditional use of the term L^* given above, but on the other hand, now that we have the Even and Mixed Type Theorems we know that all groups of finite Morley rank are L^*-groups in that earlier sense, so we are not likely to encounter the term often in that sense going forward.

What is a suitable version of Altınel's Lemma in odd type? This would concern faithful definable actions of groups containing a non-trivial 2-torus (divisible abelian 2-group) on groups of degenerate type. Unfortunately, we cannot simply rule these out, since the multiplicative groups of a field acts on its additive group. A more appropriate analog of Altınel's Lemma in the odd type setting would be the absence of faithful definable actions of groups of odd type on *simple* groups of degenerate type. This is conjectural and difficult, and the question has been a subject of considerable interest in one form or another (also in the broader context of definable involutive automorphisms of degenerate type [DJ16]).

Automorphism Conjecture. A simple group of finite Morley rank of degenerate type has no definable odd type group of automorphisms; in other words, no non-trivial divisible abelian 2-group can be contained in a definable group acting faithfully on a simple group of degenerate type.

We will consider the class of L^*-groups whose sections satisfy this version of Altınel's conjecture.

Definition 1.3. A group G of finite Morley rank satisfies the condition NTA_2 if every definable section of G which acts faithfully and definably on a definable simple section of G of degenerate type, is itself of degenerate type.

Here the notation NTA_2 is intended to suggest the phrase "no 2-toral automorphisms" in a definable setting.

How reasonable is the condition NTA_2? Given that as yet no connected simple groups of degenerate type have been constructed, there are no formal constraints on what may be conjectured.

This particular condition strikes us as having some considerable degree of plausibility. It is similar in some ways to the conjecture that the Algebraicity Conjecture itself holds in Prüfer rank 1, with inner automorphisms replaced by outer automorphisms. Each of these is a very hard problem.

The algebraicity conjecture in degenerate type is viewed as doubtful. The automorphism conjecture seems distinctly more robust, and it can be proved in some noteworthy, though quite special, cases. In particular, the Automorphism Conjecture holds for connected simple bad groups.

A *bad group* is a non-solvable connected group of finite Morley rank all of whose proper definable connected subgroups are nilpotent. An argument of Delahan and Nesin, given in [BN94, Prop. 13.4], shows that bad group admits no definable automorphism of order 2; Jaligot [Jal01, Prop. 6.1] notes that the argument applies to a somewhat broader class of minimal connected simple groups (full Frobenius). In [BB08] it is shown that a similar argument applies within linear groups to degenerate type sections. This assumes more than linearity of the section itself; still, linearity is a condition which sometimes arises from the internal structure of a group.

Involutory automorphisms and actions by 2-tori are also a central theme in the analysis of [DJ16]. For connected minimal simple groups of degenerate type, a connected group of odd type acting definably and faithfully contains no 4-group, and if it contains an involution then the centralizer of that involution is a self-normalizing Borel subgroup of G. In other words, when one restricts attention to minimal connected simple groups of degenerate type, the configurations which arise are quite similar to those which arise in minimal simple groups of Prüfer rank 1, and the two cases can be put into a common framework. There is nothing here that suggests an approach to a proof of the Automorphism Conjecture in general, but the problem lies naturally within an already existing framework.

In the present paper we will show that an odd type simple L^*-group of finite Morley rank and generic type satisfying condition NTA_2 is either algebraic or has a strongly embedded group. In other words, minimal obstructions to the Odd Type conjecture fall into one of the following classes.

(1) Simple groups of degenerate type with a definable group of automorphisms of odd type.
(2) Groups of Prüfer rank at most 2.
(3) Groups with a strongly embedded subgroup.

We expect to refine this in subsequent articles so that in Prüfer rank 2 we come down either to 2-rank 2 or to some exceptional configurations reminiscent of G_2 in characteristic 3 where the 2-rank is 3.

1.3. **The main theorem.** We now state our main result formally, and give the necessary technical definitions.

High Prüfer Rank Theorem (5.1). Let G be a simple L^*-group of finite Morley rank of odd type with Prüfer 2-rank at least three which satisfies NTA_2. Then one of the following applies.

(1) G is a Chevalley group over an algebraically closed field of characteristic other than 2; or
(2) G has a proper definable strongly embedded subgroup.

We note that by prior work case (2) has various special features; notably, the 2-rank and Prüfer 2-rank must coincide. We will discuss the sharper version of this known under a K^* hypothesis below.

Prior work reduces the proof of this result to the verification of a set of conditions which are reviewed in §5.3. Thus the bulk of the present paper is devoted to the verification of those conditions. Along the way the machinery (signalizer functor theory and some applications) is developed in greater generality than is required here, so as to prepare for the continuation in the case of Prüfer 2-rank 2.

Now we give precise definitions.

Definition 1.4. Let G be a group of finite Morley rank (normally, but not exclusively, connected, simple, and of odd type).

G is a *K-group* if every definable connected simple section is algebraic (more explicitly, a Chevalley group over an algebraically closed field).

G is an *L-group* if every definable connected simple section containing an involution is algebraic.

G is a *D-group* if no definable connected simple section contains an involution.

G is a *K^*-group, L^*-group,* or *D^*-group*, respectively, if the corresponding condition applies to every *proper* definable simple section.

In the absence of degenerate type simple sections the notions of L-group, L^*-group, D-group, and D^*-group reduce to more conventional notions as follows. as follows.

K-group notions and their L-group counterparts

Classical	K	K^*	solvable	minimal connected simple
Variant	L	L^*	D	D^*

Now we discuss strong embedding and some related concepts.

Definition 1.5. Let G be a group of finite Morley rank.

1. A subgroup M of G is *strongly embedded* in G if M is a proper subgroup containing an involution, and the normalizer of eery 2-subgroup of M lies in M.

2. The 2-*generated core* $\Gamma_{S,2}$ of G relative to a Sylow 2-subgroup S of G is the smallest subgroup of G containing the normalizer of every elementary abelian subgroup of S of rank 2.

The 2-generated core is well-defined up to conjugacy, so in particular the condition that the 2-generated core be a proper subgroup is independent of the choice of Sylow 2-subgroup.

Clearly if G has a definable strongly embedded subgroup then it contains a Sylow 2-subgroup of G and also the associated 2-generated core, and so G has a proper 2-generated core in this case.

The fundamental case division in the proof of Theorem 5.1 goes according as G has a proper 2-generated core or not. The first branch leads to a strongly embedded subgroup and the other branch leads to identification.

A High Prüfer Rank Theorem was proved for the K^* setting in [Bur09, BBN08]; this in turn builds on the tame case treated earlier, with the strongly embedded case eliminated via [BCJ07]. As far as the first two steps are concerned, we follow the same strategy closely in the L^* setting.[3]

The results for the K^* setting are as follows.

Theorem A. Generic Trichotomy Theorem[4] ([Bur09]). Let G be a connected simple K^*-group of finite Morley rank and odd type with Prüfer 2-rank at least 3. Then one of the following applies.

- G is a Chevalley group over an algebraically closed field of characteristic other than 2; or
- G has a proper 2-generated core.

Theorem B. Strong Embedding Theorem I ([BBN08]). Let G be a connected simple K^*-group of finite Morley rank and odd type with normal 2-rank at least 3 and Prüfer 2-rank at least 2. Suppose that G has a proper 2-generated core M. Then G is a minimal connected simple group, and M is strongly embedded.

Theorem C. Strong Embedding Theorem II ([BCJ07], cf. [ABF13, §6]). Let G be a minimal connected simple group of finite Morley rank and of odd type. Suppose that G contains a proper definable strongly embedded subgroup. Then G has Prüfer 2-rank one.

Thus in the K^* setting we wind up with the following, as discussed in [Bur09].

Odd Type Theorem, Generic Case. A connected simple K^*-group of finite Morley rank and odd type with Prüfer 2-rank at least 3 is a Chevalley group over an algebraically closed field of characteristic not 2.

The second step above, the Strong Embedding Theorem, was already carried over to the L^* context in [BC08]; here the hypothesis NTA_2 is not needed.

[3]In this paper, at least, the term "L^* setting" is used loosely, to refer to the study of simple L^*-groups of odd type satisfying the condition NTA_2.

[4]This is either a Trichotomy Theorem or a Generic Dichotomy Theorem, depending on what one thinks of the non-generic case. We leave the name as it is.

Fact 1.6 ([BC08, Corollary 4.2 and Theorem 4.3]). *Let G be a connected simple L^*-group of finite Morley rank and odd type with*

$$m_2(G) \geq 3$$

and with proper 2-generated core $\Gamma_{S,2}$. Then $M = N(\Gamma_{S,2})$ is strongly embedded in G.

Here Corollary 4.2 of [BC08] connects the 2-generated core to yet another subgroup, namely the subgroup Γ_V associated with an elementary abelian 2-subgroup of 2-rank 2, defined as

$$\langle C^\circ(i) \,:\, i \in V^\times \rangle.$$

Namely, Corollary 4.2 gives $N(\Gamma_V) = \Gamma_{S,2}$, and then Theorem 4.3 works with $N(\Gamma_V)$.[5] Theorem 6.6 of [BC08] then gives more information about the strongly embedded case, but this is an aspect we do not pursue here. It has considerable importance in its own right.

The treatment of Theorem C in [BCJ07], as well as the alternative approach given in [ABF13], involves methods appropriate to the setting of minimal simple groups, where the theory of solvable groups plays a major role. There is a large gap between the theory of solvable groups and the theory of D-type groups, so generalization is problematic.

One particularly useful, though elementary, part of the solvable theory is the fact that a unipotent p-group for a prime p must fall into the Fitting subgroup of a solvable group. It might be of interest to extend the results known in the K^* setting to the context of L^*-groups of odd type which satisfy not only NTA_2 together with that the condition that no definable simple connected section of degenerate type contains a non-trivial unipotent p-subgroup, for any prime p. A more adventurous approach would require working only with restrictions on definable actions of p-tori for all p. In any case torsion elements will certainly come into play.

In the treatments of the Strong Embedding Theorem II there is a major case division according as involutions lie in the Fitting subgroup of the strongly embedded subgroup, or not. Each case leads to a contradiction in the K^* context. The former case can also be eliminated in the L^* context [BC22d].

[5] Corollary 4.2 was stated in excessive generality and one should be aware of that when quoting it. Namely, the lemma refers to a prime p but the proof requires $p = 2$ and the lemma fails more generally. Here the relevant prime is in fact 2. One can find detailed and useful discussions of this point in [Del12, Jal11].

1.4. **Discussion.** The results presented above show that the obstacles to a complete classification of connected simple groups of finite Morley rank with an involution are of the following kinds.

(1) The strongly embedded case in Prüfer 2-rank at least 3, with the group a D^*-group.
(2) Configurations with Prüfer rank at most 2, conjecturally corresponding to Chevalley groups of rank at most 2 (again with issues around strongly embedded subgroups, along with a variety of other issues);
(3) Hypothetical simple groups of degenerate type with definable automorphism groups containing an infinite abelian 2-subgroup, and Prüfer rank 1 cases; these may have a similar flavor at times.

In case (1) the Sylow 2-group is connected, and in particular the 2-rank coincides with the Prüfer rank (Fact 2.13). Elimination of this case seems very difficult but it should be attempted.

We intend to examine case (2) further in subsequent papers, under the assumption of 2-rank at least 3 (hence Prüfer 2-rank 2, by Fact 2.10). Under that hypothesis, the strongly embedded case is ruled out, so the aim becomes outright identification of the group as PSp_4 or G_2. Our work [BC22a, BC22b, BC22c] (in preparation) is intended to show that one arrives either at the expected identification, or else at a very specific configuration associated to groups resembling the group G_2 in characteristic 3. This configuration arises also in finite group theory but is eliminated in that setting using character theory.

1.4.1. *Structure of the paper.* After some lengthy preliminaries, we give the main results in Sections 4 and 5.

In §4 we study the structure of centralizers of involutions and their quasi-simple components, arriving at Proposition 4.6. The standing hypothesis in this section is that the 2-rank is at least 3. We intend to make considerable use of this material subsequently in the setting of Prüfer rank 2.

In §5 we proceed to the proof of the main theorem. Here we require the Prüfer rank to be at least 3 throughout. In the case of Prüfer rank 2 we will be forced to turn to more laborious methods.

1.4.2. *Remark.* Given the difficulties in eliminating simple groups with strongly embedded subgroups of high Prüfer rank, one might want to consider further what happens to the theory if such groups are allowed as definable sections, along with degenerate type groups.

2. General theory (background)

Here we review some of the general theory of groups of finite Morley rank on which we will rely. Some more specialized material of a similar character will be reviewed in the following section. Proofs are given when we lack a clear reference for the result as formulated here.

Points noted without reference are generally to be found in [BN94], and reviewed in [ABC08]. The conjugacy of Sylow 2-subgroups [BN94, Theorem 10.11] is often applied without explicit mention.

2.1. Π-tori and 2-ranks.

Definition 2.1. Let G be a group of finite Morley rank.

For p a prime, a *p-torus* is a divisible abelian p-subgroup of G. Such a subgroup will have finite p-rank.

A *Π-torus* is a divisible abelian torsion subgroup of G (Π denotes the set of all primes).

A *decent torus* is the definable hull of a Π-torus.

A *good torus* is a definable connected group such that each of its definable connected subgroups is a decent torus.

Fact 2.2 ([AB08, Thm. 1])**.** *Let T be a Π-torus in a connected group G of finite Morley rank. Then $C_G(T)$ is connected.*

Note that a simple group of finite Morley rank is of odd type if and only if the connected component of a 2-Sylow subgroup is a 2-torus.

Corollary 2.3. *Let G be a connected group of finite Morley rank and odd type. Then any finite normal subgroup A of a Sylow 2-subgroup S of G is contained in the maximal 2-torus T of S.*

Proof. As the definable hull $d(T)$ is connected and normalizes A, it follows that A centralizes T.

But the Sylow 2-subgroups of $C_G(T)/T$ are finite, hence trivial by Facts 1.1 and 2.2. Thus $A \leq T$. $\square$

Fact 2.4 ([BC08, Lemma 1.6])**.** *Let G be a 2-torus and i an involution acting on T. Then either i inverts T, or $C_T(i)$ is infinite.*

Fact 2.5 ([BC09, Theorem 3])**.** *Let G be a connected group of finite Morley rank and odd type. Then any involution in G lies in some 2-torus of G.*

In particular, this gives $i \in C_G^\circ(i)$ in this case.

Definition 2.6. Let G be a group of finite Morley rank. Two involutions $i, j \in I(G)$ are said to be *co-toral* if they lie in a 2-torus of G.

Lemma 2.7. *Let G be a group of finite Morley rank and odd type, and $i, j \in I(G)$. Then the following conditions are equivalent.*

(1) i, j are co-toral.
(2) $j \in C_G^\circ(i)$.

Proof. The forward direction is clear.

For the converse, suppose

$$j \in C_G^\circ(i).$$

Then by Fact 2.5 and hypothesis, both i and j lie in $C_G^\circ(i)$. Then Fact 2.5 applies also to $C_G^\circ(i)$, and hence there are maximal 2-tori T_i, T_j of $C_G^\circ(i)$ containing i and j respectively. These 2-tori are conjugate in $C_G^\circ(i)$, and hence $i \in T_j$. Thus i, j are co-toral. □

Fact 2.8 ([Ber01, BB04, §3.3] (Tate Modules))**.** *Let T_p be a p-torus of Prüfer p-rank n in a group of finite Morley rank. Then the endomorphism ring* $\mathrm{End}(T_p)$ *can be faithfully represented as the ring of $n \times n$ matrices over the p-adic integers $\mathbb{Z}_p$.*

More precisely, if T_p is represented as the direct limit

$$T_p = \lim_{\rightarrow} T_p[p^n]$$

where $T_p[n]$ is the subgroup annihilated by multiplication by n, then we identify $\mathrm{End}(T_p)$ *with the endomorphism ring of*

$$\hat{T}_p = \lim_{\leftarrow} T_p[p^n]$$

with maps given by multiplication by p.

E.g., in the case of Prüfer rank 1, the action of $\mathbb{Z}_p = \lim\limits_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$ on $\mathbb{Z}(p^\infty) = \lim\limits_{\rightarrow} \mathbb{Z}/p^n\mathbb{Z}$ is induced by the action by multiplication of $\mathbb{Z}/p^n\mathbb{Z}$ on itself.

Definition 2.9 (2-ranks)**.** Let G be a group of finite Morley rank.

- The *2-rank* $m_2(G)$ is the maximum rank of an elementary abelian 2-subgroup of G.
- The *normal 2-rank* $n_2(G)$ is the maximal rank of a normal elementary abelian subgroup of a Sylow 2-subgroup of G.
- The *Prüfer* 2-rank $\mathrm{pr}_2(G)$ is the maximum 2-rank of a 2-torus of G.

Remarks. If S is a Sylow 2-subgroup of G then $m_2(G) = m_2(S)$, $n_2(G) = n_2(S)$, and $\mathrm{pr}_2(G) = \mathrm{pr}_2(S)$.

In odd type, these various ranks are all finite and nonzero, with

$$m_2 \geq n_2 \geq \mathrm{pr}_2,$$

since the Prüfer rank of a 2-torus is its 2-rank.

In fact when G is connected of finite Morley rank and odd type, then

$$n_2(G) = \mathrm{pr}_2(G)$$

(Corollary 2.3).

Fact 2.10 ([BC08, Thm. 1.2])**.** *Let G be a connected group of finite Morley rank and odd type with*

$$m_2(G) \geq 3.$$

Then every elementary maximal elementary abelian 2-subgroup has 2-rank at least 3*.*

This important point was overstated in [BC08]. Namely, what we state here for the prime 2 was claimed for an arbitrary prime. In the proof given, the step just following the display labeled $(*)$ requires $p = 2$. See [Del12] for further discussion.

Fact 2.11 ([BC08, Lemma 1.11])**.** *Let G be a connected group of finite Morley rank and odd type with $m_2(G) \geq 3$. Then*

$$\mathrm{pr}_2(G) \geq 2.$$

Corollary 2.12. *Let G be a connected group of finite Morley rank and odd type with*

$$m_2(G) \geq 3,$$

and let T be a maximal 2*-torus of G.*

Then every elementary abelian 2-subgroup U contained in T can be extended to an elementary abelian 2*-group A of* 2*-rank at least* 3 *which normalizes T.*

Proof. We may suppose that $U = \Omega_1(T)$. By Fact 2.11, T has 2-rank at least 3. If $m_2(T) > 2$ then we take $A = U$. So we may suppose that T and U have 2-rank 2. By Fact 2.10 there is an elementary abelian 2-subgroup A of G of 2-rank 3 which contains U.

Consider a Sylow 2-subgroup S of $C(U)$ which contains A. Then the connected component S° is a maximal 2-torus of $C(U)$, and S° is conjugate to the 2-torus T in $C(U)$. As A normalizes S°, T is normalized by a conjugate of A containing U. $\square$

Fact 2.13 ([BC08, Lemma 1.15])**.** *A connected D-group has a connected Sylow 2-subgroup.*

In particular, in this case the 2-rank and Prüfer 2-rank will coincide.

There is another notion of 2-rank which becomes of particular importance when the Prüfer 2-rank is 2 in connection with the signalizer functor theory developed below (see Proposition 4.6). In the present paper we restrict ourselves to the case of Prüfer 2-rank at least 3, but in order to develop the signalizer functor theory at an appropriate level of generality for future applications we now deal with this point.

Definition 2.14. Let G be a group of finite Morley rank and odd type. The *co-toral 2-rank* $m_2^\circ(G)$ is defined as the maximal 2-rank of an elementary abelian 2-subgroup A such that the graph on $I(A)$ whose edges are the co-toral pairs of involutions in A is connected.

Thus

$$\mathrm{pr}_2(G) \leq m_2^\circ(G) \leq m_2(G)$$

E.g., for groups of type G_2 the Prüfer 2-rank is 2 while $m_2^\circ(G) = m_2(G) = 3$.

Lemma 2.15. *Let G be a group of finite Morley rank and odd type. Suppose $m_2(G) \geq 3$. Then $m_2^\circ(G) \geq 3$.*

More precisely, if U is an elementary abelian 2-group of rank 2 which is contained in a 2-torus, then there is an elementary abelian 2-subgroup A of 2-rank 3 containing U, and an involution $i \in I(U)$ which is co-toral with every involution of A.

Proof. By Fact 2.11 we have $\mathrm{pr}_2(G) \geq 2$, so the second statement is indeed a refinement of the first. Therefore we take up the second formulation, with $U \leq \Omega_1(T)$, where T is a maximal 2-torus and U has 2-rank 2.

If the Prüfer rank is at least 3 then we take $U \leq A \leq \Omega_1(T)$.

So we suppose now

$$\mathrm{pr}_2(G) = 2.$$

Claim 1. *Suppose $U \leq A$ with A an elementary abelian 2-group of rank 3, and $i \in I(U)$ is co-toral with some involution $j \in A \setminus U$. Then i is co-toral with all involutions of A.*

Our assumption can be expressed as follows.

$$U \leq C_G^\circ(i); \qquad\qquad A \cap C_G^\circ(i) \not\leq U,$$

and thus $A \leq C_G^\circ(i)$, which gives the claim.

So it suffices to find an involution $j \in C_G(U)$ which is co-toral with an involution of U, and then take $A = \langle U, j \rangle$.

By Fact 2.10 there is an elementary abelian 2-subgroup A of rank 3 which contains U.

We take $j \in A$, $j \notin U$, and let S be a Sylow 2-subgroup of $C_G(j)$ containing A. Then $R = S^\circ$ is a maximal 2-torus of $C_G(j)$ normalized by A. If R meets U nontrivially we are done, so suppose $U \cap R = 1$.

We consider the action of U on R.

Suppose some involution $i \in I(U)$ acts nontrivially on $\Omega_1(R)$. Then i does not invert or centralize R, and hence $C_R^\circ(i)$ is a 2-torus of Prüfer rank 1. As $C_{\Omega_1(R)}(i) = \langle j \rangle$, this 2-torus contains j. Thus $j \in C_G^\circ(i)$, and i, j are co-toral; we conclude that the group A and the involution i meet our conditions.

Now we may suppose that U centralizes $\Omega_1(R)$. We cannot have every involution of U inverting R, so some $i \in I(U)$ has infinite centralizer in R, by Fact 2.4. Let j' be an involution in $C_R^\circ(i)$ and replace A by $\langle U, j' \rangle$. Then this group meets our conditions. □

2.2. **Nilpotent groups.** A number of elementary properties of nilpotent groups can be sharpened in the context of finite Morley rank by considering connected components. In particular the normalizer condition takes on the following useful form.

Fact 2.16 ([ABC08, Corollary I.5.2])**.** *Let Q be a connected nilpotent group of finite Morley rank. Then $Z(Q)$ is infinite.*

Fact 2.17 ([ABC08, Proposition I.5.3])**.** *Let Q be a nilpotent group of finite Morley rank and P a definable subgroup of infinite index. Then P has infinite index in $N_Q(P)$.*

In practice we are interested in the case in which P is connected, and the conclusion becomes $N_Q^\circ(P) > P$.

Fact 2.18 ([BN94, Thm. 6.8, 6.9])**.** *Let H be a nilpotent group of finite Morley rank. Then*

$$H = B * D$$

where B and D are definable characteristic subgroups satisfying

- *B has bounded exponent;*
- *D is divisible.*

In particular, D is connected and $B \cap D$ is finite.

Furthermore, the torsion subgroup T of D is central in H and there is a decomposition

$$D = T \oplus D_0$$

with D_0 torsion free and divisible. Hence D' is torsion-free.

Remark 2.19. A nilpotent torsion group Q is the direct sum of its p-torsion subgroups Q_p with p varying over the primes. Hence in Fact 2.18, B is a finite direct sum

$$B = \bigoplus_p B_p$$

with B_p a p-group, for various primes p.

Definition 2.20. A group of finite Morley rank is Π-*unipotent* if it is connected, solvable, and has bounded exponent. If it is a p-group, it is said to be *p-unipotent.*

In [ABC08] Π-unipotence is called *unipotence.* As we will make use of variant notions of unipotence discussed below, we refine the terminology.

Fact 2.21. [ABC08, Lemma I.5.5] *A Π-unipotent group of finite Morley rank is nilpotent.*

Notation 2.22. Let G be a group of finite Morley rank. Then $F(G)$ is the Fitting subgroup (a definable subgroup) and $F^\circ(G)$ is its connected component.

2.3. **Solvable groups.**

Fact 2.23 ([ABC08, Lemma I.8.3, Cor. I.8.4])**.** *Let G be a connected solvable group of finite Morley rank.*

Then the quotient $G/F^\circ(G)$ is divisible abelian. Hence any Π-unipotent subgroup of G is contained in the Fitting subgroup.

Notation 2.24. Let G be a group of finite Morley rank.

- $\sigma(G)$ is the solvable radical and $\sigma^\circ(G)$ is its connected component.
- $O(G)$ denotes then the largest *connected* definable normal subgroup of G of degenerate type.
  We write $OF(G)$ and $O^\sigma(G)$ for $O(F(G))$ and $O(\sigma(G))$.

If the group G has no definable simple sections of degenerate type then $O(G) = O^\sigma(G)$.

Lemma 2.25. *Let H be a connected group of finite Morley rank and A a finite central subgroup. Suppose that $\bar{H} = H/A$ is a direct product*

$$\bar{H} = \bar{H}_1 \times \bar{H}_2,$$

and let H_1, H_2 be the preimages in H of $\bar{H}_1$, $\bar{H}_2$, respectively.

Then

(1) $O^\sigma H = O^\sigma H_1 * O^\sigma H_2$.

(2) $\overline{O^\sigma H} = O^\sigma \bar{H}$.

Proof. Let $\bar{K}_i$ be $O^\sigma \bar{H}_i$ and K_i the preimage of $\bar{K}_i$ in H, for $i = 1, 2$. We have

$$O^\sigma(\bar{H}) = \bar{K}_1 \times \bar{K}_2.$$

Now $\bar{K}_i^\circ = \bar{K}_i$. Furthermore K_i° is connected and of degenerate type, hence has no involutions; so $\bar{K}_i^\circ \subseteq O^\sigma H$. It follows that $K_i^\circ = O^\sigma(H_i)$ and $K_1 K_2$ covers $O^\sigma(\bar{H})$, so $O^\sigma(H) = K_1 K_2 = O^\sigma(H_1) * O^\sigma(H_2)$. $\square$

Definition 2.26. Let G be a group of finite Morley rank acting definably on an abelian group V.

V is *G-minimal* if there is no infinite proper definable G-invariant subgroup of A.

V is *G-irreducible* if there is no nontrivial proper definable G-invariant subgroup of A.

Lemma 2.27. *Let G be a group of finite Morley rank and H a connected solvable normal subgroup that acts trivially on each G-minimal section of $F(H)$. Then $H \leq F(G)$.*

Proof. Let V be a G-minimal subgroup of $Z(F^\circ(H))$. Then $V \leq Z(H)$. Factoring out V, we may conclude inductively. $\square$

Fact 2.28 ([ABC08, Proposition 4.11])**.** *Let G be a connected group of finite Morley rank acting definably, faithfully, and irreducibly on an abelian group V. Let $T \lhd G$ be abelian.*

Then the subring of $\mathrm{End}(V)$ *generated by the action of T is a G-definable field K, and V is a finite dimensional K-vector space on which G acts linearly.*

Lemma 2.29. *Let G be a connected group of finite Morley rank. Then $[G, \sigma^\circ(G)] \leq F^\circ(G)$.*

Proof.

Claim 1. *The commutator $[G, \sigma^\circ(G)]$ acts trivially on every G-minimal abelian section V of $F^\circ(G)$.*

As $F^\circ(G)$ is nilpotent and connected, it acts trivially on V. Furthermore, the action of the abelian quotient $\sigma^\circ(G)/F^\circ(G)$ on V commutes with the action of G, since that action is either trivial or else, by Fact 2.28, it gives rise to a field K with the action of G K-linear.

This proves the claim. Now by Lemma 2.27 it follows that $[G, \sigma^\circ(G)]$ is nilpotent and hence contained in $F^\circ(G)$. $\square$

Definition 2.30. We write $U_p(G)$ for the subgroup generated by all p-unipotent subgroups of G.

Note that for simple algebraic groups, $U_p(G)$ will be G if the characteristic is p, and trivial otherwise.

Fact 2.31 ([BN94, Cor. 6.20; Theorem 9.29])**.** *Let G be a connected solvable group of finite Morley rank. Then*

- *$U_p(G) \leq F^\circ(G)$ is p-unipotent.*
- *Any maximal p-subgroup P of G is connected, and $P = U_p(G) * T$ with T a divisible abelian p-group.*

Remark 2.32. In the context of solvable groups of finite Morley rank, there is a good theory of Sylow p-subgroups (and even Hall subgroups). If one works more generally, it is not even clear what the appropriate definitions are: rather than looking at maximal p-subgroups, it seems more useful to consider maximal locally finite p-subgroups, in contexts where the notions differ. But in the solvable case these coincide.

2.4. **Notions of unipotence.**

We will require a "characteristic zero" analog of the operators U_p considered above. These are introduced in [Bur09]; a more refined variant is introduced by Frécon [Fré06].

Definition 2.33. Let A be a connected abelian group of finite Morley rank.

The group A is *indecomposable* if it has a unique maximal proper definable connected subgroup, denoted $J(A)$.

The *reduced rank* $\bar{\mathrm{r}}(A)$ of A is the Morley rank of the quotient $A/J(A)$, i.e. $\bar{\mathrm{r}}(A) = \mathrm{rk}(A/J(A))$.

For a group G of finite Morley rank, and any integer r, we define

$$U_{(0,r)}(G) = \left\langle A \leq G \;\middle|\; \begin{array}{c} A \text{ is a definable indecomposable group,} \\ \bar{\mathrm{r}}(A) = r, \text{ and } A/J(A) \text{ is torsion-free} \end{array} \right\rangle.$$

We say that G is a $U_{(0,r)}$*-group* if $U_{0,r}(G) = G$.

Set $\bar{\mathrm{r}}_0(G) = \max\{r : U_{(0,r)}(G) \neq 1\}$, the maximal (relevant) unipotence rank. One may take this to be 0 if no such rank exists; similarly, $U_{0,0}(G)$ is the trivial subgroup.

For simple algebraic groups G in characteristic zero, possibly with additional structure (but of finite Morley rank) we will have $\bar{\mathrm{r}}_0(G)$ equal to the rank of the base field (Corollary 2.45).

Fact 2.34 ([Bur09, Lemma 2.11])**.** *Let $f : G :\to H$ be a definable homomorphism between two groups of finite Morley rank. Then*

(1) (Push-forward) $f[U_{(0,r)}(G)] \leq U_{(0,r)}(H)$ is a $U_{(0,r)}$ -group.

(2) (Pull-back) If $U_{(0,r)}(H) \leq f[G]$ *then* $f[U_{(0,r)}(G)] = U_{(0,r)}(H)$.

We view the reduced rank parameter r as a *scale of unipotence*, with larger values being more unipotent. By the following fact, analogous to Fact 2.31, the "most unipotent" subgroups of a solvable group are nilpotent.

Fact 2.35 ([Bur09, Theorem 2.16])**.** *Let G be a connected solvable group of finite Morley rank. Then $U_{0,\bar{r}_0(G)}(G) \leq F(G)$.*

Fact 2.36 ([Bur06, Corollary 4.6])**.** *Let $G = HT$ be a group of finite Morley rank, with H and T definable and nilpotent, and $H \lhd G$. Suppose that T is a $U_{(0,r)}$-group for some $r \geq \bar{r}_0(H)$. Then G is nilpotent.*

If none of our notions of unipotence apply, we have tori, in a suitable abstract sense.

Fact 2.37 ([Bur04, Thm. 2.19]; [Bur09, Thm. 2.15])**.** *Let H be a connected solvable group of finite Morley rank. Suppose*

$$\begin{aligned} U_p(H) &= 1 \textit{ for all primes } p \\ U_{0,\bar{r}_0(H)}(H) &= 1 \textit{ for all } r \end{aligned}$$

Then H is a good torus, i.e. every definable subgroup of H is the definable hull of its torsion subgroup.

The operators $U_{0,r}$ can be also be used to give a refined decomposition of a definable nilpotent subgroup.

Fact 2.38 ([Bur06, Cor. 3.6]; [Bur04, Thm. 2.31])**.** *Let G be a nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups $D, B \leq G$ where D is divisible and B is connected of bounded exponent. Let T be the torsion part of D. Then we have decompositions of D and B into central products as follows.*

$$\begin{aligned} D &= d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots \\ B &= \Pi_p B_p \textit{ (p-torsion subgroups).} \end{aligned}$$

The overlap between the factors of this decomposition is far from clear, as our definitions involve a kind of radical $J(A)$ whose structure is not controlled. But a result of Frécon casts some useful light on this.

Definition 2.39 ([Fré06])**.** Let H be a connected nilpotent group of finite Morley rank, and $r \geq 1$. Then H is a *homogeneous $U_{(0,r)}$-group* iff every connected definable subgroup K of H satisfies

$$K = U_{0,r}(K).$$

This is, clearly, a more robust notion of unipotence, when it is applicable.

Fact 2.40 ([Fré06, Thm. 4.11])**.** *Let G be a connected group of finite Morley rank acting definably on a nilpotent group H with $H = U_{0,r}(H)$. Then $[G,H]$ is a definable homogeneous $U_{0,r}$-group.*

Notation 2.41. Let H be a group of finite Morley rank. For $r \geq 1$ let $U^*_{0,r}$ denote the largest normal homogeneous $(0,r)$-unipotent subgroup of H, and

$$U^*(H) = \prod_{p>2} U_p(H) \cdot \prod_{r\geq 1} U^*_{0,r}(H).$$

Let $U(H)$ denote the largest connected definable nilpotent normal subgroup of H whose torsion subgroup has bounded exponent.

Notice that $U^*(H) \leq U(H) \leq O(H)$ and $U(H) = B * D$ where B has bounded exponent and D is torsion free.

Lemma 2.42. *Let G be a connected group of finite Morley rank and odd type.*

(1) If $U^*F(G) = 1$ then G centralizes $F(G)$.
(2) In general, $[G,F(G)] \leq UF(G) \leq OF(G)$.

Proof.
Ad 1. We take $F(G) = B * D$ as in Fact 2.38. As $U^*F(G) = 1$ we find that B is finite. So G centralizes B.

By Fact 2.40, the subgroups $[G, U_{0,r}(F(G))]$ are homogeneous $U_{0,r}$-groups, and hence trivial by our hypothesis. So the decomposition of D reduces to $D = d(T)$ with T a central Π-torus. Thus G centralizes D and (1) follows.

Ad 2. We argue by induction on rank. If $U^*F(G) = 1$ then (1) applies. Otherwise, we set $\bar{G} = G/U^*F(G)$ we apply induction. Then

$$[\bar{G}, F(\bar{G})] \leq UF(\bar{G}).$$

Now $\overline{F(G)} \leq F(\bar{G})$, and the preimage H in G of $UF(\bar{G})$ is a normal solvable subgroup containing no Π-torus. Then

$$[G,F(G)] \leq (F(G) \cap H)^\circ \leq U(F(G)). \qquad \square$$

We will use the unipotence theory in connection with signalizer functor theory. This leads in particular to a consideration of the various unipotent radicals of the additive and multiplicative groups of a field.

In nonzero characteristic we need only the following.

Fact 2.43. *Let k be an infinite field of finite Morley rank of characteristic $p > 0$. Then the additive group k_+ is p-unipotent.*

This merely asserts that k_+ is connected, which is well known: indeed, the field k has Morley degree 1 and both k_+ and $k^\times$ are connected, whatever the language. (The rank may be greater than 1.)

We turn to the case of characteristic 0.

Fact 2.44 ([Poi87, Cor. 3.3])**.** *Let k be a field of finite Morley rank and characteristic 0. Then the additive group k_+ is minimal: there are no proper definable nontrivial subgroups.*

Corollary 2.45. *Let k be an field of finite Morley rank and characteristic 0, and $k_+, k^\times$ the additive and multiplicative groups. Then*

$$\bar{r}_0(k_+) = \mathrm{rk}(k) > \bar{r}_0(k^\times)$$

and k_+ is a $U_{0,\mathrm{rk}(k)}$-group.

Proof. By minimality

$$\bar{r}_0(k_+) = \mathrm{rk}(k).$$

On the other hand $k^\times$ is connected and contains torsion so

$$\bar{r}_0(k^\times) < \mathrm{rk}(k). \qquad \square$$

2.5. **"Relatively prime" actions.** We give two results on what may be called "relatively prime" group actions in the finite Morley rank context, one of which includes a "characteristic zero" variant.

Fact 2.46 ([ABC99, Prop. 2.43], [ABC08, Prop. I.9.12])**.** *Let $G = H \rtimes P$ be a group of finite Morley rank, and Q a normal subgroup of H, with Q, H, and P definable.*

Suppose that Q and T are solvable, and the following conditions hold, for some finite set of primes π.

- *P is a π-group of bounded exponent;*
- *Q is a P-invariant $\pi^\perp$-subgroup.*

Then

$$C_{H/Q}(P) = C_H(P)Q/Q$$

A case of frequent interest takes P to be generated by an involution t and Q centralized by P: then $C_{H/Q}(t) = C_H(t)/Q$.

Fact 2.47 ([ABCC03], [Bur09, Lemma 3.5])**.** *Let G be a connected solvable $p^\perp$-group of finite Morley rank, and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

If in addition G is a nilpotent $U_{0,r}$-group then $C_G(P)$ is a $U_{(0,r)}$-group.

3. Specialized topics (background)

We continue our review of useful background material with some important topics of more limited scope.

3.1. **Automorphisms of algebraic groups.**

A key tool in our analysis is the fact that a group of finite Morley rank acting faithfully as a group of automorphisms of a quasi-simple algebraic group[6] must itself be algebraic.

Definition 3.1. Given a quasi-simple algebraic group G, a maximal torus T of G, and a Borel subgroup B of G which contains T, the group Γ of *graph automorphisms* associated to T and B is the group of algebraic automorphisms of G which normalize both T and B.

Fact 3.2 ([BN94, Theorem 8.4])**.** *Let $G \rtimes H$ be a group of finite Morley rank where G and H are definable, G is a quasi-simple algebraic group over an algebraically closed field, and $C_H(G)$ is trivial. Then viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leq \mathrm{Inn}(G)\Gamma$ where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms of G, relative to a fixed choice of Borel subgroup B and maximal torus T contained in B. In other words, H acts as a group of algebraic automorphisms.*

In particular, if the group H is connected then it acts by inner automorphisms.

An algebraic group is said to be *reductive* if it has trivial unipotent radical. Such a group is a central product of semisimple algebraic groups and algebraic tori. The centralizer of an involution in a reductive algebraic group over a field of characteristic not 2 is itself reductive.

Fact 3.3 ([Ste68, Theorem 8.1])**.** *Let G be a quasi-simple algebraic group over an algebraically closed field. Let ϕ be an algebraic automorphism of G whose order is finite and relatively prime to the characteristic of the field. Then $C^\circ_G(\phi)$ is nontrivial and reductive.*

Proof. If we replace G by its universal central extension $\hat{G}$ for the category of algebraic groups, then ϕ lifts to an algebraic automorphism, by the universal property, with the same order, since the cover is again perfect. Furthermore, if K is the kernel of the map $\hat{G} \to G$ then

$$C^\circ_{\hat{G}}(\phi)K/K = C^\circ_G(\phi)$$

since K is finite.

[6]The groups we call quasi-simple algebraic groups are called "simple" in the context of algebraic group theory, but we avoid that usage.

Thus it suffices to treat the case in which G is simply connected.

Since ϕ is algebraic and has finite order, $G \rtimes \langle \phi \rangle$ is an algebraic group which contains ϕ as an inner automorphism. Since the order of ϕ is finite and relatively prime to the characteristic, ϕ is a semisimple automorphism of G. So the result follows from Theorem 8.1 of [Ste68]. □

We need some more precise information about involutory automorphisms. We use the notation of §3.2 here.

Fact 3.4 ([GLS98, Table 4.3.1, p. 145])**.** *Let L be a quasi-simple algebraic group over an algebraically closed field of characteristic other than 2, and let α be an involutory algebraic automorphism of L. Suppose*

$$L \not\simeq (\mathrm{P})\mathrm{SL}_2 .$$

Then $E(C_L(\alpha)) > 1$.

Fact 3.5 ([Bur09, Fact 2.16])**.** *Let G be a quasi-simple algebraic group over an algebraically closed field of characteristic other than 2, and let V be a 4-group acting algebraically, though not necessarily faithfully, on G. Suppose*

$$L \not\simeq (\mathrm{P})\mathrm{SL}_2 .$$

Then $L = \langle E(C_G(\alpha)) : \alpha \in I(V)$.

3.2. **Quasi-simple components.**

Definition 3.6. Let H be a group of finite Morley rank.

A *component* of H is a quasi-simple subnormal subgroup ([BN94, p. 118 (2)]).

$E(H)$ is the product of the quasi-simple components of H.

$E_{alg}(H)$ denotes the product of the algebraic quasi-simple components of H°.

The *generalized Fitting subgroup* $F^*(H)$ is the product of $F(H)$ and $E(H)$.

This leaves us with the awkward notation $F^{*\circ}(H)$ for $F^\circ(H)E^\circ(H)$; this is the more important subgroup.

Remarks.

1. All components are normal in H° by [BN94, Lemma 7.1iii].

2. By [BN94, Lemmas 7.9, 7.10, 7.13], the product $F^*(H) = F(H)E(H)$ is a central product, the group $E(H)$ itself is a central product of finitely many normal quasi-simple factors, each definable in G, and $C_H(F^*(H)) = Z(F^*(H)) = Z(F(G)) \le F^*(H)$.

In consequence, $H/Z(F(G))$ embeds definably into $\mathrm{Aut}(F^*(H))$

3. $E^\circ(H)$ is the product of the connected quasi-simple components. If H is connected then $E(H) = E^\circ(H)$.

We make use of the theory of central extensions of quasi-simple algebraic groups in the finite Morley rank category, in the following form.

Fact 3.7 ([AC99])**.** *Let G be a group of finite Morley rank, and L a quasi-simple component of G with $L/Z(L)$ algebraic. Then L is an algebraic component of G.*

In particular, in this situation $Z(L)$ is finite.

We mention that the treatment in [AC99] is brisk around the point where the connection to K-theory is described, and a fuller account exists but remains unpublished. It may find its way to arXiv.

Fact 3.8 ([BC08, Lemma 1.13])**.** *Let H be a connected group of finite Morley rank, and $K = C_H^\circ(E_{alg}(H))$. Then*

$$H = E_{alg}(H) * K; \qquad\qquad E_{alg}(K) = 1.$$

We remark that this follows directly from Fact 3.2.

3.3. **Structure of L-groups with NTA$_2$.** As the hypothesis NTA$_2$ involves degenerate simple sections rather than degenerate quasi-simple sections, we insert the following.

Remark 3.9. If an automorphism of a quasi-simple group L acts trivially on $L/Z(L)$, then it acts trivially on L.

Indeed, with α the automorphism and $a, b \in L$, we find

$$\alpha([a,b]) = [\alpha(a), \alpha(b)] = [a,b]$$

The following structural analysis will be our workhorse.

Proposition 3.10. *Let H be a connected L-group of finite Morley rank and odd type satisfying the condition* NTA$_2$*. Suppose that*

$$UF(H) \le Z(H).$$

Then

$$H = E_{alg}(H) * K \text{ where } K \text{ is connected with } K/Z^\circ(K) \text{ of degenerate type.}$$

Hence the Sylow 2-subgroup of K is central in H, and connected.

Proof. We set $K = C_G^\circ(E_{alg}(H))$. By Fact 3.8 we have

$$H = E_{alg}(H) * K; \qquad\qquad E_{alg}(K) = 1.$$

By Lemma 2.42 we have

$$[K, F(K)] \leq UF(K) \leq Z(K).$$

Let $a \in F(K)$. Then commutation with a induces a homomorphism from K into $UF(K)$. If S is a 2-Sylow subgroup of K, it follows that S centralizes $F(K)$.

By NTA_2 and Remark 3.9, S° centralizes $E(K)$. Thus S° centralizes $F^*(K)$. So $S^\circ \leq F(K)$.

It follows that $S^\circ \leq Z(K)$. Thus $K/Z^\circ(K)$ is of degenerate type, and being connected, this quotient has no involutions. That is, $S \leq Z^\circ(K) \leq Z^\circ(H)$.

In particular, S is connected, and all claims have been proved. $\square$

Lemma 3.11. *Let H be a connected L-group of finite Morley rank and odd type satisfying* NTA_2 *and let $\bar{H} = H/OF(H)$.*

Then $OF(\bar{H}) \leq Z(\bar{H})$. Hence

$$\bar{H} = E_{alg}(\bar{H}) * \bar{K} \text{ where } \bar{K} \text{ is connected and } \bar{K}/Z^\circ(\bar{K}) \text{ has degenerate type.}$$

Proof. Let H_0 be the preimage in H of $OF(\bar{H})$. Then $H_0 \leq O^\sigma(H)$. By Lemma 2.29, $[H, H_0] \leq F(H)$ and as $[H, H_0]$ is connected it follows that $[H, H_0] \leq OF(H)$. Thus $OF(\bar{H}) = \bar{H}_0$ is central in $\bar{H}$.

The rest then follows from Proposition 3.10. $\square$

Lemma 3.12. *Let H be a connected D-group of finite Morley rank and odd type satisfying* NTA_2*. Then $H/UF(H)$ has a unique, central, 2-Sylow subgroup.*

Proof. Let $\overline{H} = H/F^\circ(H)$. By Lemma 2.29 we have

$$F^\circ(\overline{H}) \leq Z(\overline{H}).$$

By Proposition 3.10 we find that $\overline{H}/Z^\circ(\overline{H})$ is of degenerate type. That is, $\overline{H}$ has a unique, central, 2-Sylow subgroup.

Let T be a Sylow 2-subgroup of H and $H_1 = F^\circ(H)d(T)$. Then H_1 is normal in H, any 2-Sylow subgroup of H lies in H_1, and $F^\circ(H_1) = F^\circ(H)$.

Let $\bar{H}_1 = H_1/UF(H)$. By Lemma 2.42 $\overline{F(H_1)} \leq Z(\bar{H}_1)$. So $\bar{H}_1$ is commutative and hence has a unique, central, 2-Sylow subgroup. Therefore the same applies to $H/UF(H)$. $\square$

3.4. **Generation theorems.** As indicated in the introduction, the following will play an essential role in our analysis.

Definition 3.13. Let G be a group of finite Morley rank, and V an elementary abelian 2-subgroup.

We set

$$\Gamma_V = \langle C^\circ(U) \,:\, U \leq V,\, [V:U]=2\rangle$$

We have the following generation principle for L-groups.

Fact 3.14 ([BC08, Theorem 2.1])**.** *Let G be a connected L-group of finite Morley rank and odd type. Let V be an elementary abelian 2-group acting definably on G.*

Then $\Gamma_V = G$.

The generality of this result is overstated in [BC08], in terms of a prime p which should in fact be 2, as is the case here. Cf. [Del12]. The case in which V is cyclic is vacuous and will not come into play.

We mention a considerably more elementary version of this generation result which we prefer to quote directly whenever it is applicable. For our purposes, the prime p will again be 2.

Fact 3.15 ([Bur09, Fact 3.7])**.** *Let p be prime, and let H be a solvable $p^\perp$-group of finite Morley rank. Let A be a finite elementary abelian p-group acting definably on H. Then*

$$H = \langle C_H(V) : V \leq A,\, [A:V]=p\rangle.$$

It would be very useful to extend Fact 3.14 from L-groups to L^*-groups. We have the following.

Fact 3.16 ([BC08, Thm. 4.3])**.** *Let G be a connected simple L^*-group of finite Morley rank and odd type with*

$$m_2(G) \geq 3.$$

Suppose that $\Gamma_V(G) < G$ for some elementary abelian 2-subgroup V of rank 2. Then the normalizer

$$N(\Gamma_V(G))$$

is a strongly embedded subgroup.

3.5. **Signalizer functors.**

Definition 3.17. Let G be a group of finite Morley rank and A an elementary abelian 2-subgroup.

An *A-signalizer functor* is a function θ defined on the set of involutions $I(A)$ whose values are definable A-invariant $2^\perp$-subgroups of G, such that θ satisfies the following additional conditions.

(1) $\theta(i) \lhd C_G(i)$ for $i \in I(A)$;
(2) ("Balance") $\theta(i) \cap C_G(j) = \theta(j) \cap C_G(i)$ for $i, j \in A$.

(One can gives definition for general primes p and elementary abelian p-groups as well, and this is often done in the literature.)

We say that the signalizer functor θ is *connected, nilpotent,* or *trivial* iff all of its values are connected, nilpotent, or trivial, respectively. In particular, θ is nontrivial if *some* value of θ is nontrivial.

We write $\theta(A)$ for the group $\langle \theta(i) : i \in I(A) \rangle$.

An A-signalizer functor θ is said to be *complete* if

- $\theta(A)$ is a solvable $2^\perp$-group, and
- $\theta(i) = \theta(A) \cap C(i)$ for all $i \in I(A)$.

The "canonical" signalizer functor would be defined by $\theta(i) = O(G) \cap C_G(i)$ and the terminology reflects the idea that θ carries a signal suggesting that $O(G)$ should be nontrivial. Here G is typically simple, so something will have to block the signal.

In the context of groups of finite Morley rank, the critical signalizer functors are the connected nilpotent ones (so the intuition replaces $O(G)$ by $OF(G)$), and in this setting other signalizer functors are considered as first approximations to nilpotent ones.

The basis for this is approach is a very general result originating with Borovik. With our conventions, the prime p referred to will be taken to be 2, but the result is general.

Fact 3.18 ([Bur09, Thm. A.2], cf. [BN94, Thm. B.30])**.** *Let G be a group of finite Morley rank, p a prime, and $A \leq G$ a finite elementary abelian p-group of rank at least 3. Let θ be a connected nilpotent A-signalizer functor. Then θ is complete and the group $\theta(A)$ is nilpotent*

The next result is not actually in the literature in the general form we wish to give here, but the missing element is supplied by [BC08]. We will go through the details.

Proposition 3.19 (cf. [Bur09, Thm. 1.28])**.** *Let G be a connected simple L^*-group of finite Morley rank and odd type. Suppose that for some rank 3 elementary abelian 2-subgroup A of G, there is a nontrivial connected nilpotent A-signalizer functor θ satisfying the naturality condition*

$$(\star) \qquad \theta(i)^g = \theta(i^g) \text{ when } g \in G \text{ and } i, i^g \in I(A).$$

Then G has a proper definable strongly embedded subgroup.

The philosophy here is that θ signals that G should have a nontrivial proper normal subgroup, but as this possibility is excluded by hypothesis, we arrive at a different and quite extreme conclusion.

We prepare for the proof of this proposition with a lemma which shall be reused below.

Definition 3.20. Let G be a group of finite Morley rank and A a subgroup.

$\Gamma_{A,2}(G)$ denotes the definable hull of the subgroup

$$\langle N_G(U) : U \leq A \text{ is elementary abelian with } m_2(U) = 2 \rangle.$$

Lemma 3.21. *Let G be an L^*-group of finite Morley rank and odd type, and A an abelian 2-subgroup of rank at least 3. Then for $i \in I(A)$ we have*

$$C_G^\circ(i) \leq \Gamma_{A,2}(G).$$

Proof. Set $H = C_G(i)$.

Take $U \leq A$ with $m_2(U) = 2$ and $i \notin U$. For $j \in I(U)$ we have

$$C_H(j) = C(i, j) \leq N(\langle i, j \rangle) \leq \Gamma_{A,2}(G).$$

By Fact 3.14 H° is generated by the centralizers $C_{H^\circ}(j)$ ($j \in I(U)$), and we conclude. □

Now we return to the proof of Proposition 3.19.

Proof of Proposition 3.19. We consider the definable subgroups

$$\Gamma_U(G) = \langle C_G^\circ(i) : i \in I(U) \rangle$$

for $U \leq A$ with $m_2(U) = 2$. Our goal is to show that $\Gamma_U(G) < G$, and to apply Fact 3.16.

For $U \leq A$ we set

$$\theta(U) = \langle \theta(i) : i \in I(U) \rangle.$$

Claim 1. *For $U \leq A$ of rank 2 we have*

$$\theta(U) = \theta(A).$$

By Fact 3.18 $\theta(A)$ is a solvable $2^\perp$-group. Fact 3.15 and completeness apply to give

$$\begin{aligned} \theta(A) &= \langle C_{\theta(A)}(i) : i \in I(U) \rangle = \langle \theta(i) : i \in I(U) \rangle \\ &= \theta(U). \end{aligned}$$

Claim 2.

$$\Gamma_{A,2}(G) \leq N_G(\theta(A)).$$

By the naturality condition, for $U \leq A$ and $g \in N(U)$ we have $\theta(U)^g = \theta(U)$, so when $m_2(U) = 2$ the previous claim yields $N_G(U) \leq N_G(\theta(A))$ and the claim follows.

In view of Lemma 3.21 and the simplicity of G, for $U \leq A$ with $m_2(U) = 2$ we have

$$\Gamma_U(G) \leq \Gamma_{A,2}(G) < G.$$

Then Fact 3.16 says that $N_G(\Gamma_U(G))$ is strongly embedded. $\square$

3.6. **Root SL_2-subgroups and Coxeter groups.** We turn now to recognition theorems for groups of high Prüfer 2-rank.

If L is a quasi-simple algebraic group and T a maximal torus, then the *root* SL_2*-subgroups* of L associated with T are the Zariski closed subgroups of G which are normalized by T and are isomorphic to $(\mathrm{P})\mathrm{SL}_2$; in other words, those generated by pairs of opposite root subgroups relative to T.

We quote the main facts in the form used for identification in the context of groups of finite Morley rank.

Fact 3.22 ([BB04, Fact 2.1], [Bur09, Fact 3.2])**.** *Let G be a quasi-simple algebraic group over an algebraically closed field. Let T be a maximal torus in G and let K, L be* Zariski closed *subgroups of G that are isomorphic to* SL_2 *or* PSL_2 *and are normalized by T. Then*

1. *Either K and L commute or $\langle K, L\rangle$ is a quasi-simple algebraic group of type A_2, C_2, or G_2.*
2. *The subgroups K and L are root* SL_2*-subgroups of $\langle K, L\rangle$.*
3. *If $\langle K, L\rangle$ is of type G_2, then $G = \langle K, L\rangle$.*

Berkman and Borovik refer to the full classification of semisimple subsystem subgroups [Sei83, 2.5] for the proof; by definition, these are the semisimple subgroups normalized by a maximal torus T. The argument in [Bur09] is more elementary but also uses properties of subsystem subgroups.

Fact 3.23 ([Car93, p. 19], [Bur09, Fact 3.4])**.** *Let G be a semisimple algebraic group over an algebraically closed field, and T a maximal algebraic torus of G. Then the following hold.*

1. *G is generated by the root* SL_2*-subgroups associated with T.*
2. *The intersection $T \cap K$ of the torus T with a root* SL_2*-subgroup K associated to T is a maximal algebraic torus of K.*

Fact 3.24 ([Bur09, Fact 3.5])**.** *Let L be a quasi-simple algebraic group over an algebraically closed field of characteristic not 2. Let $\mathbb{T}$ be a maximal algebraic torus of L, and let K be a root* SL_2*-subgroup of L normalized by $\mathbb{T}$.*

Let T be the 2-torsion subgroup of $\mathbb{T}$ and let $T_K^\perp$ denote $C_T^\circ(K)$.[7] *Then*

$$K = E(C_L(T_K^\perp)).$$

Fact 3.25 ([Bur09, Fact 3.7])**.** *Let L be a quasi-simple algebraic group over an algebraically closed field of characteristic not 2, and let T be a maximal 2-torus of L. Then $C_L(T)$ is a maximal algebraic torus of L.*

We add a result which is useful for pinning down the structure of Weyl groups. This relies on a careful examination of the classification of complex reflection groups.

Fact 3.26 ([BBBC08, Prop. 3.1])**.** *Let W be a finite group and $I \subseteq W$. Assume the following conditions.*

(1) I is a set of involutions which generates W and is closed under conjugation in W.
(2) The graph on I with edges the pairs (i, j) of non-commuting involutions $i, j \in I$ is connected.
(3) For all sufficiently large prime numbers p, the group W has a faithful representation V_p over the prime field $\mathbb{F}_p$ in which the elements of I operate as "complex reflections" (i.e., fixing a hyperplane) with no common fixed vectors.

Then one of the following occurs.

(a) The group W is a dihedral group acting in dimension $n = 2$, or cyclic of order two.
(b) The group W is isomorphic to an irreducible crystallographic Coxeter group, that is, one of A_n, B_n, C_n, D_n with $n \geq 3$, E_n with $n = 6$, 7, or 8, or F_4.
(c) The group W is a semidirect product of a quaternion group of order 8 with the symmetric group Sym_3 acting naturally, represented in dimension 2.

Suppose in addition that the group W has an irreducible representation of dimension at least 3 over some field, in which the elements of I act as reflections. Then case (b) applies.

One elementary point involved in the wording of the foregoing lemma is worth isolating further.

Lemma 3.27. *Let W be a finite group and $I \subseteq W$ a subset satisfying the following conditions.*

[7] [Bur09] uses $C_T(K)$ but it seems more natural to take the connected component, and the proof is the same.

(1) The set I generates W, consists of involutions, and is closed under conjugation in W;
(2) The graph on I with edges the pairs (i,j) of non-commuting involutions $i,j \in I$ is connected.

Let W act linearly on the vector space V over a field whose characteristic p does not divide the order of W, with the elements of I acting as complex reflections, and without a common fixed point. Then W acts irreducibly on V.

Proof. As $p \nmid |W|$ the representation is completely reducible.

If the representation splits as $V = V_1 \oplus V_2$ then the (-1)-eigenspace of each involution is contained in either V_1 or V_2 and the action on the other space is trivial.

If all of the (-1)-eigenspaces lie in one factor then the other factor is fixed by W.

If the (-1)-eigenspaces are split between V_1 and V_2, then I is partitioned correspondingly into two sets of involutions which must commute with each other, violating the connectivity hypothesis.

The result follows. □

The reflections in a Coxeter group correspond to roots in the associated root system ([Hum90, Lemma 5.7]), and hence there are at most two conjugacy classes of such reflections, depending on how many root lengths occur.

Fact 3.28 ([Hum78, 10.4 Lemma C])**.** *A finite irreducible reflection group of type A_n, D_n, E_6, E_7, or E_8 has only one conjugacy class of reflections. A finite irreducible reflection group of type B_n, C_n, F_4, and G_2 has two conjugacy classes of reflections, corresponding to the short and long roots.*

When there are two root lengths, then as the roots of a fixed length are closed under the action of the Coxeter group they form the root system for a proper subgroup.

Fact 3.29. *The subgroup of a reflection group of type B_n, C_n, F_4, or G_2 which is generated by the reflections associated to roots of a fixed length is a proper subgroup.*

4. Components in centralizers of involutions

Our concern here is with the general structure of $O^\sigma(C_G(i))$ for i an involution, and with the existence of components in $C_G(i)$. More particularly, the line of argument shows that certain well-behaved components of $C_G(i)/O^\sigma C_G(i)$ must "descend" to $E(C_G(i))$.

We will work with the following hypothesis.

Hypothesis 4.1. Let G be a group of finite Morley rank. We suppose the following.

(1) G is a connected simple L^*-group of odd type satisfying the condition NTA_2.
(2) $m_2(G) \geq 3$.
(3) G has no proper nontrivial definable strongly embedded subgroup.

Eventually we will add to this hypothesis the assumption of *Prüfer 2-rank* at least 3.

The hypothesis NTA_2 enters into our analysis via Proposition 3.10, in the proof of Lemma 4.4 below (via the preparatory Lemma 4.3). We will be explicit about the operative hypotheses as we proceed.

4.1. The structure of $O^\sigma C(i)$.

We aim at Proposition 4.6 below, which kills certain subgroups of $O^\sigma(C_G(i))$ for i an involution, namely the subgroups $U_p O^\sigma(C_G(i))$ and certain subgroups $U_{0,r} O^\sigma C_G(i)$, for r sufficiently large. Recall that $O^\sigma C_G(i)$ is connected and solvable, by definition.

Definition 4.2. Let G be a group of finite Morley rank and i an involution of G.

1. We associate to the involution i the following reduced ranks.

$$r_{0,i} = \bar{\mathrm{r}}_0(O^\sigma C_G(i));$$
$$r_{f,i} = \max(\bar{\mathrm{r}}_0(k^\times) : \ k \text{ the base field of a component of } E_{alg}(C_G(i)/O^\sigma C_G(i))).$$

Here the subscript "f" stands for "field."

If is possible to have $r_{f,i} = 0$, and not just in the absence of algebraic components: this would mean that the multiplicative group of any base field involved is a good torus.

2. Let A be an elementary abelian 2-subgroup of G.

We set

$$r_{0,A} = \max(r_{0,i} : i \in I(A)); \qquad r_{f,A} = \max(r_{f,i} : i \in I(A)).$$

Similarly we define

$$r_{f,G} = \max(r_{f,i} \, : \, i \in I(G)).$$

3. For i an involution, and ρ either a prime or a symbol $(0, r)$, we set

$$\theta_\rho(i) = U_\rho(O^\sigma C_G(i))$$

Note that if ρ is the prime 2 then $\theta_\rho(i) = (1)$. So we restrict our discussion to odd primes below.

Lemma 4.3. *Let G be an L^*-group of finite Morley rank of odd type, satisfying* NTA_2. *Let i, k be commuting involutions of G.*

Let $H = C_G^\circ(i)$, $\bar{H} = H/O^\sigma(C_G(i))$, $\bar{K} = C_{\bar{H}}(E(\bar{H}))$, and let K be the preimage of $\bar{K}$ in H.

Then the following hold.

(1) *$\bar{H} = E_{alg}(\bar{H}) * \bar{K}$ with $\bar{K}/Z(\bar{K})$ of degenerate type.*
(2) *$\overline{O^\sigma C_G(k) \cap C_G(i)} \leq O^\sigma C_{\bar{H}}(\bar{k})$.*
(3) *$O^\sigma C_{\bar{H}}(\bar{k}) = O^\sigma C_{E_{alg}(\bar{H})}(\bar{k}) * O^\sigma C_{\bar{K}}(\bar{k})$.*

Proof.

Ad 1.

This holds by Proposition 3.10 (which assumes NTA_2).

Ad 2.

As i normalizes $O^\sigma C_G(k)$, by Fact 2.47 the group $O^\sigma C_G(k) \cap C_G(i)$ is connected, hence coincides with $O^\sigma C_G(k) \cap H$.

By Fact 2.46 we have $C_{\bar{H}}(\bar{k}) = \overline{C_H(k)}$. Accordingly $\overline{O^\sigma C_G(k) \cap H}$ is normal in $C_{\bar{H}}(\bar{k})$. As this intersection is also connected, we have $\overline{O^\sigma C_G(k) \cap H} \leq O^\sigma C_{\bar{H}}(\bar{k})$.

Point (2) follows.

Ad 3.

] Let $A = E_{alg}(\bar{H}) \cap \bar{K}$, a finite central subgroup of $\bar{H}$. Let $\bar{H}_1 = C_{\bar{H}}(\bar{k})^\circ$ and let $\bar{H}_2$ be the connected component of the preimage in $\bar{H}$ of the centralizer of $\bar{k}$ in $\bar{H}/A$. Then $\bar{H}_1 \subseteq \bar{H}_2$. On the other hand $[\bar{H}_2, \bar{k}] \leq A$ and this group is connected, hence trivial. Thus $\bar{H}_1 = \bar{H}_2$. In particular the image of $\bar{H}_1$ in $\bar{H}/A$ is the direct product

$$C_{E(\bar{H})A/A}(\bar{k}) \times C_{KA/A}(\bar{k}).$$

Now by Lemma 2.25 applied to $C_{\bar{H}}^\circ(\bar{k})$, point (3) follows. $\square$

Lemma 4.4. *Let G be an L^*-group of finite Morley rank of odd type, satisfying* NTA_2

Let i, j, k be three commuting involutions in G and let ρ be either an odd prime, or a symbol $(0, r)$ satisfying the conditions

$$r > r_{f,i};\ r \geq r_{0,i}.$$

Suppose the following.

(1) i and j are co-toral in G.
(2) $\theta_\rho(k) \cap C_G(j) \leq \theta_\rho(j)$.

Then

$$\theta_\rho(k) \cap C_G(i) \leq \theta_\rho(i).$$

Proof. We use the notation of the preceding lemma: $H = C_G^\circ(i)$, $\bar{H} = H/O^\sigma(C_G(i)) = E(\bar{H}) * \bar{K}$. That lemma gives in particular

$$U_\rho(O^\sigma C_{\bar{H}}(\bar{k})) \leq U_\rho(O^\sigma C_{E_{alg}(\bar{H})}(\bar{k}))U_\rho(O^\sigma C_{\bar{K}}(\bar{k})).$$

Claim 1.

$$U_\rho(O^\sigma C_{E_{alg}(\bar{H})}(\bar{k})) = 1.$$

By Fact 3.3, $C^\circ_{E_{alg}(\bar{H})}(\bar{k})$ is reductive, and therefore $O^\sigma C_{E_{alg}(\bar{H})}(\bar{k})$ is contained in a product R of algebraic tori over various algebraically closed fields.

For p a prime, the p-unipotent subgroups of R are trivial. Furthermore, for $r > r_{f,i}$, we have $U_{0,r}(k^\times) = 1$ for the fields k associated with the factors of R. So by Fact 2.34, we have $U_{0,r}(R) = 1$ as well. Thus the claim follows.

Claim 2.

$$\theta_\rho(k) \cap C_G(i) \leq K.$$

First, $\theta_\rho(k) \cap C_G(i)$ is a U_ρ-group by Fact 2.47. Now apply the previous claims.

We have $\theta_\rho(k) \cap C_G(i) = \theta_\rho \cap H \leq U_\rho(O^\sigma C_H(k))$. Modulo $O^\sigma(C_G(i))$ we have $\overline{\theta_\rho(k) \cap H} \leq U_\rho(O^\sigma C_{\bar{H}}(\bar{k})) \leq \bar{K}$ and thus the claim follows.

Claim 3.

$$\theta_\rho(k) \cap C_G(i) \leq O^\sigma C_G(i).$$

Let

$$Q = \theta_\rho(k) \cap C_G(i).$$

As j commutes with i and k, it acts on Q.

As i, j are co-toral, and $\bar{K}$ has a central 2-Sylow subgroup, $\bar{\jmath}$ centralizes $\bar{K}$, and in particular $\bar{\jmath}$ centralizes $\bar{Q}$. By Fact 2.46 we have

$$\overline{C_Q(j)} = C_{\bar{Q}}(\bar{\jmath}) = \bar{Q}.$$

By hypothesis

$$C_Q(j) = Q \cap C_G(j) \leq \theta_\rho(k) \cap C_G(j) \leq O^\sigma C_G(j).$$

Thus $C_Q(j) \leq O^\sigma C_G(j) \cap K$. Again by Fact 2.46 we have

$$\begin{aligned} \overline{C_K(j)} &= C_{\bar{K}}(\bar{\jmath}) = \bar{K}; \\ \bar{Q} &= \overline{C_Q(j)} \leq \overline{O^\sigma C_G(j) \cap K} \leq \overline{O^\sigma C_K(j)} \\ &\leq O^\sigma(C_{\bar{K}}(\bar{\jmath})) = O^\sigma \bar{K} = 1. \end{aligned}$$

The claim follows.

From the last claim, the statement of the Lemma follows using Fact 2.47. □

Lemma 4.5. *Let G be an L^*-group of finite Morley rank and odd type satisfying* NTA$_2$ *with*

$$m_2(G) \geq 3$$

Let A be an elementary abelian 2-subgroup of G of 2-rank 3.

Suppose that the graph on $I(A)$ with edges (i, j) for co-toral pairs of involutions is a connected graph. Let ρ be either a prime or a symbol of the form $(0, r)$ with $r \geq r_{0,A} > r_{f,A}$.

Then $\theta_\rho(i) = U_\rho(O^\sigma C_G(i))$ defines a connected nilpotent signalizer functor on A.

Proof. By definition $\theta_\rho(i)$ is connected, and by Facts 2.31 and 2.35 $\theta_\rho(i)$ is nilpotent.

The only other nontrivial condition is the balance condition: for $i, k \in A$ we claim that

$$\theta_\rho(k) \cap C_G(i) \leq \theta_\rho(i).$$

We argue by induction on the distance $d(i, k)$ in the co-torality graph on A. If the distance is 0 then $i = k$ and the claim is clear.

Now consider a pair of involutions i, k in A at positive distance $d(i, k) > 0$ and choose $j \in I(A)$ co-toral with i, and with

$$d(j, k) = d(i, k) - 1$$

Then by induction $\theta_\rho(k) \cap C_G(j) \leq \theta_\rho(j)$ and by Lemma 4.4 the claim follows.

□

We now derive the main result of this section as a corollary to the above.

Proposition 4.6. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let A be an elementary abelian 2-subgroup of 2-rank 3.*

Suppose that the graph on $I(A)$ with edges (i,j) for co-toral pairs of involutions is a connected graph.

Then for $i \in I(A)$ an involution, we have

$$U_p(O^\sigma C_G(i)) = 1 \textit{ for all primes } p;$$
$$U_{(0,r)}(O^\sigma C_G(i)) = 1 \textit{ for all } r > r_{f,A}.$$

Proof. Assuming the contrary, then by Lemma 4.5 we get a nontrivial connected nilpotent A-signalizer functor θ_ρ, where ρ is either a prime p or the symbol $(0, r_{0,A})$.

Then we may apply Proposition 3.19 to get a proper definable strongly embedded subgroup, contradicting our hypothesis. □

4.2. **Existence of components.**

Definition 4.7. Let G be a group of finite Morley rank, H a definable subgroup, and $r \geq 0$.

$\Delta_r(H)$ denotes the definable subgroup of H generated by all p-unipotent subgroups with p prime, and all connected abelian subgroups of reduced rank strictly greater than r.

For A a subgroup of G, $\mathcal{E}_A$ is the set of all quasi-simple algebraic components of any of the subgroups $\Delta_r(C_G(i))$ as i varies over $I(A)$ and $r \geq r_{f,A}$.

Remark 4.8. With notation as above, for $r \geq r_{f,A}$ we have

$$\Delta_r(C_G(i)) \lhd \Delta_{r_{f,A}}(C_G(i))$$

and hence the quasi-simple components of $\Delta_r(C_G(i))$ are also quasi-simple components of $\Delta_{r_{f,A}}(C_G(i))$; in other words, we are free to set $r = r_{f,A}$ in the definition of $\mathcal{E}_A$.

Lemma 4.9. *Let L be a quasi-simple algebraic group of finite Morley rank (in any language), with base field k. Fix $r \geq 0$.*

Then the following are equivalent.

(1) $\Delta_r(L) > 1$.
(2) $\Delta_r(L) = L$.
(3) $\mathrm{rk}(k) > r$, *or the characteristic of k is nonzero.*

Proof. We prove (1) $\implies$ (3) $\implies$ (2), beginning with the latter.

If k has characteristic $p > 0$ then the root subgroups are p-unipotent and they generate L, so $\Delta_r(L) = L$ and we may set this case aside.

If k has characteristic zero and $\mathrm{rk}(k) > r$, then as L is generated by its unipotent subgroups and $\bar{\mathrm{r}}(k_+) = \mathrm{rk}(k)$, we again find $\Delta_r(L) = L$.

For the converse, suppose now that $\Delta_r(L) > 1$ and that k has characteristic zero. Consider an indecomposable connected abelian subgroup A of L with $\bar{\mathrm{r}}(A) > r$.

If A is contained in a unipotent subgroup then its Zariski closure is a vector subgroup and it follows easily that $A \simeq k_+$. If A is not contained in a unipotent subgroup then it has a nontrivial image in a quotient $B/F(B)$ with B a Borel subgroup, and it follows easily that

$$\mathrm{rk}(k) > \bar{\mathrm{r}}_0(k^\times) \geq \bar{\mathrm{r}}(A).$$

So in either case $\mathrm{rk}(k) > r$. □

Our goal is to show under Hypothesis 4.1 that suitable quasi-simple algebraic components exist for A of 2-rank 3 such that the co-torality graph on $I(A)$ is connected.. To begin with, we work in $C_G(i)/O^\sigma C_G(i)$ for suitable involutions i.

Lemma 4.10. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1, and let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank at least 3 such that the co-torality graph on $I(A)$ is connected.*

Then for some $i \in I(A)$,

$$E_{alg}(C_G(i)/O^\sigma C_G(i)) > 1.$$

Proof. Suppose the contrary. Then by definition we have

$$r_{f,A} = 0.$$

Claim 1. *$O^\sigma(C(i))$ is a good torus for $i \in I(A)$.*

By Proposition 4.6 we find $U_p(O^\sigma C(i)) = U_{(0,r)}(O^\sigma C(i)) = 1$ for all primes p and all $r \geq 1$. Then Fact 2.37 yields the claim.

In particular $O^\sigma(C(i))$ is central in $C^\circ(i)$.

Claim 2. *$\sigma^\circ C(i) = F^\circ C(i)$ for $i \in I(A)$.*

By Lemma 2.42 we find

$$[C^\circ(i), FC^\circ(i)] \leq ZC^\circ(i).$$

Then by Fact 2.23 $\sigma^\circ C(i)$ is nilpotent, proving the claim.

We could go on to show that $\sigma^\circ C(i)$ is divisible abelian but this will suffice for our purposes.

By Proposition 3.10 and the supposed absence of algebraic components modulo $O^\sigma C(i)$, a maximal 2-torus T in $C(i)$ lies in $\sigma^\circ C(i)$, hence in $F^\circ C(i)$. It follows that T is central in $C^\circ(i)$, and in particular the 2-Sylow subgroup is unique, and is normalized by A.

By Fact 2.4, we can find $j \neq i$ in $I(A)$ so that $C^\circ_T(j)$ is nontrivial. So a maximal 2-torus T_0 of $C^\circ(i,j)$ is central in $C^\circ(i)$, $C^\circ(j)$, and $C^\circ(ij)$. Hence with $U = \langle i, j\rangle$ we have

$$\Gamma_U(G) \leq C(T_0) < G.$$

Applying Fact 3.16 once more, we arrive at a proper definable strongly embedded subgroup, and a contradiction. □

Next we consider the structure of the subgroups $\Delta_r(C_G(i))$ for $r = \bar{\mathrm{r}}_0(O^\sigma C_G(i))$. First, a preparatory lemma.

Lemma 4.11. *Let H be a group of finite Morley rank for which*

$$U_p(O^\sigma(H)) = 1 \text{ for all primes } p.$$

Let $\rho = \bar{\mathrm{r}}_0(O^\sigma(H))$. Then

$$[\Delta_\rho(H), F(O^\sigma(H))] = 1.$$

We remark that for $r > \rho$ the same conclusion then applies a fortiori to $\Delta_r(H)$.

Proof. Let U be either a p-unipotent subgroup of H for some prime p, or a $U_{0,r}$-subgroup with $r > \rho$.

Then the group $F(O^\sigma(H)) \cdot U$ is solvable and by Facts 2.31 and 2.35 we have $U \leq F(UF(O^\sigma(H)))$. Thus $F(O^\sigma(H)) \cdot U$ is nilpotent. By Fact 2.38 U centralizes $F(O^\sigma(H))$.

As such subgroups U generate $\Delta_\rho(H)$, the result follows. □

Lemma 4.12. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let i be an involution of G and $r \geq r_{0,i}$.*

Then

$$\Delta_r(C(i)) = E_{alg}(\Delta_r(C(i))) * K,$$

where $K = \Delta_r(K)$ is a group whose Sylow 2-subgroup is central, and

$$E_{alg}(\Delta_r(C(i))) = \Delta_r(E_{alg}(C(i))).$$

Proof. Set $H = C^\circ(i)$.

By Proposition 4.6 $U_p(O^\sigma H) = 1$ for all primes p. By Lemma 4.11

$$[\Delta_r(H), F(O^\sigma(H)]) = 1.$$

As $\Delta_r(H)$ is characteristic in H we have $OF(\Delta_r(H)) \leq OF(H) \leq FO^\sigma(H)$ and thus $OF(\Delta_r(H)) \leq Z(\Delta_r(H))$.

By Proposition 3.10

$$\Delta_r(H) = E_{alg}(\Delta_r(H)) * K$$

with $K/Z(K)$ of degenerate type.

As the center of $E_{alg}(\Delta_r(H))$ is finite it follows easily that

$$\Delta_r(K) = K.$$

Now $E_{alg}(\Delta_r(H))$ is a central product of quasi-simple components L of H with finite centers, so each satisfies $L = \Delta_r(L) \leq \Delta_r(E_{alg}(H))$. On the other hand, $\Delta_r(E_{alg}(H))$ is a connected normal subgroup of H contained in $E_{alg}(H)$, hence is a product of components L of $E_{alg}(H)$ with $\Delta_r(L) = L$. It follows that

$$E_{alg}(\Delta_r(H)) = \Delta_r(E_{alg}(H)).$$

This completes the proof. $\square$

Lemma 4.13. *Let G be a group satisfying Hypothesis 4.1.*
Let i be an involution of G and $\rho = r_{f,G}$. Then

$$\Delta_\rho(C_G(i)) = \Delta_\rho(E_{alg}(C_G(i)))$$

is the product of algebraic components of $C_G(i)$ whose base field k either has non-zero characteristic or satisfies $\mathrm{rk}(k) > \rho$.

Proof. We use the notation of Lemma 4.12 and continue the analysis.

$$\Delta_\rho(C_G(i)) = E_{alg}(\Delta_\rho(C(i))) * K$$

with $K = \Delta_\rho(C_{C_G(i)}(\Delta_\rho(E_{alg}(C_G(i)))))$. We need to show that K is trivial.

By the definition of ρ we find that K centralizes

$$E_{alg}(C_G(i)/OFC_G(i)).$$

Claim 1. *If T is a 2-torus in $C_G(i)$ then T centralizes K.*

In view of Lemma 3.11, we have $[T, K] \leq OF(C_G(i))$.

Let U be an abelian $U_{0,r}$-subgroup of K with $r > \rho$, or a connected p-unipotent subgroup for some prime p. Then $[T, U] \leq OF(C_G(i))$ and thus T normalizes $OF(C_G(i)) \cdot U$. As

$$U = U_{0,r}(UOF(C_G(i)) \text{ or } U_p(UOF(C_G(i)), \text{ respectively,}$$

we find that T normalizes U.

If T does not centralize U then some $t' \in T$ acts on U as a non-trivial involutory automorphism and U decomposes as $U^+ \times U^-$ under this action, with both factors being $U_{0,r}$-subgroups or connected p-unipotent subgroups, correspondingly. But $U^- = [t', U^-] \leq OFC_G(i)$ and hence $U^- = 1$, and t' centralizes U. The claim follows.

Claim 2. $K \leq \Delta_\rho(C_{C_G(j)}(\Delta_\rho(E_{alg}(C_G(j)))))$ *for j co-toral with i.*

We have i, j in a 2-torus T of $C_G(i)$. In particular $K \leq C_G(j)$.

With U either an abelian $U_{0,r}$-subgroup of K and $r > \rho$, or a p-unipotent subgroup of K, as U centralizes T the induced action on $E_{alg}(C_G^\circ(j)/OFC_G(j))$ is either trivial or acts on some component like a subgroup of the multiplicative group of the base field. This would contradict the definition of ρ. This proves the claim.

Now take a 4-group V containing i and contained in a maximal 2-torus T. If we define

$$K_t = \Delta_\rho(C_{C_G(t)}(\Delta_\rho(E_{alg}(C_G(t)))))$$

for t an involution then our last claim states that $K_j \leq K_j$ for co-toral involutions. Thus for t an involution in V, the group K_t is independent of t.

Hence $\Gamma_V \leq N(K)$ and thus $K = 1$. □

Proposition 4.14. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected. Then $\mathcal{E}_A$ is nonempty.*

Proof. By Lemma 4.10 there is at least one algebraic component in $C(i)/O^\sigma C(i)$, for some $i \in I(A)$.

Take an involution $i \in A$ with $r_{f,i} = r_{f,A}$ for which there is an algebraic component $\bar{L}$ of $C(i)/O^\sigma C(i)$ whose base field $k = k_{\bar{L}}$ satisfies

$$\bar{\mathrm{r}}_0(k^\times) = r_{f,A}.$$

Set $H = C^\circ(i)$ and $\rho = r_{f,A}$. Let L be the preimage of $\bar{L}$ in H.

By Proposition 4.6 $\rho \geq \bar{\mathrm{r}}_0(H)$. Thus by Lemma 4.12 we have

$$\Delta_\rho(H) = E_{alg}(\Delta_\rho(H)) * K$$

with $K = \Delta_\rho(K)$ and the Sylow 2-subgroup of K central.

By Lemma 4.9 and Corollary 2.45 we have

$$\bar{L} = \Delta_\rho(\bar{L}).$$

By Fact 2.34 $\Delta_\rho(L)$ covers $\bar{L}$. In particular $\Delta_\rho(L)$ is a normal subgroup of $\Delta_\rho(H)$ which, taken modulo its solvable radical, is simple and has a nontrivial 2-Sylow subgroup.

If we had

$$\Delta_\rho(L) \cap E_{alg}(H) = 1$$

then $\Delta_\rho(L)$ would be isomorphic to a subgroup of K, and have a central Sylow 2-subgroup. So we must have $\Delta_\rho(L) \leq E_{alg}(H)$.

Accordingly $\Delta_\rho(L)$ is a quasi-simple algebraic component of H with $\bar{L}$ as a quotient (by a finite center contained in $O^\sigma(H)$). In particular

$\Delta_\rho(L)$ has the same base field as $\bar{L}$ up to definable isomorphism, and belongs to $\mathcal{E}_A$. $\square$

The following variation on the previous lemma is also useful.

Lemma 4.15. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected, and let i be an involution of A. Let L be a definable quasi-simple algebraic subgroup of $C_G(i)$ over a base field k which is either of characteristic p or has $\bar{\mathrm{r}}_0(k^\times) = r_{f,A}$. Then $L \leq E_{alg}(C_G^\circ(i))$.*

Proof. Set $H = C_G^\circ(i)$, $\bar{H} = H/O^\sigma(H)$, and $\rho = r_{f,A}$. Then as in the preceding argument

$$\begin{aligned} \bar{L} = \Delta_\rho(\bar{L}) &\leq \Delta_\rho(H) \\ &= E_{alg}(\Delta_\rho(H)) * \bar{K} \end{aligned}$$

where the Sylow 2-subgroup of $\bar{K}$ is central.

As $\bar{L}$ is quasi-simple and contains a noncentral 2-torus, we have $\bar{L} \leq E_{alg}(\Delta_\rho(\bar{H}))$. So $\bar{L}$ is contained in a product of algebraic components of $E_{alg}(\bar{H})$ with the same base field, and these are covered by isomorphic algebraic components of $E_{alg}(H)$. Thus $L \leq \hat{L} * O^\sigma(H)$ where $\hat{L}$ is a product of algebraic components of H. It follows that $L \leq \hat{L} \leq E_{alg}(H)$. $\square$

4.3. **Generation by components.** Now we aim at the following.

Proposition 4.16. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let A be an elementary abelian 2-group of 2-rank 3 such that any pair of involutions in A are co-toral. Set*

$$\rho = r_{f,A}.$$

Then G is generated by the groups

$$E_{alg}(\Delta_\rho(C_G(i))) \text{ (for } i \in I(A)\text{)},$$

or even by the restricted family corresponding to $i \in I(U)$, where $U \leq A$ is any elementary abelian subgroup of 2-rank 2.

We work with components of type $\mathcal{E}_A$ (Definition 4.7).

Lemma 4.17. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected, and let $U \leq A$ be a subgroup of 2-rank 2 which is contained in a 2-torus.*

Then there is an involution $i \in U$ such that $C(i)$ contains a quasi-simple component belonging to $\mathcal{E}_A$. Furthermore, for any quasi-simple *algebraic component L belonging to $\mathcal{E}_A$, there is an involution $i \in U$ such that L is contained in a product of quasi-simple components belonging to $\mathcal{E}_U$.*

Proof. Let $\rho = r_{f,A}$ and fix an involution $j \in I(A)$ and a quasi-simple component L of $C(j)$ with $L \in \mathcal{E}_A$ (Proposition 4.14).

Case 1. Some $i \in I(U)$ does not normalize L.

In this case, consider $\hat{L} = \langle L, L^i \rangle = L * L^i$. Then $C_{\hat{L}}(i)$ contains the diagonal subgroup $\hat{L} = \{(a, a^i) : a \in L\}$ which is a central quotient of L. By Lemma 4.15 this is contained in $E_{alg}(C_G(i))$, hence lies in a product of components with the same base field as L up to definable isomorphism.

Case 2. U normalizes L.

Then by Fact 3.2 U induces algebraic automorphisms of L. By Fact 3.3 their centralizers are reductive (with the same base field, up to definable isomorphism).

Thus any components of these centralizers will meet the conditions of the lemma.

If there are no such components, then by Fact 3.4 the group L is of the form (P)SL_2. Then there are no graph automorphisms so U induces inner automorphisms.

Now U is contained in a maximal 2-torus T of G. Then $T \cap L$ is a maximal 2-torus of L, and the Zariski closure of $T \cap L$ is an algebraic torus R of L. As U acts trivially on T it acts like a subgroup of R on L. Therefore the action is not faithful, and some involution $i \in U$ centralizes L. We then conclude via Lemma 4.15. □

Corollary 4.18. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3 such that the co-torality graph on $I(A)$ is connected., and let $U \leq A$ be a subgroup of 2-rank 2 which is contained in a 2-torus. Then*

$$r_{f,U} = r_{f,A}.$$

Now we require an analog of the groups Γ_V.

Definition 4.19. Let G be a group of finite Morley rank, U an elementary abelian 2-subgroup of of 2-rank 2, H a U-invariant subgroup of G, and $r \geq 0$. Then

$$\tilde{\Gamma}_{U,r}(H) = \langle E_{alg}(\Delta_r(C_H(i))) : i \in I(U) \rangle$$

Lemma 4.20. *Let G be a group of finite Morley rank satisfying Hypothesis 4.1. Let $A \leq G$ be an elementary abelian 2-subgroup of 2-rank 3. Suppose that every pair of involutions in A are co-toral. Let $\rho = r_{f,A}$.*

Then for $U, V \leq A$ of 2-rank 2 we have

$$\tilde{\Gamma}_{U,\rho}(G) = \tilde{\Gamma}_{V,\rho}(G)$$

Proof. Let $i \in U$ and let L be an algebraic component of $C_G(i)$ with $\Delta_\rho(L) = L$. We claim that

$$L \leq \langle E_{alg}(\Delta_\rho(C_G(v))) \,:\, v \in I(V)\rangle$$

We highlight the following point as it depends on the co-torality hypothesis.

Claim 1. *V normalizes L.*

By the co-torality hypothesis, $A \leq C_G^\circ(i)$. As the group L is normal in $C_G^\circ(i)$, the claim follows.

If L is not of type $(\mathrm{P})\mathrm{SL}_2$ then by Facts 3.2 and 3.5,

$$L = \langle E(C_L(v) \,:\, v \in I(V)\rangle = \langle E_{alg}(C_L(v)) \,:\, v \in I(V)\rangle$$

Now $E_{alg}(C_L(v))$ is a product of quasi-simple groups with nontrivial Sylow 2-subgroups, and with the same base field as L, and it follows that $E_{alg}(C_L(v)) \leq E_{alg}(\Delta_\rho(C(v)))$. The result follows easily in this case.

On the other hand, if L is of type $(\mathrm{P})\mathrm{SL}_2$, then as argued previously, some $v \in I(V)$ centralizes L, and we conclude similarly. □

Proof of Proposition 4.16. Set

$$G_0 = \langle E_{alg}(\Delta_\rho(C_G(i))) \,:\, i \in I(A)\rangle.$$

It follows from Lemma 4.20 that

$$G_0 = \tilde{\Gamma}_{U,\rho}$$

for $U \leq A$ elementary abelian of 2-rank 2. Hence

$$\Gamma_{A,2} \leq N(G_0).$$

But then by Lemma 3.21, for $U \leq A$ elementary abelian of 2-rank 2 we have

$$\Gamma_U \leq N(G_0).$$

So by Fact 3.16 G has a strongly embedded subgroup, a contradiction. □

5. The High Prüfer Rank Theorem

We arrive at the main result.

Theorem 5.1 (High Prüfer Rank Theorem)**.** *Let G be a simple L^*-group of finite Morley rank of odd type with Prüfer 2-rank at least three which satisfies* NTA_2*. Then one of the following applies.*

- *G is a Chevalley group over an algebraically closed field of characteristic other than 2; or*
- *G has a proper definable strongly embedded subgroup.*

The full proof would be rather long, but the bulk of it has been covered by previous work. We follow [Bur09] which aims to bring [BB04] to bear. In fact, in [Bur09], sufficient conditions for the argument in [BB04] were given. These conditions are collected in Hypothesis 5.10 below.

From that point onward, one can follow the line of [BB04] to analyze the Weyl group and then apply the form of the Curtis-Tits theorem given as Fact 5.23 below. The differences between the treatment in the K^* and L^* contexts become minor, the principal difference being that we must now explicitly require all quasi-simple components considered to be algebraic.

See Fact 5.11, p. 50 for a statement of this last point, which could conclude the present paper, though we continue afterward with a review of how the argument proceeds from that point.

5.1. **Hypotheses and notation.**

Our operative hypotheses will be those of Hypothesis 4.1, together with the assumption of Prüfer 2-rank at least 3. We will work relative to a fixed maximal 2-torus T. We codify this as follows.

Hypothesis 5.2. Let G be a group of finite Morley rank. We suppose the following.

1. G is a connected simple L^*-group of odd type.
2. G satisfies NTA_2.
3. $\mathrm{pr}_2(G) \geq 3$.
4. G has no proper nontrivial definable strongly embedded subgroup.

Furthermore, as a matter of notation we suppose

$$(*) \qquad T \text{ is a fixed maximal 2-torus in } G.$$

Notation 5.3. Let G be a group of finite Morley rank satisfying Hypothesis 5.2.

For any involution $i \in I(T)$ and any definable connected quasi-simple algebraic $L \leq E_{alg}(C\_G^\circ(i))$ which is normalized by T, we set

$$\mathbb{T}_L = C_L(T).$$

Lemma 5.4. *Let G be a group of finite Morley rank satisfying Hypothesis 5.2. For any $i \in I(T)$ and any definable connected quasi-simple algebraic $L \leq E(C_G^\circ(i))$ which is normalized by T, we have*

1. *$T = (T \cap L) \cdot C_T^\circ(L)$, and*

$$\mathrm{pr}_2(G) = \mathrm{pr}_2(T) = \mathrm{pr}_2(C_T^\circ(L)) + \mathrm{pr}_2(T \cap L).$$

2. *$\mathbb{T}_L = C_L(T \cap L)$ is a maximal algebraic torus of L,*

Proof.

Ad 1. Since T normalizes L, the intersection $T \cap L$ is a maximal 2-torus of L.

By Fact 3.2, the connected definable group $d(T)$ acts by inner automorphisms on L. These inner automorphisms centralize $T \cap L$ so those in T are induced by elements of $T \cap L$. The claim follows.

Ad 2. By (1)

$$\mathbb{T}_L = C_L(T \cap L)$$

By Fact 3.25, this is a maximal algebraic torus of L. □

Our focus will be on the components in the family $\mathcal{E}_T$ (Definition 4.7).

5.2. **Abstract root SL_2-subgroups.** We develop an abstract notion of *root* SL_2*-subgroup* slightly extending the algebraic theory.

Definition 5.5.

1. If L is a quasisimple algebraic group and $\mathbb{T}$ a maximal algebraic torus of L, a *root* SL_2*-subgroup* is any Zariski closed subgroup of L which is normalized by $\mathbb{T}$ and isomorphic to $(\mathrm{P})\mathrm{SL}_2$.

2. Let G be a group of finite Morley rank satisfying Hypothesis 5.2.

A subgroup K of G is a *root* SL_2*-subgroup* of G with respect to T if there is a component L in $\mathcal{E}_T$ such that K is a root SL_2-subgroup of the algebraic group L with respect to its maximal torus $\mathbb{T}_L$.

3. $\Sigma = \Sigma_T$ denotes the set of all root SL_2-subgroups of G with respect to T.

Remark 5.6. Note that in the foregoing, if L is a component in $\mathcal{E}_T$, then L is in particular a component of some $C_G^\circ(i)$ with an i an involution of T, and since $T \leq C_G^\circ(i)$, T will normalize L and $\mathbb{T}_L$ will indeed be a maximal torus of L.

We note also that the restriction to components in $\mathcal{E}_T$ is purely technical, and one could certainly consider a broader definition. But given that we aim at an identification theorem, at which point there would only be one base field, it is at least plausible that our definition of root SL_2-subgroup is sufficiently broad for practical purposes.

Lemma 5.7. *Let G be a group of finite Morley rank satisfying Hypothesis 5.2. Then G is generated by its root SL_2-subgroups.*

Proof. The root SL_2-subgroups of any component K in $\mathcal{E}_T$ generate K by Fact 3.23 (1), and by Proposition 4.16 these components generate G. □

Lemma 5.8. *Let G be a group of finite Morley rank satisfying Hypothesis 5.2, and $\Sigma = \Sigma_T$. For any $K \in \Sigma$, we have the following.*

1. *K is normalized by T.*
2. *If $g \in N_G(T)$ then $K^g \in \Sigma$.*
3. *$K = E(C_G(C_T^\circ(K)))$.*
4. *K is a Zariski closed subgroup of any definable algebraic quasi-simple subgroup $M < G$ which contains K, and which is normalized by T.*

The term *algebraic* is used here in our customary sense (in the context of quasi-simple groups): *Chevalley over an algebraically closed field.* And at this stage in the analysis, we must still allow for the possibility that the base field may vary from one component to another.

Proof.

Ad 1. This point was already observed in Remark 5.6.

Ad 2. The set $\mathcal{E}_T$ is normalized by $N(T)$, hence the same applies to Σ.

Ad 3. Set $T_K^\perp = C_T^\circ(K)$. This is a 2-torus of Prüfer 2-rank one less than that of T. Take an involution $j \in T_K^\perp$. By Lemma 4.15 we have $K \le E(C_G(j))$.

We consider the projection K_i of K onto simple factors of

$$E(C_G(j))/ZE(C_G(j)).$$

Then $T_K^\perp$ centralizes K_i. If there are two such projections which are nontrivial then the Prüfer 2-ranks of T and $T_K^\perp$ differ by at least 2, a contradiction. So K is contained in one of the quasisimple factors L of $E(C_G(j))$. As $L \lhd C_G^\circ(j)$ we find $C_L(T_K^\perp) \lhd C_G^\circ(T_K^\perp)$.

By Fact 3.24, $K = E(C_L(T_K^\perp))$. So $K \lhd C_G^\circ(T_K^\perp)$ and $K \le E(C_G^\circ(T_K^\perp))$. But again $E(C_G^\circ(T_K^\perp))$ has Prüfer 2-rank at most 1, so $K = E(C_G^\circ(T_K^\perp))$.

Ad 4. Let $K \leq M < G$ where M is definable, algebraic, quasi-simple, and normalized by T. The group $T_K^\perp \leq T$ acts on M by inner automorphisms by Fact 3.2, so $K = E(C_M(K^\perp))$ is Zariski closed. □

Lemma 5.9 (cf. [BB04, Lemma 3.1])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 5.2. Let $K, L \in \Sigma$ be distinct, and set*

$$M = \langle K, L \rangle .$$

Then

1. *$(C_T(K) \cap C_T(L))^\circ \neq 1$.*
2. *Either K and L commute or M is an algebraic group of type A_2, $B_2 = C_2$, or G_2.*
3. *K and L are root SL_2-subgroups of M normalized by $\mathbb{T}_M$.*
4. *The maximal tori $\mathbb{T}_K$, $\mathbb{T}_L$ associated with K and L commute.*
5. *$T \cap M = (T \cap K) * (T \cap L)$ is a Sylow° 2-subgroup of M.*

Proof. Set

$$K^\perp = C_T^\circ(K); \qquad\qquad L^\perp = C_T^\circ(L).$$

Ad 1. Here we use the hypothesis of Prüfer 2-rank at least 3.

Since T normalizes K and L by Lemma 5.8 (1), and K, L are of type $(\mathrm{P})\mathrm{SL}_2$ over their respective base fields, we know

$$\mathrm{pr}_2(K^\perp), \mathrm{pr}_2(L^\perp) = \mathrm{pr}_2(T) - 1$$

and it follows easily that

$$\mathrm{pr}_2(K^\perp \cap L^\perp) \geq \mathrm{pr}_2(T) - 2 \geq 1.$$

Thus $(C_T(K) \cap C_T(L))^\circ \neq 1$

Ad 2. Let $i \in I(K^\perp \cap L^\perp)$. By Lemma 4.15, K and L are contained in algebraic quasi-simple components of $C_G(i)$.

If they belong to different components of $C_G^\circ(i)$, then they commute.

Suppose K and L both belong to the same algebraic component $H \in C_G^\circ(i)$. As $K, L \in \Sigma$ we find $H \in \mathcal{E}_T$. Thus H is a quasi-simple algebraic group normalized by T.

By Lemma 5.8 (3), K and L are Zariski closed in H. By Fact 3.22 (1), $M = \langle K, L \rangle$ is an algebraic group of type A_2, $B_2 = C_2$, or G_2.

For points (3–5), if K and L commute then everything is clear. So we suppose $[K, L] > 1$ and M is a quasi-simple algebraic group.

Then points (3,4) follow by Facts 3.22 (2), and 3.23 (2), respectively, and point (5) follows from (4). □

5.3. **An axiomatic setting; The graph on Σ.** At this point we reach the axiomatic setting of [Bur09], which suffices for the identification argument of [BB04] to be carried through. We can drop several of the hypotheses on G which brought us to this point and replace them by the properties of the root SL_2-subgroups.

Hypothesis 5.10. *G is a connected simple group of finite Morley rank and odd type with*

$$\mathrm{pr}_2(G) \geq 3.$$

T_2 is a maximal 2-torus of G.

Σ is a family of subgroups of G of type $(\mathrm{P})\mathrm{SL}_2$.

For $L \in \Sigma$, $\mathbb{T}_L = C_L(T_2)$ is a maximal algebraic torus of L.

We assume that Σ satisfies all conditions given in Lemmas 5.8 and 5.9.

We also assume that

$$(\star) \qquad \left\langle \bigcup \Sigma \right\rangle = G.$$

This hypothesis is in effect to the end of the article, and—in a departure from our earlier practice—will not be systematically repeated below.

Fact 5.11 ([Bur09, §3, cf. Hyp. 3.12, Fact 3.25 et seq.])**.** *Under Hypothesis 5.10 the group G is algebraic (i.e., Chevalley, over an algebraically closed field).*

Furthermore, the argument of [BB04]*, as presented in* [BBBC08]*, applies.*

In the remainder of this article we review the proof of Fact 5.11, following and occasionally elaborating on the discussion from §3 of [Bur09].

Definition 5.12. We give Σ a graph structure by placing an edge between $L, K \in \Sigma$ when $[L, K] \neq 1$.

Lemma 5.13. *The graph Σ is connected, and all groups in Σ are defined over the same base field (the base fields are definably isomorphic). In particular, the rank of the base field is the same in each case.*

Proof. Since G is simple and $\bigcup \Sigma$ generates G, the graph Σ is connected.

By Lemma 5.9 (2), any adjacent pair $L, K \in \Sigma$ are algebraic groups over the same algebraically closed base field. By connectedness the same follows for all pairs K, L. $\square$

Whether the definable isomorphisms referred to in the preceding lemma are *canonical* depends on the structure of Σ. If this graph is acyclic this is more or less the case, selecting one of the copies of the base field as a point of reference.

5.4. **The Weyl group.** We continue under Hypothesis 5.10. We turn our attention to the construction and identification of the Weyl group of G.

Definition 5.14. The *natural torus* $\mathbb{T}$ for G is the group defined by

$$\mathbb{T} = \langle \mathbb{T}_L \,:\, L \in \Sigma \rangle .$$

Lemma 5.15.

(1) *The natural torus* $\mathbb{T}$ *is divisible abelian.*
(2) *For* $L \in \Sigma$*, we have* $\mathbb{T}_L = \mathbb{T} \cap L = C_L(\mathbb{T})$*.*
(3) $N_L(\mathbb{T}_L) \le N_G(\mathbb{T})$*.*

Proof.

Ad 1. By Lemma 5.9 (5), the algebraic tori $\mathbb{T}_K$ for $K \in \Sigma$ all commute,

Ad 2. By the definition of $\mathbb{T}$ and (1) we have $\mathbb{T}_L \le \mathbb{T} \cap L \le C_L(\mathbb{T})$. Hence

$$C_L(\mathbb{T}) \le C_L(\mathbb{T}_L) = \mathbb{T}_L$$

and (2) follows.

Ad 3. Let $w \in L$ normalize $\mathbb{T}_L$, and let $K \in \Sigma$. The claim is that w normalizes $\mathbb{T}_K$.

If K and L commute this is clear. Otherwise, $M = \langle K, L\rangle$ is a quasi-simple algebraic group, and K, L are root SL_2-subgroups. In this case the claim may be verified within M. □

Definition 5.16 (Restricted Weyl Group)**.** Set

$$W = N_G(\mathbb{T})/C_G(\mathbb{T});$$
$$W_L = N_L(\mathbb{T}_L)/\mathbb{T}_L \text{ for } L \in \Sigma.$$

By Lemma 5.15 (2,3) there is a canonical embedding of W_L into W.

The group W_L has order 2. Let r_L denote the involution of W corresponding to a generator of W_L.

Define the *restricted Weyl group* W_0 as

$$\langle r_L \,:\, L \in \Sigma \rangle .$$

Recall that W is finite, and hence W_0 is also finite.

Lemma 5.17 (cf. [BB04, Lemma 3.5])**.** *For any* $L, K \in \Sigma$*,* $[K, L] = 1$ *if and only if* $[r_K, r_L] = 1$*.*

Proof. We may suppose that $[K, L] \neq 1$, in which case $M = \langle K, L \rangle$ is algebraic by Fact 3.22, and K, L are root SL_2-subgroups of M with respect to $\mathbb{T}_M$.

In this setting the Weyl group acts on the root groups as it acts on the roots, and the commutation relation corresponds to orthogonality of roots in both cases. □

Lemma 5.18 (cf. [BB04, Lemmas 3.6 & 3.7])**.**

(1) T_2 is the Sylow 2-subgroup of $\mathbb{T}$.

(2) $C_G(T_2) = C_G(\mathbb{T})$

In particular, the restricted Weyl group W_0 acts faithfully on T_2.

Proof.

Ad 1. By Lemma 5.15, $\mathbb{T}$ is divisible abelian, so its Sylow 2-subgroup is connected.

Let $D = \mathbb{T} \cap T_2$. Suppose toward a contradiction that

$$D < T.$$

For all $K \in \Sigma$, we have $[T_2, K] \leq K$ and

$$[T_2, r_K] \leq T_2 \cap K \leq \mathbb{T}_K,$$

so r_K acts trivially on T_2/D, and thus W_0 acts trivially on T_2/D.

Let $b \in T_2$ be of order at least 4 modulo D, and let $a \in T_2$ satisfy $a^{|W_0|} = b$. Then

$$c = \Pi_{w \in W_0} a^w$$

satisfies $b/D = c/D$. Hence the order of c is at least 4.

Now for $K \in \Sigma$ we have

$$T_2 = C_{T_2}(K) * (T_2 \cap K);$$
$$C_{T_2}(r_K) = C_{T_2}(K) * C_{T_2 \cap K}(r_K).$$

Hence

$$[C_{T_2}(r_K) : C_{T_2}(K)] \leq 2$$

We also have $c \in C_{T_2}(r_K)$, and thus

$$c^2 \in C_{T_2}(K)$$

for all $K \in \Sigma$. So $c^2 \in C_G(\langle \bigcup \Sigma \rangle) = Z(G) = 1$, a contradiction.

Thus T_2 is the Sylow 2-subgroup of $\mathbb{T}$.

Ad 2. Since $T_2 \leq \mathbb{T}$ we have $C_G(\mathbb{T}) \leq C_G(T_2)$.

For the reverse direction, consider $x \in C_G(T_2)$. Then for every $L \in \Sigma$, x centralizes $C_{T_2}(L)$. So x normalizes $L = E(C_G(C_{T_2}(L)))$ by Lemma

5.8 (2). Since x centralizes the maximal 2-torus $\mathbb{T}_L$, x must act on L as an element of $\mathbb{T}_L$ by Fact 3.2. Thus $x \in C_G(\mathbb{T})$ and $C_G(T_2) \leq C_G(\mathbb{T})$. □

Now we use the action of the restricted Weyl group W_0 on the group T_2 to obtain a complex representation of W_0.

Lemma 5.19 (cf. [BB04, §3.3])**.** *W_0 has a faithful irreducible complex representation of dimension $d = \mathrm{pr}_2(G) \geq 3$ in which the involutions r_L act as complex reflections for $L \in \Sigma$.*

For the construction we employ a Tate module over the 2-adic integers (Fact 2.8).

Proof of Lemma 5.19. By Lemma 5.18, W_0 acts faithfully on T_2. By Fact 2.8, W_0 has a faithful representation over the ring of 2-adic integers $\mathbb{Z}_2$ on a free module of dimension $d = \mathrm{pr}_2(T_2) \geq 3$. Embedding $\mathbb{Z}_2$ into $\mathbb{C}$, we view this as a faithful complex representation of dimension d.

For $L \in \Sigma$, r_L acts by inversion on $T_2 \cap L$ and trivially on $C_{T_2}(L)$. The intersection of these two 2-tori, though finite, is not necessarily trivial. However in the dual Tate module $\hat{T}_2$ this corresponds to a direct sum decomposition and r_L is represented as a reflection over the ring $\mathbb{Z}_2$ and hence over $\mathbb{C}$.

It remains to prove the following.

Claim 1. *This complex representation V is irreducible.*

By Lemma 3.27 we need to show that the reflections r_K for $K \in \Sigma$ have no common fixed point. For this, it suffices to consider the action on the Tate module. In terms of the dual action on T_2 the question is whether there is a nontrivial 2-torus $R \leq T_2$ centralized by all of these reflections.

If $K \in \Sigma$ and r_K centralizes R, then $R \leq K^\perp = C_{T_2}(K)$. Since $\bigcup \Sigma$ generates G, if R commutes with all the r_K then $R \leq Z(G)$, a contradiction. □

Lemma 5.20 (cf. [BB04, §3.4])**.** *For primes p not dividing the order of W_0, and not the characteristic of the (common) base field of the groups in Σ, the group W_0 has a faithful irreducible representation over $\mathbb{Z}/p\mathbb{Z}$ such that the involution r_L acts by a reflection for any $L \in \Sigma$.*

Proof. Consider the finite elementary abelian p-group E_p generated by all elements of order p in $\mathbb{T}$. The whole Weyl group $W = N_G(\mathbb{T})/C_G(\mathbb{T})$ acts on E_p.

Claim 1. *$W = N_G(\mathbb{T})/C_G(\mathbb{T})$ acts faithfully on E_p.*

Suppose $w \in W$ centralizes E_p.

If $L \in \Sigma$ then $L \cap L^w$ contains E_p, but L and L^w either commute or are root SL_2-subgroups in some quasi-simple algebraic group. Hence $L = L^w$. As L is of type $(\mathrm{P})\mathrm{SL}_2$, the element w acts on L as an inner automorphism by Fact 3.2, and hence acts on $\mathbb{T} \cap L$ as an element of $N_L(\mathbb{T} \cap L)$. That is, w either centralizes or inverts $\mathbb{T}_L$. Since w centralizes E_p and p is not the characteristic of the base field, w centralizes $\mathbb{T}_L$ for each $L \in \Sigma$.

But then w centralizes $\mathbb{T}$, that is $w = 1$ as an element of $N_G(\mathbb{T})/C_G(\mathbb{T})$.

Claim 2. *The involutions r_K act as reflections on E_p for $K \in \Sigma$.*

For every $L \in \Sigma$, $[E_p, r_L] \leq \mathbb{T}_L$ so $[E_p, r_L] = E_p \cap L$, a cyclic group inverted by the involution r_L.

Claim 3. *W_0 acts irreducibly on E_p.*

By Lemma 3.27 it suffices to check that the subgroup V of E_p fixed by all of the involutions r_L is trivial.

For any $L \in \Sigma$, V acts on L by inner automorphisms (Fact 3.2). As $V \leq E_p$, V centralizes $\mathbb{T}_L$. Accordingly, for each $v \in V$ there is $t = t(v, L)$ such that vt centralizes L. As r_L commutes with v, the element t centralizes r_L, so t is at worst an involution and v^2 centralizes L. As $\bigcup \Sigma$ generates G we find $v^2 \in Z(G)$, $v^2 = 1$, and $V = 1$. □

Proposition 5.21 (cf. [BB04, Lemma 3.11])**.** *There exists an irreducible root system I of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, or F_4 on which W_0 acts as a crystallographic reflection group.*

Proof. Apply Fact 3.26 to W_0, with $I = \{r_L : L \in \Sigma\}$.

The set

$$I = \{r_L : L \in \Sigma\}$$

generates W_0, and is closed under conjugation even in W since Σ is $N(T_2)$-invariant. The noncommuting graph on this set is connected by Lemma 5.17.

Lemma 5.20 provides suitable linear actions over $\mathbb{F}_p$. Lemma 5.19 provides a representation of dimension at least 3 as required in the final clause of Fact 3.26. □

To complete the discussion of the restricted Weyl group, we show that all reflections in this group come from the root SL_2-subgroups in Σ.

Lemma 5.22 (cf. [BB04, Lemma 3.12])**.** *Every $r \in W_0$ which is a reflection in the representation R over $\mathbb{C}$ has the form r_K for some $K \in \Sigma$.*

Proof. By Fact 3.28, there are at most two conjugacy classes of reflections in $I(W_0)$, corresponding to the short and long roots. So we may assume that W_0 has more than one root length, i.e. W_0 is of type B_n, C_n, or F_4, and that the set $S = \{r_L : L \in \Sigma\}$ consists of only one of these conjugacy classes.

But then by Fact 3.29, $\langle S \rangle < W_0$, a contradiction. $\square$

5.5. **Identification.** We continue under Hypothesis 5.10 to the final identification of the group G, following the discussion at the end of [BBBC08, §4].

With the Weyl group identified, we can refine our notion of "root SL_2-subgroup" and arrive at the hypotheses of a version of the Curtis-Tits Theorem, based on a result given by Timmesfeld [Tim04].

The result required says that if we have a suitable collection of "root SL_2-subgroups" which is actually attached to an irreducible root system of spherical type, then we have, essentially, the root SL_2-subgroups of the expected Chevalley group. The precise formulation we use runs as follows.

Fact 5.23 ([BBBC08, Prop. 2.3])**.** *Let Φ be an irreducible root system of spherical type and rank at least 3, and let Π be a system of fundamental roots for Φ. Let X a group generated by subgroups X_r for $r \in \Pi$, Set $X_{rs} = \langle X_r, X_s \rangle$. Suppose that for $r, s \in \Pi$ we have*

> *X_{rs} is a group of Lie type Φ_{rs} over an infinite field, with X_r and X_s corresponding root SL_2-subgroups with respect to some maximal torus of X_{rs}.*

Then $X/Z(X)$ is isomorphic to a group of Lie type via a map carrying the subgroups X_r to root SL_2-subgroups.

Proof of Theorem 5.1. By Proposition 5.21 and Lemma 5.22, our restricted Weyl group W_0 is the Coxeter group associated with an irreducible root system of spherical type and the set of distinguished involutions

$$I = \{r_L : L \in \Sigma\}$$

corresponds to the set of reflections in the associated real representation of W_0. Furthermore *apart from root lengths,* the associated Dynkin diagram Δ is determined by W_0, and corresponds to a subset of I which we call I_0.

Claim 1. *The groups L_i for $i \in I_0$ generate G.*

Let $G_0 = \langle L_i : i \in I_0 \rangle$.

The reflections r_i for $i \in I_0$ generate W_0 and the groups L_i contain representatives of these elements. If $L \in \Sigma$ then r_L is conjugate under W_0 to one of the reflections r_i with $i \in I_0$, so there is a group $L^* \in \Sigma$ which is conjugate to L_i under G_0 and satisfies

$$r_L = r_{L^*}.$$

But as L, L^* lie in Σ, if $L \neq L^*$ this contradicts the structure of $\langle L, L^* \rangle$. Thus $L = L^* \leq G_0$.

Now we have largely recovered the hypotheses of Fact 5.23, with Π corresponding to I_0. But we need to examine more carefully the key condition, which we repeat.

> For $r, s \in I_0$, X_{rs} is a group of Lie type Φ_{rs} over an infinite field, with X_r and X_s corresponding root SL_2-subgroups with respect to some maximal torus of X_{rs}.

This is approximately the content of Lemma 5.9. To this we may add that when two root SL_2-subgroups $K, L \in \Sigma$ do not commute, the structure of $\langle K, L\rangle$ can be inferred from the order of $r_K r_L$, and that even when K, L do commute the base fields can be identified, as the Dynkin diagram is connected.[8]

At this point, we have managed to verify the required information for Fact 5.23 (without actually determining root lengths, which may perhaps be remarkable).

Accordingly G, which is centerless, is a simple Chevalley group over an algebraically closed field. □

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The strategy follows the treatment of the K^* case given in [Bur09], supplemented primarily by some technical points that emerged from ideas about the study of torsion in groups of finite Morley rank. In particular, before the hiatus mentioned, the paper [BC08] developed the generation theorem needed for the odd type L^*-group theory (without additional hypotheses on definable automorphisms). Modulo that foundational work, we are able to follow the existing argument for the K^* case quite closely.

On a technical note, we do not refer here explicitly to the "B-conjecture," but that would be an appropriate abstract framework for the core argument.

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[8] Even canonically, as the Dynkin diagram is a tree. This fact is not used here but it does come into the proof of Lemma 5.23).

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