L^*-GROUPS OF ODD TYPE WITH RESTRICTED 2-TORAL ACTIONS III: IDENTIFICATION OF PSp_4

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Abstract. We give an identification theorem for PSp_4 as part of an investigation of simple K^*-groups of finite Morley rank of odd type having Prüfer 2-rank 2 and 2-rank at least 3. More generally, rather than taking a K^*-hypothesis, we assume an L^*-hypothesis, so that degenerate type simple sections are allowed, but we also place restrictions on their definable automorphism groups.

A prior paper analyzed algebraic components in centralizers of involutions, isolating the expected configurations corresponding to PSp_4 or G_2. Here we pursue the line which leads to PSp_4. This is more tractable than the other line, which we will discuss further elsewhere.

In the present paper the first half of the analysis, dealing with the Weyl group, covers both the PSp_4 and G_2 configurations. The identification of PSp_4 then involves the construction of a BN-pair.

Contents

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1. Introduction

1.1. **Toward the quasi-thin Algebraicity Conjecture.** The Algebraicity Conjecture for groups of finite Morley rank states that an infinite simple group of finite Morley rank is algebraic, or more explicitly, a Chevalley group over an algebraically closed field. This has been proved for groups of infinite 2-rank. In the remaining cases the connected component of a Sylow 2-subgroup is a 2-torus (a divisible abelian 2-group) of finite Prüfer 2-rank (i.e., the 2-rank of its socle). When the Algebraicity Conjecture holds this will also be the Lie rank.

One approach to take is inductive. This runs into the difficulty that the analysis is easier if the Prüfer 2-rank is large, so that if one argues in an inductive setting one may prove that the *first* counterexample to the Algebraicity Conjecture must occur in low Prüfer rank, but not that *all* counterexamples have low Prüfer rank. Still it is valuable to attempt a census of the configurations involved essentially in potential counterexamples to the Algebraicity Conjecture. We aim to do this in a way which separates the issues connected specifically with groups of degenerate type from the other issues involved when involutions are present.

The Prüfer rank of a *minimal* counterexample to the Algebraicity Conjecture is at most 2. This is proved in [Bur09] after a long series of developments, notably the identification theorem of [BB04]. This is stated as follows. (See also Fact 1.6 below.)

Fact 1.1. *A simple K^*-group of finite Morley rank with Prüfer 2-rank at least three is algebraic.*

The K^* hypothesis incorporates the inductive assumption: every proper definable simple connected section of the group is algebraic. By analogy with the terminology used in finite group theory, one might call the Prüfer 2-rank two case the *quasi-thin* case.

The proof of Fact 1.1 is quite uniform. We cannot expect such a uniform treatment in Prüfer 2-rank two. Instead, one anticipates (that is, one aspires to) three separate identification theorems, one for each of the groups SL_3, PSp_4, and G_2. These must be supplemented by some prior analysis that produces a suitable point of departure for each of the three cases.

As far as the case of 2-rank at least 3 is concerned, that preliminary analysis was carried out in [BC22b]. The following contains a good deal of the information resulting from that analysis. A more detailed and more general version will be given afterward as Fact 2.1.

Fact 1.2 ([BC22b, Theorem 1.2])**.** *Let G be a K^*-group of finite Morley rank and odd type, and Prüfer 2-rank two. Suppose further that the 2-rank is at least three. Then there is an involution i in G whose centralizer has a component of type SL_2, and one of the following occurs.*

(1) There are two conjugacy classes of involutions, the 2-rank is 4, and the centralizer $C_G(i)$ is not connected.

(2) There is one conjugacy class of involutions, the 2-rank is 3, and the centralizer $C_G(i)$ is connected.

This gives us three useful dividing lines—conjugacy classes of involutions, 2-rank, or connectivity of involution centralizers—which are mutually equivalent. It is most helpful to take the number of conjugacy classes of involutions as the initial point of departure. The prior analysis also gives fairly detailed information about the structure of centralizers of involutions in these two settings.

In each case there is only one conjugacy class of involutions i whose centralizer contains a component of type SL_2, and in that case $E(C_G(i))$ is of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$. Furthermore, when $C_G(i)$ is disconnected some involution in $C_G(i)$ swaps the two components, and otherwise, the components are normal in $C_G(i)$.

When there is a second conjugacy class of involutions, for such an involution t we have

$$E(C_G(t)) \simeq \mathrm{PSL}_2(k)$$

where k is also the base field for the components of type SL_2. Further one sees that the Sylow 2-subgroup is determined in each case, and has the expected form as in PSp_4 or G_2 respectively.

A preliminary statement (for the K^* setting) of our main result runs as follows.

Theorem. *Let G be a K^*-group of finite Morley rank and odd type, with Prüfer 2-rank two and 2-rank at least three, and having precisely two conjugacy classes of involutions. Then G has the form $\mathrm{PSp}_4(k)$ for some algebraically closed field k.*

However we will limit the use of the inductive hypothesis so as to get a clearer view of the actual difficulties which may arise from smaller configurations.

1.2. **Beyond K^*-groups.** To get a sharper view of what the minimal obstructions to the Algebraicity Conjecture may be we will work in a broader inductive framework.

Definition 1.3. The groups of finite Morley rank and finite 2-rank are subdivided as follows.

(1) Groups without involutions: *degenerate type*;

(2) Groups with involutions (and finite 2-rank): *odd type*.

In the case of odd type, the Prüfer rank is non-zero [BBC07]. So the degenerate/odd type distinction provides a fairly strong dividing line.

Relatively little is known about the possible structure of simple groups of finite Morley rank of degenerate type—that is, we have no Feit-Thompson theorem, and for that matter no character theory. In odd type, the K^* hypothesis rules out all sections of degenerate type by assumption. But one can get by with substantially weaker inductive assumptions. We will make use of the following two.

Definition 1.4. Let G be a group of finite Morley rank and finite 2-rank.

(1) G is an L^**-group* if every proper definable section of non-zero Prüfer rank is algebraic.

(2) G satisfies the condition NTA_2 if very connected definable section which acts definably and faithfully on a simple section without involutions also has no involutions. (This may be read as: "no 2-toral actions," for short.)

Similar notions played a very striking role in the case of infinite 2-rank, where they actually led to a full proof of the corresponding part of the Algebraicity Conjecture. One reason this works so well in that setting is that the analog of the condition NTA_2 does not need to be taken as a hypothesis (and even holds in a stronger form), because in that setting there is a direct proof of it ("Altınel's Lemma"). We are not going to describe the that version of L^*-group theory in any more detail here. It is necessary to distinguish the meaning of the term "L^*-group" in the two settings, but in view of the successful proof of the Algebraicity Conjecture in the case of infinite 2-rank, one does not foresee much further use for the version used in the case of infinite 2-rank (that is, even and mixed types).

One should also keep a third condition in view, in general.

Definition 1.5. Let G be a group, M a proper subgroup. Then M is *strongly embedded* in G if M contains an involution while the intersection of M with any conjugate M^x, $x \notin M$, contains no involution.

The particular case in which our group has a definable strongly embedding subgroup can present difficulties, though not in the present paper, as we shall see. Such groups resist inductive analysis (regardless of Prüfer rank) though they may be amenable to other techniques.

In particular, the generalized version of Fact 1.1 runs as follows.

Fact 1.6 ([BC22a, Theorem 5.1]). *A simple L^*-group of finite Morley rank with Prüfer 2-rank at least three is either algebraic or has a strongly embedded subgroup and a connected Sylow 2-subgroup.*

In particular, in the latter case, the Prüfer 2-rank and the 2 rank are equal.

A satisfactory treatment of the strongly embedded case has been given for K^*-groups in Prüfer rank at least 2. So with that in mind, Fact 1.6 can be viewed as a generalization of Fact 1.6. There is presently no clear strategy for extending the treatment of strongly embedded subgroups to the class of L^*-groups with NTA_2.[1]

On the other hand, our concern here is with the case of Prüfer rank equal to two and 2-rank at least three, so in the present paper we will not encounter difficulties with strongly embedded subgroups.

Fact 1.2 has also been proved with the K^* hypothesis weakened to an L^*-hypothesis together with the condition NTA_2. This provides the starting point for the present paper and will be discussed in considerable detail below.

1.3. **The main result.** The main result to be proved here takes on the following form.

Theorem 1.7. *Let G be a group of finite Morley rank and odd type. Suppose that G is a simple L^*-group satisfying NTA_2, with Prüfer 2-rank two and 2-rank at least three, and having precisely two conjugacy classes of involutions. Then G has the form $\mathrm{PSp}_4(k)$ for some algebraically closed field k.*

The main differences between the K^* setting and our more general setting are found in the prior work in this series (and the literature). For what follows little will be lost (or gained) by thinking in terms of the K^* setting.

We proceed as follows, aiming at identification via the theory of BN-pairs. With the "component analysis" of [BC22b] in hand—that

[1]It is an interesting topic, with some unpublished partial results. For the present, the strongly embedded case must be added to the list of problematic configurations (with, unfortunately, no bound on Prüfer rank).

is, with the structure of centralizers of involutions pinned down to the extent described in Fact 2.1— one next defines the *Weyl group* and a well controlled subgroup N lifting the Weyl group into the given group G. At this point one would like to look also at a maximal unipotent subgroup. Here a more abstract substitute, denoted U, is introduced. In fact U becomes considerably less abstract as we proceed; cf. Lemma 3.20. This brings us to a putative BN-pair. Proving that the desired conditions are in fact satisfied involves working rather closely with the action of the Weyl group and establishing a qualitative version of the Chevalley commutator formula.

The component analysis left us with two configurations to consider: the case of two conjugacy classes of involutions, which leads here to identification of PSp_4, and the case of one conjugacy class of involutions, which conjecturally should lead to identification of G_2. These two configurations are delineated more thoroughly in Fact 2.1.

The present paper continues the analysis for a time in parallel for both configurations, at least up to the point of determination of the Weyl group, and somewhat beyond that point. When we come to the actual verification of the BN-pair axioms, the details will depend on the precise structure of the Weyl group, and some significant differences arise in the two cases, with respect to the length, the difficulty, and the degree of success of the analyses. And in the end the G_2 analysis will lead into some further byways.

Accordingly, as far as the verification of the BN-pair axioms is concerned, we limit ourselves here to the case of two conjugacy classes of involutions, and we will return elsewhere to a discussion of the remaining case.

One noteworthy point in the case of a single conjugacy class of involutions is that the treatment of some version of a "maximal unipotent subgroup" poses particular difficulties in that setting, and in one case, associated with fields of characteristic 3, those difficulties remain unresolved.

2. Preliminaries

2.1. **The two configurations.** From [BC22b, Thm. 1.2] we have the following. We go off a little into the weeds of characteristic zero unipotence theory here, but the main point is to have a general sense of the structure of centralizers of involutions.

Fact 2.1. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer 2-rank 2 and*

$$m_2(G) \geq 3.$$

Then there are at most two conjugacy classes of involutions, and one of the following applies.

(1) There are two conjugacy classes of involutions.

Then the 2-rank of G is 4; and the Sylow 2-subgroup is as in PSp_4.

One conjugacy class of involutions satisfies

$$C_G^\circ(i) \simeq \mathrm{PSL}_2(k) \times K,$$

with K isomorphic to a subgroup of $k^\times$, and the other class satisfies

$$C_G(i) \simeq \mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k),$$

with the two components of $\mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k)$ conjugate (and all three base fields the same in the sense that they are definably isomorphic).

(2) There is one conjugacy class of involutions, and these satisfy

$$C_G(i) = \mathrm{SL}_2(k_1) *_2 \mathrm{SL}_2(k_2)$$

where the base fields k_1, k_2 have the same characteristic. Furthermore, in characteristic zero, we have

$$\bar{\mathrm{r}}_0(k_1^\times) = \bar{\mathrm{r}}_0(k_2^\times)$$

in the sense of characteristic zero unipotence theory (§2.3).

In particular $C_G(i)$ is connected. Furthermore, $C_G(i)$ contains a Sylow 2-subgroup of G, isomorphic to that of $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ (in characteristic other than 2).

One point to note in case (1) is the use of the notation $*_2$ for a central product with central involutions identified.

We add a bit more information, which lies more on the side of preliminary analysis, but may be worth keeping in mind. This was also included in [BC22b, Thm. 1.2].

Fact 2.2. *Under the assumptions of Fact 2.1, in the case of two conjugacy classes of involutions, we have the following.*

(1) For i an SL_2-involution, i will be the only SL_2-involution in $C_G^\circ(i)$.

(2) The following are equivalent for involutions t.

(a) t is a PSL_2-involution.

(b) t lies in a component of a PSL_2-involution.

(c) t lies in a subgroup of G of type PSL_2.

(3) For t a PSL_2-involution, and $L_t = E_{alg}(C_G(t))$, the involutions of $C_G^\circ(t)$ are those of $L\langle t\rangle$. Those in $L_t \cup \{t\}$ are PSL_2-involutions, and the rest are SL_2-involutions.

We should perhaps add in Lemma 5.2 as well as it gives a fuller sense of the situation. Much of the above is covered in that more precise statement.

For the most part we will be working in the following setting, going forward.

Hypothesis 2.3. Let G be a group of finite Morley rank. We suppose

(1) G is a connected simple L^*-group of odd type which satisfies the condition NTA_2;

(2) The Prüfer 2-rank of G is two, and the 2-rank is at least three.

With regard to the hypothesis NTA_2, we note that the prior work cited here relies heavily on that assumption. In addition, the condition NTA_2 is used again in the proofs of Lemmas 3.12 and 3.16.

We use the following terminology.

Definition 2.4. Let G be a group of finite Morley rank, i an involution. We say that i is an SL_2-*involution* or a PSL_2-*involution* if $C_G(i)$ contains a component (a factor of $E_{alg}C_G(i)$) of the corresponding type. We also use the expressions SL_2-*type* or PSL_2-*type* in the same sense—so these expressions may apply either to components or to involutions.

2.2. **BN-pairs.** Our main tool for identification will be the construction of a suitable BN-pair.

Definition 2.5 (BN-pair)**.** Let G be a group. A *BN-pair* for G consists of two subgroups B and N satisfying the following conditions, where $T = B \cap N$.

(BN1) $G = \langle B, N \rangle$ and $T \lhd N$.

(BN2) The group $W_{BN} := N/T$ is generated by a (specified) nonempty set I of involutions.

(BN3) For $v, w \in N$ and $wT \in I$ we have
$$vBw \subseteq BvB \cup BvwB.$$

(BN4) $wBw \neq B$ for all $w \in N$ with $wT \in I$.

We call W_{BN} the *Weyl group* of the BN-pair.
We say that a BN-pair is :

- *spherical* if the Weyl group W_{BN} is *finite*,
- *irreducible* if W_{BN} is not a direct product of proper subgroups, and
- *split* if $B = U \cdot T$ for some normal nilpotent subgroup U of B.

Fact 2.6 ([Ten04])**.** *Let G be a group with an irreducible spherical BN-pair of Tits rank two, where B contains a normal nilpotent subgroup U with $B = UT$. Then the associated generalized n-gon Γ is a Moufang n-gon and G/R contains its little projective group, where R denotes the kernel of the action of G on Γ.*

Remarks 2.7.
1. The effect of Fact 2.6, together with the classification of Moufang polygons of finite Morley rank, is that for G of finite Morley rank and Prüfer rank 2, identification will follow if we produce an irreducible spherical BN-pair of Tits rank two.
2. An earlier version of Fact 2.6 found in [TVM03] also suffices, because $|W_{BN}| = 8$ or 12 in our cases and [Ten04] was needed to treat the case $|W_{BN}| = 16$.

The final identification is given by the following.

Fact 2.8 ([TVM03])**.** *An infinite simple group of finite Morley rank with a spherical Moufang BN-pair of Tits rank two is one of* $\mathrm{PSL}_2(\mathbb{F})$, $\mathrm{PSp}_4(\mathbb{F})$ *or* $\mathrm{G}_2(\mathbb{F})$ *for some interpretable field* $\mathbb{F}$.

2.3. **Unipotence theory.** We review some aspects of the theory of unipotence in groups of finite Morley rank.

Definition 2.9. A *unipotence parameter* is either a prime p or a pair $(0, r)$ with $r \geq 0$.

For a prime p, a p-unipotent group is a definable connected nilpotent p-group, and the p-unipotent radical of a group H of finite Morley rank is the largest definable normal p-unipotent subgroup. One of the useful points is that in a solvable group H of finite Morley rank, every p-unipotent subgroup lies in $U_p(H)$ (hence in $F(H)$).

The "characteristic zero" unipotence theory uses the full range of parameters $(0, r)$, with the intuition being that larger values of r correspond to "more unipotent" subgroups, smaller values to "more semisimple" ones. One has a notion of $(0, r)$-unipotence and the $U_{0,r}$-radical. It is not the case in general that all $(0, r)$-unipotent subgroups belong to the Fitting subgroup of a solvable group of finite Morley rank, but this does hold for the most unipotent subgroups: that is, for the $(0, r)$-unipotent subgroups corresponding to the largest value of r for which non-trivial $(0, r)$-unipotent subgroups exist. One therefore focuses attention on the parameter $\bar{\mathrm{r}}_0(H)$, defined as the largest such value of r.

Generally speaking the $(0, r)$-unipotence theory runs parallel to the more straightforward theory of p-unipotence. One recurring point of the $U_{0,r}$-theory is that by definition $(0, r)$-unipotent groups are generated by connected abelian $(0, r)$-unipotent groups, so that when entering into details one frequently returns to the abelian case.

In particular, when k is a field of finite Morley rank, one has the following points.

(1) If the characteristic is non-zero then $\bar{\mathrm{r}}_0(k^\times) = \bar{\mathrm{r}}_0(k_+) = 0$.

(2) If the characteristic is zero then $\bar{\mathrm{r}}_0(k_+) = \mathrm{rk}(k_+) > \bar{\mathrm{r}}_0(k^\times)$. Thus, as one might hope, the additive group is more unipotent than the multiplicative group.

Definition 2.10. Let k be a field of finite Morley rank. The *Morley characteristic* $\chi_M(k)$ is defined as follows.

(1) If the characteristic is $p > 0$, the Morley characteristic is also p.

(2) If the characteristic is zero, the Morley characteristic is the pair $(0, \mathrm{rk}(k))$.

In other words, the Morley characteristic associates an abstract notion of unipotence to the field.

We also require a partial order on Morley characteristics.

For π and π' Morley characrteristics, "$\pi' \geq_M \pi$" means the following.

(1) If $\pi = p > 0$: then $\pi' = \pi$.

(2) If $\pi = (0, r)$: then $\pi' = (0, r')$ and $r' \geq r$, or $\pi' = p > 0$.

In this connection we have also the following, slightly rephrasing the foregoing.

Fact 2.11. *Let k be a field of finite Morley rank, π its Morley characteristic, and $\pi' \geq \pi$. Then $U_{\pi'}(k^\times) = 1$.*

Here (and throughout) one treats the classical case $\pi = p > 0$ and the case $\pi = (0, r)$ separately.

In the next lemma one is interested mainly in the special case of simple algebraic groups, where the definability hypotheses are automatically satisfied.

Lemma 2.12. *Let L be an affine algebraic group over an algebraically closed field k of characteristic zero, equipped with its structure as an algebraic group (with the field as an additional sort), and possibly additional structure. Let $r \geq \mathrm{rk}(k)$. Then a $(0, r)$-unipotent subgroup of L is a unipotent subgroup (non-trivial only if $r = \mathrm{rk}(k)$).*

Proof. Certainly k_+ is a $(0, r)$-unipotent group and therefore any unipotent subgroup of L is, as well.

For the converse it suffices to consider an abelian $(0, r)$-unipotent subgroup of L. This is then a product of a unipotent group and a torus and as we have a $(0, r)$-unipotent group, the torus is trivial. □

A similar result, in a less transparent notation, is the following.

Fact 2.13 ([BC22b])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 2.3.*

Let t be an involution of G. Then

$$\Delta_\rho(C_G(t)) = E_{\mathcal{E}}(C_G(t))$$

Here ρ is the maximum "reduced rank" of the multiplicative group of the base field of a component of an involution. The fact implies that for r at least the maximum rank of any such base field of characteristic zero, any $(0, r)$-unipotent subgroup of $C_G(t)$ will be contained in $EC_G(t)$ (hence unipotent).

One important property of this parameter, coming from signalizer functor theory, is the following.

Fact 2.14. *Assuming Hypothesis 2.3, for any involution i we have*

$$\bar{\mathrm{r}}_0(OFC_G(i)) \leq \rho.$$

In addition $U_pC_G(i) = 1$ for all primes p.

2.4. **Auxiliary results and notation.** We record some useful principles.

Fact 2.15 ([AB08, Theorem 1])**.** *If G is a connected group of finite Morley rank and T is a p-torus of G, then $C_G(T)$ is connected.*

Fact 2.16 ([ABC99, Prop. 2.43], [ABC08, Prop. I.9.12])**.** *Let $G = H \rtimes T$ be a group of finite Morley rank, $Q \lhd H$, and π a set of primes, such that Q, H, T are definable and*

- *Q and T are solvable;*
- *T is a π-group of bounded exponent;*
- *Q is a T-invariant $\pi^\perp$-subgroup.*

Then

$$C_{H/Q}(T) = C_H(T)Q/Q$$

Fact 2.17 ([BC22a, Lemma 3.11])**.** *Let H be a connected L-group of finite Morley rank and odd type satisfying* NTA_2 *and let $\bar{H} = H/OF(H)$.*

Then

$$\bar{H} = E_{alg}(\bar{H}) * \bar{K} \text{ where } \bar{K} \text{ is connected and}$$
$$\bar{K}/Z^\circ(\bar{K}) \text{ has degenerate type.}$$

Fact 2.18 ([ABCC03], [Bur09, Lemma 3.5])**.** *Let G be a connected solvable $p^\perp$-group of finite Morley rank, and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

If in addition G is a $(0, r)$-unipotent group then $C_G(P)$ is a $(0, r)$-unipotent group.

Remark 2.19. The assumption of finiteness on P can be weakened to local finiteness since the centralizer will be the centralizer of a finite subgroup. This is useful in dealing with 2-tori.

Fact 2.20 ([BC08, Theorem 2.1])**.** *Let G be a connected L-group of finite Morley rank and odd type. Let V be an elementary abelian 2-group acting definably on G.*

Then $\Gamma_V = G$.

The following emerges from component analysis.

Fact 2.21 ([BC22b, Lemmas 7.8 and 7.14])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let L be a component of $E_{\mathcal{E}}C_G(i)$ with i an* SL_2*-involution. If H is a proper definable subgroup of G containing L, and $\hat{L}$ the normal closure of L in H. Then at least one of the following occurs.*

(1) $H \leq C_G(i)$.

(2) $\hat{L} = OF(\hat{L}) \cdot L$ *with* $OF(\hat{L}) > 1$.

(3) There is one conjugacy class of involutions, and $\hat{L} = E_{alg}(H)$ *is of type* SL_3.

(4) There are two conjugacy classes of involutions, and $\hat{L} = E_{alg}(H)$ *is of type* PSp_4.

Actually with Fact 2.1 in hand the last case above does not arise, as it puts Γ_V into $\hat{L}$ for a 4-group in $\hat{L}$.

Fact 2.22 ([LW93, Theorem 4] Linearization, characteristic 0)**.** *Suppose that there is an infinite definable set S of automorphisms of the abelian, torsion free group A, such that A is S-minimal, and the structure (A, S) together with the action has finite Morley rank. Then there is a subgroup $A_1 \leq A$ and a field K such that $A_1 \simeq K_+$ definably. Furthermore, S embeds into a matrix ring over K.*

Implicit in this statement is the definable structure of a K-vector space on A.

Fact 2.23 ([Bor20, Theorem 3] Linearization, nonzero characteristic)**.** *Let K be an algebraically closed field of characteristic $p > 0$ and G the group of points over K of a simple algebraic group defined over K. Assume that G acts definably and irreducibly on an elementary abelian p-group V of finite Morley rank. Then V can be given the structure of a finite dimensional K-vector space V_K in a manner compatible with the action of G, and G becomes a Zariski closed subgroup of* $\mathrm{GL}(V_K)$

.

Notation 2.24. We make heavy use of the Fitting subgroup $F(H)$ and its odd part $OF(H)$, and of course the subgroup $E_{alg}(H)$, the product of algebraic components. We also make occasional use of the connected solvable radical, denoted $\sigma^\circ(H)$.

3. The Weyl group and unipotent subgroups

We now move towards the construction of a BN-pair by constructing a maximal unipotent subgroup and analyzing the normalizer $N_G(T)$ of a maximal 2-torus T.

First we consider the Weyl group.

3.1. **Weyl groups.**

We now introduce the Weyl group of G and prove that it is a dihedral group of order 8 or 12, according as the number of conjugacy classes of involutions in G is two or one, respectively. We will focus afterward on the case of two conjugacy classes.

Notation 3.1. Let G be a group of finite Morley rank of odd type.

For T a maximal 2-torus of G and j an involution in T set

$$W_T = N_G(T)/C_G(T); \qquad W_j = N_{C(j)}(T)/C_{C(j)}(T).$$

We call these the Weyl group of G, and of $C(j)$, respectively (with respect to T). One may replace T by $d(T)$ here, so these groups are quotients of definable groups.

By conjugacy of maximal 2-tori the Weyl group $W = W_T$ of G is well-defined up to conjugacy.

Notation 3.2. We continue the preceding with some additional notation and clarifying remarks.

W_j may be identified with a subgroup of W_T, namely the image of $N_{C(j)}(T)$ in W_T.

In particular, any element of W_j will be considered also as an element of W_T.

Remarks 3.3.

1. For i an SL_2-involution and $E = E_{alg}(C_G(i))$ the Weyl group W_i contains

$$W_E \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

There is a unique element of W_i inverting T, and we denote this element by $\bar{w}_i$. W_i coincides with W_E if there is one conjugacy class of involutions (that is, there is no element conjugating the two components).

The element $\bar{w}_i$ is represented by an involution $w_i \in E$ (not unique). The components of w_i in the factors of E (which are well-defined elements up to multiplication by i) are elements of order 4.

2. For j a PSL_2-involution, we have $W_j \simeq \mathbb{Z}/2\mathbb{Z}$, that is

$$W_j = \langle \bar{w}_j \rangle$$

where $w_j \in L_j = E_{alg}(C_G(j))$ is an involution inverting $T \cap L_j$.

Lemma 3.4. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Suppose that G has two conjugacy classes of involutions.*

Fix an SL_2-involution i, and a maximal 2-torus T in $E_i = E_{alg}(C_G(i))$.

Then the Weyl group W of G is a dihedral group D_8 of order 8, with generators $\bar{w}_1, \bar{j}$, where $w_1 \in L_{i,1}$ is a Weyl group element (of order 4) and $j \in C_G(i)$ is an involution swapping the components of W.

Proof. We have $N_G(T) \leq C_G(i)$ since i is the unique SL_2-involution in T. Therefore $W_T = W_i$.

As $C_G(i) = E\langle j \rangle$ with E of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ and j swapping the components, the Weyl group has the form $W_E \langle j \rangle$ with $W_E \simeq (\mathbb{Z}/2\mathbb{Z})^2$, where each involution centralizes one 2-torus of Prüfer rank one, and inverts another. The extension by j is easily recognized as a dihedral group. □

Now we examine the Weyl group in the case of one conjugacy class of involutions, with every involution an SL_2-involution.

Notation 3.5. When G has just one conjugacy class of involutions, we label the components of $E_i = E_{alg}(C_G(i))$ as $L_{i,1}$ and $L_{i,2}$, taking the labels consistent with conjugation: $L_{i^g,1} = L_{i,1}^g$. In other words, we label the two conjugacy classes of components of type SL_2.

(We will also use a similar labeling when the components are conjugate, but without any particular coherence conditions.)

Lemma 3.6. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Suppose that G has one conjugacy class of involutions.*

Then the Weyl group of G is a dihedral group of order 12, with generators $\bar{w}_1, \bar{w}_2, \bar{\sigma}$ where w_ℓ is an element of $N_{L_{i,\ell}}(T)$ of order 4, and $\sigma \in N(T)$ is an element of order three acting on $\Omega_1(T)$ as the 3-cycle (i, j, ij).

Proof. Let T be a maximal 2-torus of G. As $N(T)$ controls fusion in T, the Weyl group acts transitively on $I(T)$ and we have

$$W = W_i \langle \bar{\sigma} \rangle$$

where $\bar{\sigma}$ has order 3. Then $\bar{\sigma}$ lifts to a 3-element σ and σ^3 is a 3-element acting trivially on $\Omega_1(T)$. In particular $\sigma^3 \in C_G(i)$. It follows that $\sigma^3 = 1$.

The Weyl group W_i may be computed in $E_i = E_{alg}(C_G(i))$ as $(\mathbb{Z}/2\mathbb{Z})^2$. Thus $|W| = 12$.

The kernel W_0 of the action of W on $V = \Omega_1(T)$ is given by the element of W_i inverting T.

The action of W/W_0 on V gives the full symmetric group on $I(V)$, hence $W/W_0 \simeq \mathrm{Sym}_3$, with W_0 central of order 2.

So W has the structure $\mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with the second factor acting non-trivially on the first; this is the dihedral group of order 12. $\square$

3.2. **The group N.**

Definition 3.7. Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let T be a maximal 2-torus of G. We define a subgroup N of $N(T)$, with the definition depending on the structure of the Weyl group, as follows.

1. If the Weyl group is D_8, let S be a Sylow 2-subgroup of G, and set

$$N = d(S).$$

2. If the Weyl group is D_{12} we proceed as follows.

For i an involution of G, let x_i be a representative for the Weyl group $W_{L_{i,1}}$ in $L_{i,1}$. That is, $x_i \in L_{i,1}$ has order 4 and inverts $T \cap L_{i,1}$ while centralizing $L_{i,2}$. Let w_i be an involution of $C_G(i)$ inverting T.

Take two involutions $i, j \in T$ and set

$$N = d(T)\,\langle x_i, x_j, w_i\rangle$$

Remark 3.8. In the second case, the action of $x_i x_j$ on $\Omega_1(T)$ is a 3-cycle (i, j, ij). Consider the action of $(x_i x_j)^3$ on $T \cap L_{i,1}$. For $a \in L_{i,1}$ we have $a^{x_j} \in L_{ij,1}$ and $a^{x_j x_i} \in L_{j,1}$, so for $a \in T \cap L_{i,1}$ we have

$$a^{(x_i x_j)^2} = ((a^{-1})^{x_j x_i})^{x_j} = ((a^{-1})^{x_j x_i})^{-1} = a^{x_j x_i};$$
$$a^{(x_i x_j)^3} = a^{x_j x_i x_i x_j} = a^{x_j i x_j} = a^{x_j^2 i^{x_j}} = a^{j(ij)} = a^i = a.$$

Thus $(x_i x_j)$ centralizes T. That is, $x_i x_j$ represents an element of order 3 in W.

Lemma 3.9. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let N be defined as above.*

Then

$$C_N(T) = N^\circ = d(T); \qquad N(T) = C(T) \cdot N; \qquad N/d(T) \simeq W_T.$$

Proof. By construction $d(T) \leq N$. Since $N/d(T)$ is finite, in view of Fact 2.15 we have $C_N(T) = C_N^\circ(T) = d(T)$. So $C_N(T) = N^\circ = d(T)$.

The statements $N(T) = C(T) \cdot N$ and $N/d(T) \simeq W_T$ are equivalent. When $W \simeq D_8$, this holds since S covers a Sylow 2-subgroup of the quotient.

When $W \simeq D_{12}$ the group N induces Sym_3 on $\Omega_1(T)$, and the element inverting T has been included. □

Lemma 3.10. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let N be defined as above.*

Then the following hold.

(1) *Any proper definable connected simple algebraic section L of G is of type* PSL_2 *or* PSL_3.

(2) *There is no proper connected definable subgroup of G containing N.*

Proof.

Ad 1.

Otherwise, the section L is of type PSp_4 or G_2. Let $L = H/K$ with K normal and definable in H.

By Fact 2.17 $E_{alg}(H/OF(H)) \simeq L$. So we may suppose $L = H/OF(H)$. By Fact 2.16 the centralizer of an involution in H covers the centralizer in L.

We consider an involution i of H covering an SL_2-involution of L. Then $C_H(i)$ has a quotient of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ and it follows that $C_G(i)$ is contained in H. Then by Fact 2.21 and the remark following it, we arrive at a contradiction.

This proves the first point.

Ad 2.

Suppose H is a proper connected definable subgroup of G containing N. Then the Sylow 2-subgroup and Weyl group of H agree with that of either PSp_4 or G_2, so H has a definable section of type PSp_4 or G_2, for a contradiction. □

3.3. **Unipotent subgroups and tori.** For L a component of the centralizer of an involution in G, with base field k, we define the Morley characteristic $\chi_M(L)$ as the Morley characteristic of the base field in the sense of Definition 2.10.

Lemma 3.11. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let T be a maximal 2-torus of G.*

Suppose that L is a component of the centralizer of a PSL_2*-involution $t \in T$. Let $\pi \geq_M \chi_M(L)$ and let U be a nilpotent U_π-group normalized by t. Set $U_0 = C_U(t)$. Then U_0 is a unipotent subgroup of L (either trivial, or a root subgroup).*

Proof. By Fact 2.18 $C_U(t)$ is again a nilpotent U_π-group.

We have $C_G^\circ(t)$ definably isomorphic to the affine algebraic group $L_t \times k^\times$ with k the base field, and Lemma 2.12 applies. □

Lemma 3.12. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let T be a maximal 2-torus of G, and let U be a maximal definable T-invariant connected nilpotent group.*

Then the following hold.

(1) U is maximal among definable connected nilpotent subgroups of G.

(2) $N^\circ(U)/\sigma^\circ(N(U))$ is of degenerate type.

Proof. Let

$$H = N_G^\circ(U).$$

Claim 1. *$F^\circ(H) = U$.*

Certainly $U \leq F^\circ(H)$. Also $F^\circ(H)$ is T-invariant, so the claim follows by maximality.

Set $\bar{H} := H/U$. Since G satisfies NTA_2, Fact 2.17 implies that

$$\bar{H} \cong E_{alg}(\bar{H}) * \bar{K},$$

where $\bar{K}/Z^\circ(\bar{K})$ has degenerate type.

Claim 2. $E_{alg}(\bar{H}) = 1$.

Suppose on the contrary that $\bar{L} = E_{alg}(\bar{H}) > 1$, and let L be the preimage of $\bar{L}$. The group $\bar{T}$ is a maximal 2-torus of $\bar{H}$ and $T \cap L$ is a maximal 2-torus of L.

Let $\bar{B}$ be a Borel subgroup of $E_{alg}(\bar{H})$ containing $\bar{T}$, and let B be its preimage in H. Then $B' \leq F(B)$ and B' covers $(\bar{B})'$, which is the unipotent radical of $\bar{B}$. So $F(B) > U$ and $F(B)$ is T-invariant, contradicting the maximality of U. This proves the claim.

Thus $\bar{H} = \bar{K}$. This already gives point (2).

Now $\bar{T} \leq Z(\bar{H})$, or $[T, H] \leq U$. So any definable connected nilpotent subgroup of H containing U will be normalized by T, so U must be maximal such in H.

Therefore U is also maximal definable connected nilpotent in G. This completes the proof. □

Lemma 3.13. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let T be a maximal 2-torus of G, and let U be a maximal definable connected T-invariant nilpotent subgroup of G.*

Let π be either a prime different from the characteristic of any base field of a component of the centralizer of an involution, or a symbol $(0, r)$ with r greater than the Morley rank of the base field of any component of the centralizer of an involution. Then

$$U_\pi(U) = 1.$$

Proof. By Facts 2.18 and 2.20 the group $U_\pi(U)$ is generated by U_π-subgroups of centralizers of involutions in T. These are T-invariant nilpotent U_π-subgroups of $C_G(i)$ with i an involution of T.

The claim then follows from the structure of these centralizers. □

Notation 3.14. Let G be a group of finite Morley rank satisfying Hypothesis 2.3. let T be a maximal 2-torus of G. Let $i \in I(G)$ be an SL_2-involution, with the components of $E_i = E_{alg}(C_G(i))$ denoted $L_{i,1}$ and $L_{i,2}$.

Then $\mathbb{B}_\ell$ denotes some Borel subgroup of L_ℓ normalized by T for $\ell = 1, 2$, and $\mathbb{B} = \mathbb{B}_1\mathbb{B}_2$. We write $\mathbb{X}_\ell$ for the unipotent radical of $\mathbb{B}_\ell$ and we set $\mathbb{X} = \mathbb{X}_1\mathbb{X}_2$.

Let U be a fixed maximal definable connected nilpotent subgroup of G which contains $\mathbb{X}$ and is T-invariant.

Lemma 3.15. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. With notation as in 3.14, we have*

$$U \cap d(T) = 1.$$

Proof. T is contained in a product $\mathbb{T}$ of algebraic tori of $L_{i,1}$ and $L_{i,2}$ such that for $T_1 \le \mathbb{T}$ non-trivial, UT_1 is not nilpotent. Since $d(T) \le \mathbb{T}$ the result follows. □

Lemma 3.16. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Let $i \in I(G)$ be an SL_2-involution and $E_i = E_{alg}(C_G(i))$, $L = L_{i,1} *_2 L_{i,2}$. In the event that the base fields k_1, k_2 are of characteristic zero, choose notation so that*

$$\mathrm{rk}(k_1) \ge \mathrm{rk}(k_2).$$

Let $\pi = \chi_M(k_1)$.

Then with notation as in 3.14, we may choose U to contain a maximal definable nilpotent U_π-subgroup of G.

Proof. Let $U_0 = U_\pi(\mathbb{X})$ (i.e., $\mathbb{X}_1$ if $\mathrm{rk}(k_1) > \mathrm{rk}(k_2)$, and $\mathbb{X}$ otherwise).

Let U be chosen maximal subject to the following conditions.

(1) U is a definable connected nilpotent U_π-group containing U_0.

(2) U is normalized by T.

(3) If $\pi \ne \chi_M(k_2)$ then U centralizes $\mathbb{X}_2$.

It suffices to show that U is a maximal definable connected nilpotent U_π-subgroup of G; we may then extend $U\mathbb{X}_2$ further by an application of Lemma 3.12.

As previously, we may work in $H = N^\circ(U\mathbb{X}_2)$, which contains T. We have $U \le F(H)$. Furthermore $U_\pi(F(H)) = U$ by the maximality of U; note that if $\pi \ne \chi_M(k_2)$ then $U_\pi(F(H))$ centralizes $\mathbb{X}_2$.

If $H/OF(H)$ has an algebraic component $\bar{K}$ with $\chi_M(\bar{K}) = \pi$, we arrive at a contradiction as in the proof of Lemma 3.12.

In the contrary case, $E_{alg}(H/U)$ contains no non-trivial U_π-subgroup.

Now suppose $H/OF(H)$ contains an abelian U_π-subgroup $\bar{A}$ commuting with $\bar{T}$. Pulling back to a subgroup $A \le H$, we find that T normalizes $U_\pi(A)$ and $U_\pi(A)$ centralizes $\mathbb{X}_2$ if $\pi \ne \pi_2$. So the maximality of U in H, and in G, follows. □

Notation 3.17. Let G be a group of finite Morley rank satisfying Hypothesis 2.3. Fix a maximal 2-torus T of G.

Then we let (correspondingly) $\mathbb{T} = C_G(T)$.

Remark 3.18. N normalizes $\mathbb{T}$.

This is clear by the construction of N.

We think of $\mathbb{T}$ as the "algebraic" torus containing T. This is somewhat justified by the following.

Lemma 3.19.

(1) $\mathbb{T} = \mathbb{T}_1\mathbb{T}_2$ *with* $\mathbb{T}_\ell$ *the algebraic torus of* $L_{i,\ell}$ *containing* $T \cap L_{i,\ell}$.

(2) For $t \in I(T)$, *and* K *a component of* $E_{alg}(C_G(t))$, *the group* $\mathbb{T} \cap K$ *is a maximal torus of* K.

(3) $U \cap \mathbb{T} = 1$.

Proof.
Ad 1. $C_G(T) = C_G^\circ(T) \le C_G^\circ(i) = E_i$, so $\mathbb{T} = C_{E_i}^\circ(T) = \mathbb{T}_1\mathbb{T}_2$.
Ad 2. Similar.
Ad 3. Since U is nilpotent and contains $\mathbb{X}$, we have $U \cap \mathbb{T} = 1$. □

Lemma 3.20. $\mathbb{T}$ *normalizes the group* U.

Furthermore, U *is generated by unipotent subgroups of algebraic components of centralizers of involutions* $t \in I(T)$. *In particular, if there are two conjugacy classes of involutions, then* U *is a* U_π*-group.*

Proof. Let $V = \Omega_1(T) = \langle i, j\rangle$.

By Fact 2.20 we have

$$U = \langle C_U^\circ(t) \,:\, t \in I(V)\rangle .$$

Set $U_t = C_U^\circ(t)$ (which is also $C_U(t)$).

We will show that $\mathbb{T}$ normalizes U_t and that U_t is generated by unipotent subgroups of algebraic components of $C_G(t)$.

Case 1. t is an SL_2-involution.

In the case the argument takes place inside $E_t = C_G(t) = E_{alg}(C_G(t))$, and reduces to the two factors. We give the details.

Then $\mathbb{T}$ is a product of tori from the components of E_t, and T is the maximal 2-torus of $\mathbb{T}$.

The 2-torus T normalizes U_t, which is a nilpotent subgroup of E_t with trivial intersection with $\mathbb{T}$.

If T centralizes U_t then $U_t \subseteq \mathbb{T}$ and hence $U_t = 1$, and there is nothing more to prove.

Accordingly we may suppose that $T_1 = T \cap L_{t,1}$ normalizes but does not centralize U_t and as U_t is nilpotent, the commutator subgroup $[T_1, U_t]$ is a non-trivial unipotent subgroup U_1 of $L_{t,1}$ normalized by $\mathbb{T}$. It then follows (again by nilpotence) that $U_t \leq U_1 L_{t,2}$. In other words, $U_t = U_1 U_2$ with $U_2 = U \cap L_{t,2}$.

A second application of the same argument then shows that U_2 is trivial if it centralizes T_2, and unipotent and normalized by $\mathbb{T}$ otherwise.

Thus in this case $\mathbb{T}$ normalizes U_t, which is a product of unipotent subgroups of the components.

Case 2. t is a PSL_2-involution.

That is, $C_G(t) = L_t \times k^\times$ with $L_t \simeq \mathrm{PSL}_2(k)$ (and all components have the same base field).

Furthermore the copy of $k^\times$ occurring in $C_G(t)$ also occurs as a torus in $C_G(i)$ for some SL_2-involution, so $U_t \cap k^\times = 1$.

As U_t is T-invariant it must then lie in L_t. As in Case 1, U_t does not centralize T, and lies in a unipotent subgroup of L_t normalized by $\mathbb{T}$.

Thus the desired conclusion applies in this case, for much the same reason.

From this, the same statements follow for U. $\square$

4. $\mathbb{B}$ and $\mathbb{N}$

From this point onward our analysis aims toward the construction of a BN-pair. The verification of the main axiom will be quite detailed.

Notation 4.1. For the remainder of the paper we operate systematically within the framework of Hypothesis 2.3 and we make relatively free use of those assumptions and the associated notation without further explicit mention.

In particular, we will keep a particular maximal 2-torus T fixed throughout. The group U is maximal definable nilpotent connected, contains root groups of the components of $C_G(i)$, and is T-invariant.

The groups N, U, and $\mathbb{T}$ should all be kept in mind, though we now replace N by a more "algebraic" counterpart.

We continue to work with both of the relevant configurations for the remainder of this section, which has a very general character, dealing with the reduction of the problem to the verification of the property (BN3) for a reasonably specific choice of subgroups. At that point each of the two configurations would require separate consideration. We continue in the next section to complete the analysis in the case of two conjugacy classes of involutions, and leave the more difficult case of one conjugacy class for further consideration elsewhere. (We mention that our discussion of the group U is less satisfactory in that setting, and will be replaced by something more precise.)

Notation 4.2.
$\mathbb{N} = \mathbb{T} \cdot N$ and $\mathbb{B} = U \cdot \mathbb{T}$.

Our goal is to show that $\mathbb{B}$ and $\mathbb{N}$ give a BN-pair with $\mathbb{B} \cap \mathbb{N} = \mathbb{T}$, and as indicated above, we will carry this through in this section and the next for the configuration corresponding to PSp_4. Once one reaches that point, then since $\mathbb{B}$ splits as $U\mathbb{T}$, we can combine Facts 2.6 and 2.8 and conclude that our group G is algebraic (namely, PSp_4).

Lemma 4.3. *$\mathbb{N} \cap \mathbb{B} = \mathbb{T}$.*

Proof. By definition $\mathbb{N} \cap \mathbb{B} = \mathbb{T}N_0$ with $N_0 = \mathbb{N} \cap U$. This last group normalizes $\mathbb{T}$ and is normalized by it. Since $U \cap \mathbb{T} = 1$, the group N_0 centralizes $\mathbb{T}$. But $C_N(\mathbb{T}) \leq C_N(T) \leq \mathbb{T}$. □

Here $\mathbb{B}$, $\mathbb{N}$, and $\mathbb{T}$ will play the role of the groups denoted by B, N, and T in the context of of BN-pairs. So we need to verify the conditions (BN1–BN4) from §2.2. Also, to apply Fact 2.6 we need to take note of the condition $\mathbb{B} = U\mathbb{T}$ with U nilpotent, and with trivial intersection, which is part of our initial setup.

We turn now to condition (BN1) from Definition 2.5, which consists of two conditions,

$$(BN1.1)\ G = \langle \mathbb{B}, \mathbb{N} \rangle . \qquad\qquad (1.2)\ \mathbb{N} \cap \mathbb{B} \lhd \mathbb{N}.$$

Condition (1.2) is given by Lemma 4.3.

Condition (1.1) is less clear, and we come back to this below, but at this point we have the following.

Lemma 4.4.

(1) $H = \langle \mathbb{B}, \mathbb{N} \rangle$ is a definable subgroup of G.

(2) If $\mathbb{N} \leq H^\circ$ then $H = G$.

Proof.
Ad 1. Let H_0 be the normal closure of $\mathbb{B}$ in $\langle \mathbb{B}, \mathbb{N} \rangle$. Then H_0 is connected and definable, and $\langle \mathbb{B}, \mathbb{N} \rangle = \langle H_0, \mathbb{N} \rangle = H_0\mathbb{N}$ is a finite extension of H_0, so $\langle \mathbb{B}, \mathbb{N} \rangle$ is definable.
Ad 2. Lemma 3.10. □

In (BN2) one must specify a generating set of involutions I to which conditions (BN3, BN4) will be applied. We know that the associated Weyl group $W = \mathbb{N}/\mathbb{T}$ is dihedral of order 8 or 12, so is generated by a (specified) pair of involutions $I = \{\bar{w}_1, \bar{w}_2\}$. There are representatives for these generators in G which are either involutions or of order 4.

The remaining conditions refer to the specified generating set I.

(BN3) For $v, w \in \mathbb{N}$ and $\bar{w} \in I$ we have

$$v\mathbb{B}w \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vwB.$$

(BN4) $w\mathbb{B}w \neq \mathbb{B}$ for all $w \in \mathbb{N}$ with $\bar{w} \in I$.

In our present context, verification of these two conditions will entail condition (1.1) as well.

Lemma 4.5. *Assuming $(BN3, BN4)$ for our choice of $\mathbb{B}$, $\mathbb{N}$, and I, $(BN1)$ follows.*

Proof. As we have seen, this comes down to $\mathbb{N} \leq H^\circ$ with $H = \langle \mathbb{B}, \mathbb{N} \rangle$, and this comes down further to

$$w \in H^\circ$$

with $\bar{w}$ one of the distinguished involutions in I.

But we assume $w\mathbb{B}w \neq \mathbb{B}$ and $w\mathbb{B}w \subseteq \mathbb{B}w\mathbb{B} \cup \mathbb{B}$, so $\mathbb{B}w\mathbb{B}$ meets $w\mathbb{B}w$ and thus $w \in \mathbb{B}\mathbb{B}^w\mathbb{B} \leq \langle \mathbb{B}, \mathbb{B}^w \rangle$, which is a connected subgroup of H. □

Now we verify part of (BN4).

Lemma 4.6. *With w a Weyl group element of $L_1 = L_{1,i}$ or $L_2 = L_{2,i}$, we have*

$$w\mathbb{B}w \neq \mathbb{B}.$$

Proof. We may suppose $w \in L_1$.

U contains a maximal unipotent subgroup $\mathbb{X}_1$ of L_1 normalized by T, and $\mathbb{X}_1^{w_1}$ is the opposite unipotent subgroup.

As $\mathbb{X}_1, \mathbb{X}_1^{w_1}$ generate L_1, $\mathbb{X}_1^{w_1}$ is not contained in the solvable group $\mathbb{B}$. □

This leaves only condition (BN3) to be considered, and at this point we turn to the specific configuration associated with PSp_4.

5. Identification of PSp_4

Now we take up the proof of the identification theorem for PSp_4, Theorem 1.7, which has previously been reduced to the verification of the property (BN3) for the groups $\mathbb{B} = U\mathbb{T}$, $\mathbb{N} = \mathbb{T}N$, and chosen generators I for $\mathbb{N}/\mathbb{T}$.

We rephrase it as follows.

Proposition 5.1. *Let G be a group of finite Morley rank satisfying Hypothesis 2.3, with two conjugacy classes of involutions.*

Then $G \simeq PSp_4(k)$ for some algebraically closed field k.

Note that in this case the base fields k of all components of centralizers of involutions are definably isomorphic. We set $\pi = \chi_M(k)$, the Morley characteristic (to be used as a notion of unipotence).

The following notation is fixed throughout. A maximal 2-torus T is chosen, and i is the SL_2-involution of T, which in the present context is unique. $E_i = C_G^\circ(i) = L_{i,1} *_2 L_{i,2}$ with $L_{i,1}$, $L_{i,2}$ definably isomorphic to $\mathrm{SL}_2(k)$. As we pay special attention to i we may also write E, L_1, and L_2, rather than E_i, $L_{i,1}$, $L_{i,2}$, when the context allows.

An involution j is chosen which swaps the components of L and t is an involution in the diagonal copy of PSL_2 with respect to the action of j (there remains some choice to be made of the type of j, which we return to below).

The group U is maximal connected definable nilpotent, $\mathbb{T}$-invariant, and contains a product of root groups $\mathbb{X}_1\mathbb{X}_2$ with $\mathbb{X}_\ell \le L_\ell$. We specify further that $\mathbb{X}_2 = \mathbb{X}_1^j$. U is a U_π-group.

$\mathbb{T} = C_G(T)$, $\mathbb{B} = U\mathbb{T}$, $\mathbb{N} = \mathbb{T}N$ where $N/d(T) \simeq W$, the Weyl group.

$I = \{\bar{w}_2, \bar{\jmath}\}$ where $w_2 \in L_{i,2}$ has order 4 and inverts $\mathbb{T} \cap L_{i,2}$, and j is a PSL_2-involution swapping the two components of E_i.[2]

We also write t for the involution of $T \cap L$ centralized by j (a "diagonal" element) and w for the involution $w_2^t w_2$. We have the maximal elementary abelian group $A = \langle i, t, w, j\rangle$, of less importance in itself, but the generators and relations between them are of considerable importance. Our main concern is with the type of the involutions (i.e., their conjugacy class) and the co-torality relation between them.

Lemma 5.2. *With hypotheses and notation as above, we may choose j to be either an SL_2-involution or a PSL_2-involution. If we take j to be an SL_2-involution then we have the following.*

[2]There is no good reason for using w_2 rather than w_1 but as that choice was made at some point we are leaving it alone.

(1) *The* SL_2*-involutions of* A *are* i, j, *and* ijt' *with* t' *an involution of* $\langle t, w\rangle$.

(2) *The co-toral pairs of involutions* t_1, t_2 *are those for which* $\langle t_1, t_2\rangle$ *contains exactly one* SL_2*-involution:*

(i, t') $t' \in I(\langle i, t, w\rangle)$; (j, t') $t' \in I(\langle j, t, w\rangle)$;
(ijt, t') $t' \in I(\langle ijt, t, wi\rangle)$; (ijw, t') $t' \in I(\langle ijw, ti, w\rangle)$;
$(ijtw, t')$ $t' \in I(\langle ijtw, ti, wi\rangle)$.

Proof. Note that a maximal 2-torus contains a unique SL_2-involution.

Claim 1. *The involutions of* L_t *are* PSL_2*-involutions, the involutions of* $L_t t$ *other than* t *are* SL_2*-involutions, and* j *could be taken to lie in either class.*

The involution i is co-toral with the involutions of $A \cap E_i = \langle i, t, w\rangle$ and no others in A. In particular the involutions of $\langle i, t, w\rangle$ other than i are PSL_2-involutions.

L_t has two conjugacy classes of involution other than the central involution t, and contains the involutions i, it, one of each type. So these classes represent the two types of involution in G.

On the other hand j can be replaced by jt and this will switch the types. For the moment (only), take j to be a PSL_2-involution. Then L_j is the diagonal subgroup of E_i with respect to j. Its involutions are PSL_2-involutions, so the rest (apart from j) are SL_2-involutions. This applies equally to L_t.

The claim follows.

With the claim in hand, we will now change our preference and take j to be an SL_2-involution, as in the statement of the Lemma.

Ad 1: SL_2-involutions.

The unique SL_2-involution of $\langle i, t, w\rangle$ is i, and the unique SL_2-involution of $\langle j, t, w\rangle$ is j. It remains to consider $ij\,\langle t, w\rangle$.

Here t, w, and tw are interchangeable so we consider $ij\,\langle t\rangle$ and look at $C_G(t)$. Then $i, j \in L_t t$, so $ij \in L_t$ and $ijt \in L_t t$ It follows that ij is a PSL_2-involution and ijt is an SL_2-involution.

Since the same analysis applies to w or tw, we conclude.

Ad 2: Co-torality

The involution i is co-toral with the involutions of $A \cap E_i = \langle i, t, w\rangle$ and no others in A; a similar statement applies to j. So it suffices to consider involutions of A co-toral with ijt' for t' an involution of $\langle t, w\rangle$, and we may take $t' = t$.

As i, j are co-toral with t, so is ijt.

As ijt is not co-toral with i or j it is not co-toral with it or jt.

It remains to consider involutions in $\langle i, j, t\rangle\, w$.

Now write $t = t_1 t_1^j$ with $t_1 \in L_{i,1}$ of order 4. Then $j^{t_1} = j(t_1^{-1})^j t_1 = jt_1 t t_1 = ijt$. So j and ijt are conjugate in $N(A)$. Therefore we can apply conjugation to check the list for ijt and similarly for ijw, $ijtw$. □

Let us fix this convention going forward.

($\star_j$) The involution j swapping components of $C_G(i)$ is taken to be an SL_2-involution.

Accordingly the involution ij is a PSL_2-involution with the same action on $C_G(i)$.

We continue on in much the same vein.

Lemma 5.3. *With hypotheses and notation as above, $N(A)/A$ acts faithfully on the five SL_2-involutions as Sym_5.*

Proof. Clearly the pointwise stabilizer of the five involutions i, j, ijt, ijw, $ijtw$ is trivial.

Toward the end of the proof of Lemma 5.2 we made use of the relation

$$j^{t_1} = ijt$$

where the element t_1 fixes i and sends w to wi, hence fixes ijw and $ijtw$. That is, this a transposition on the SL_2-involutions fixing i. There is a similar transposition fixing j (and moving i) and so the action of $N(A)/A$ on the five SL_2-involutions is transitive and contains a transposition. Therefore it induces Sym_5. □

5.1. **The group U.**

Lemma 5.4. *$U \not\le C_G(i)$.*

Proof. If $U \le C_G(i)$ then $U = \mathbb{X}$ is a maximal unipotent subgroup of L.

We write Δ_i for the diagonal subgroup $C(i, j)$ of E_i, of type PSL_2, and U_Δ for the intersection $U \cap \Delta_i$, a maximal unipotent subgroup of Δ_i. Let $T_\Delta = T \cap \Delta_i$.

By lemma 5.3 there is an element τ of G (more specifically, an element of $N(A)$) which interchanges i and j and fixes t, w. In particular τ acts on Δ_i like an element of A and after adjusting by an element of $\langle t, w\rangle$ we may suppose τ centralizes Δ_i. In particular T_Δ normalizes U^τ.

Set

$$H = \langle U, U^\tau\rangle \le C^\circ_G(U_\Delta)$$

and let $\bar{H} = H/OF(H)$. By Lemma 2.17 we have

$$\bar{H} = \bar{E}_H * \bar{K}$$

with $\bar{E}_H = E_{alg}(\bar{H})$ and $\bar{K}/Z^\circ(\bar{K})$ of degenerate type.

T_Δ normalizes H.

Claim 1. *$[T_\Delta, \bar{H}] \lhd \bar{E}_H$ is nontrivial.*

We have $[T_\Delta, \bar{K}] = 1$ so $[T_\Delta, \bar{H}] \le \bar{E}_H$.

If the commutator is trivial, this means

$$[T_\Delta, H] \le OF(H).$$

But $[T_\Delta, U] = U$ and $[T_\Delta, U^\tau] = U^\tau$ so this would force $\langle U, U^\tau\rangle$ to be nilpotent. Then by maximality of U, we find $U^\tau = U$ and $U \le C_G(j)$, which is not the case.

The claim follows.

We have $T_\Delta \cap H \le C_{T_\Delta}(U_\Delta) = 1$. So H has Prüfer 2-rank 1. Thus we have

$$[T_\Delta, \bar{H}] = E_{alg}(\bar{H}).$$

Now $U_\Delta \le F(H)$, so j inverts $\bar{U}$. Hence ijt centralizes $\bar{U}$. Similarly ijt centralizes $\bar{U}^\tau$. So ijt centralizes $\bar{H}$.

Therefore $C_H(ijt)$ covers $\bar{H}$. So $E_{alg}C_H(ijt) \simeq E_{alg}(\bar{H})$.

If $E_{alg}C_H(ijt)$ is of type SL_2 then $ijt \in H \le C(U_\Delta)$, a contradiction. Thus $E_{alg}C_H(ijt)$ is of type PSL_2.

Claim 2. *If s is a PSL_2-involution then $C^\circ_G(E_{alg}C_G(s)) \le C_G(s)$.*

Let i_1, j_1 be commuting SL_2-involutions with $E_{alg}C_G(s) = C_G(i_1, j_1)$. Then $C^\circ_G(E_{alg}C_G(s))$ is generated by its intersection with the connected centralizers of i_1, j_1, and i_1j_1. The connected centralizer in $C_G(i_1)$ or $C_G(j_1)$ is trivial so $C^\circ_G(E_{alg}C_G(s)) \le C_G(i_1j_1)$. Since $E_{alg}C_G(s) = E_{alg}C_G(i_1j_1)$ and $s \in C^\circ_G(E_{alg}C_G(s))$, this then forces $i_1j_1 = s$ and the claim follows.

Now $U_\Delta \le C(E_{alg}C_G(ijt))$ gives $U_\Delta \le C_G(ijt)$, a contradiction.

The result follows. $\square$

Lemma 5.5.

(1) *The involution* $w_t = tj$ *is in* L_t *and generates the Weyl group relative to* $\mathbb{T} \cap L_t$.

(2) *The involution* $w_{ti} = twj$ *is in* L_{ti} *and generates the Weyl group relative to* $\mathbb{T} \cap L_{ti}$.

(3) *The involution* j *acts on* L_t *as the Weyl group element* w_t, *and centralizes* L_{ti}.

(4) *The involution* ijw *acts on* L_{ti} *as the Weyl group element* w_{ti}, *and centralizes* L_t.

We should comment on the symmetry-breaking here. We could replace j by ijw as the involution swapping L_1 and L_2. Selecting j gives a particular isomorphism between L_1 and L_2 and a particular diagonal subgroup Δ_i, and then a particular choice of t. Replacing j by ijw interchanges t and ti, L_t and L_{ti}.

Proof.

Ad (1)

By Lemma 5.2 (2), j, t are co-toral and $\langle j, t\rangle$ contains the unique SL_2-involution j. Thus tj is a PSL_2-involution in $C_G^\circ(t)$. We have also noticed that the involutions of L_t are PSL_2-involutions and the involutions of $L_t t$ other than t are SL_2-involutions. So $tj \in L_t$.

In its action on T, tj centralizes $\mathbb{T} \cap \Delta_i$, an algebraic torus containing t, and inverts a an algebraic torus containing ti. As tj normalizes $\mathbb{T} \cap L_t$ which does not contain t, tj inverts $\mathbb{T} \cap L_t$ and generates the corresponding Weyl group.

So (1) holds.

Ad (2)

With w_1 a Weyl group element in $L_{i,1}$ relative to $\mathbb{T} \cap L_{i,1}$, we have $j^{w_1} = ijw$ and $t^{w_1} = ti$, so (2) follows from (1).

Ad (3,4)

By definition j acts on L_t like w_t.

On the other hand wi inverts $\mathbb{T}$ and normalizes L_t, so wi also acts like w_t on L_t. Hence ijw centralizes L_t.

Now conjugating by w_1, ijw acts on L_{ti} like w_{ti} and j centralizes L_{ti}.

(3,4) are verified. □

Lemma 5.6. *$U = \mathbb{X}\langle \mathbb{Z}_1, \mathbb{Z}_2\rangle$, where the following hold.*

(1) $\mathbb{Z}_1$ is a root subgroup of L_t.

(2) $\mathbb{Z}_2$ is a root subgroup of L_{ti}.

(3) $\mathbb{Z}_2 = \mathbb{Z}_1^{w_2}$ where w_2 represents a Weyl group element in $L_{i,2}$.

(4) $\mathbb{X}\mathbb{Z}_2$ is abelian.

Furthermore, the subgroup

$$\mathbb{Y} = U_\pi(N_U(\mathbb{X})) = \langle \mathbb{X}, \mathbb{Z}_2\rangle = \mathbb{X}\mathbb{Z}_2$$

is abelian, normal in U, and normalized by j.

Proof. Let $\mathbb{Y} = U_\pi(N_U(\mathbb{X}))$. As U is not contained in $C_G(i)$, we have $\mathbb{Y} > \mathbb{X}$. Set $\mathbb{Y}_1 = C^\circ_{\mathbb{Y}}(t)$, $\mathbb{Y}_2 = C^\circ_{\mathbb{Y}}(ti)$. By Fact 2.20 we have

$$\mathbb{Y} = \langle C^\circ_{\mathbb{Y}}(t') \,:\, t' \in I(\langle i, t\rangle)\rangle = \langle \mathbb{X}, \mathbb{Y}_1, \mathbb{Y}_2\rangle .$$

By Fact 2.18 the groups $\mathbb{Y}_1$, $\mathbb{Y}_2$ are nilpotent U_π-subgroups. By Fact 2.1, $\mathbb{Y}_\ell \le E_{alg}(C^\circ_G(t_\ell))$ for $t_1 = t$ and $t_2 = ti$. Furthermore $\mathbb{Y}_\ell$ is a unipotent subgroup of the corresponding component, invariant under the action of $\mathbb{T}$. If $\mathbb{Y}_\ell$ is non-trivial, it will be a root subgroup of that component.

Claim 1. $\mathbb{Y}_1 = 1$

Otherwise, $\mathbb{Y}_1$ is a root subgroup of L_t and $\mathbb{Y}_1 \subseteq N_G(\mathbb{X})$. Applying w_t it follows that $L_t \le N_G(\mathbb{X})$. It then follows from Facts 2.22 and 2.23 that the action of L_t (of type PSL_2) on $\mathbb{X}$ must be trivial.

In particular tj acts trivially on $\mathbb{X}$. But tj acts by inversion on $\mathbb{X}$, so this is a contradiction. This proves the claim.

It follows that $\mathbb{Y}_2$ is non-trivial and hence is a root subgroup of L_{ti} normalized by $\mathbb{T} \cap L_{ti}$. We now set

$$\mathbb{Z}_2 = \mathbb{Y}_2.$$

Thus

$$\mathbb{Y} = \mathbb{X}\mathbb{Z}_2.$$

Claim 2. $\mathbb{Y}^j = \mathbb{Y}$.

.

The involution j normalizes $\mathbb{X}$ and centralizes L_{ti}, so this follows.

Now define

$$\mathbb{Z}_1 = \mathbb{Z}_2^{w_2},$$

where w_2 is an $L_{2,i}$-component of w (well-defined up to multiplication by i). Then $\mathbb{Z}_1$ is a maximal unipotent subgroup of L_t normalized by $\mathbb{T}$. As w_2 and $\mathbb{Z}_2$ centralize $\mathbb{X}_1$, the group $\mathbb{Z}_1$ centralizes $\mathbb{X}_1$.

Claim 3. $\mathbb{Y}$ *is abelian.*

As t centralizes $\mathbb{Z}_2$, we have

$$C_{\mathbb{Y}}(t) = C_{\mathbb{X}\mathbb{Z}_2}(t) = (\mathbb{X} \cap \Delta_i)\mathbb{Z}_2.$$

As this group is $\mathbb{T}$-invariant, it follows easily that $Z(C_{\mathbb{Y}}(t)) \geq \mathbb{X} \cap \Delta_i$, and hence $(\mathbb{X} \cap \Delta_i)\mathbb{Z}_2$ is abelian.

Since $\mathbb{X}$ is abelian, we find $\mathbb{X} \cap \Delta_i \leq Z(\mathbb{Y})$. So if $\mathbb{Y}$ is not abelian then $Z(\mathbb{Y}) = \mathbb{X} \cap \Delta_i$.

In the latter case, we still have $\mathbb{X} \leq \mathbb{Z}_2(\mathbb{Y})$. We may decompose $\mathbb{X}$ with respect to the action of t. For $x \in \mathbb{X}$ inverted by t and $z \in \mathbb{Z}_2$, we have $[x, z] \in Z(\mathbb{Y}) = C_{\mathbb{X}}(t)$ and so

$$\begin{aligned}
&[x, z] = [x, z]^t = [x^{-1}, z] = [x, z]^{-1};\\
&[x, z] = 1,
\end{aligned}$$

and so $\mathbb{X} \leq Z(\mathbb{Y})$ after all.

This proves the claim.

Claim 4. $U > \mathbb{Y}$.

Suppose toward a contradiction that $U = \mathbb{Y}$.

Let $H = N_G^\circ(\mathbb{X}_1)$ and $\bar{H} = H/OF(H) = \bar{E}_H * \bar{K}$ with $\bar{E}_H = E_{alg}(\bar{H})$ and $\bar{K}/Z^\circ(\bar{K})$ of degenerate type.

We have $L_{i,2} \leq H$ and so $\bar{L}_{i,2} \leq \bar{E}_H$. In particular $\bar{\imath} \in Z(\bar{E}_H)$ and so $E_i \cap H$ covers $\bar{E}_H$. Hence $\bar{E}_H = \bar{L}_{i,2}$.

Now we show that $\bar{\mathbb{Y}} = \bar{\mathbb{X}}_2$. Otherwise, $\bar{\mathbb{Y}}$ meets $\bar{K}$ non-trivially. But the intersection centralizes the 2-torus $\bar{T}$ and the corresponding subgroup of $\mathbb{Y}$ centralizes T modulo $OF(H)$. On the other hand $[T, \bar{\mathbb{Y}}] = \bar{\mathbb{Y}}$, so this is impossible. So $\bar{\mathbb{Y}} = \bar{\mathbb{X}}_2$.

Therefore $\mathbb{Y} \leq OF(H)\mathbb{X}_2$. So $\mathbb{Y} \leq U_\pi(F(H))\mathbb{X}_2$, and the latter is a nilpotent group.

On the other hand, we are supposing $U = \mathbb{Y}$. So by maximality we now have

$$U = U_\pi(F(H))\mathbb{X}_2.$$

As $\mathbb{Z}_1$ centralizes $\mathbb{X}_1$, it lies in H. Now i operates on $\mathbb{Z}_1$ like the involution ti of L_t, that is by multiplication by -1. That is, i inverts $\mathbb{Z}_1$. It follows that $\mathbb{Z}_1 \leq OF(H)$. But then $\mathbb{Z}_1 \leq U$, a contradiction.

This proves the claim.

Now we have

$$U = \langle C_U(v) \,:\, v \in I(\langle i,t\rangle)\rangle = \langle \mathbb{X}, C_U(t), C_U(ti)\rangle\,,$$

and by definition $\mathbb{Z}_2 = C^\circ_U(ti)$, a maximal unipotent subgroup of L_{ti}. The group $C^\circ_U(t)$ must be non-trivial, hence is a maximal unipotent subgroup $\tilde{\mathbb{Z}}_1$ of $\bar{L}_t$.

In particular, with the notation used above, $\tilde{\mathbb{Z}}_1$ normalizes $U_\pi(F(H))\mathbb{Z}_2$ and by maximality of U we find $U = U_\pi(F(H))\mathbb{Z}_2\tilde{\mathbb{Z}}_1$. In particular $\mathbb{Z}_1 \leq U$, and thus $\tilde{\mathbb{Z}}_1 = \mathbb{Z}_1$.

This completes our analysis. □

5.2. **(BN3,BN4).**

Lemma 5.7. $\mathbb{B}^j \neq \mathbb{B}$.

Proof. We have $\left\langle \mathbb{Z}_1, \mathbb{Z}_1^j \right\rangle = L_t$ and hence $\mathbb{Z}_1^j \not\leq \mathbb{B}$. □

Lemma 5.8. *The pair* $(\mathbb{B},\mathbb{N})$ *satisfies condition (BN4) with respect to the generating set* I.

Proof. Lemma 4.6 and 5.7. □

We turn to condition (BN3), which we repeat in our present notation, taking as distinguished generators j and w_2.

(BN3) For $v, w \in \mathbb{N}$ and $w = w_2$ or j, we have

$$v\mathbb{B}w \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw\mathbb{B}.$$

We will need to make a detailed calculation involving the structure of U. Our claim reduces at once to

$$\text{(BN3')} \qquad vUw \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw\mathbb{B}.$$

We take representatives for the eight elements v of W_{BN}.

$$\begin{array}{cccc} 1 & w_1 & w_1 = w_2^j & w = w_1 w_2 \\ j & jw_1 & jw_2 & jw = wj \end{array}$$

Condition (BN3) certainly holds when $v = 1$ so we may leave that case aside.

One should bear in mind the structure of U as described in Lemma 5.6. In addition we have the following.

Lemma 5.9.

(1) $\mathbb{X}^j = \mathbb{X}$.

(2) $\mathbb{X}_1^{w_2} = \mathbb{X}_1$, $\mathbb{X}_2^{w_1} = \mathbb{X}_2$.

(3) $\mathbb{X}_1^{w_1} \subseteq \{1\} \cup \mathbb{B}w_1\mathbb{X}$, $\mathbb{X}_2^{w_2} \subseteq \{1\} \cup \mathbb{B}w_2\mathbb{X}$.

(4) $\mathbb{Z}_1^j = \mathbb{Z}_1^w \subseteq \{1\} \cup \mathbb{B}j\mathbb{B}$.

(5) $\mathbb{Z}_1^{w_2} = \mathbb{Z}_2$.

(6) $\mathbb{Z}_1^{jw} = \mathbb{Z}_1$; $\mathbb{Z}_2^j = \mathbb{Z}_2$.

Proof.
Ad 1. Recall that our notation is taken so that $\mathbb{X}_1^j = \mathbb{X}_2$. Thus $\mathbb{X}^j = \mathbb{X}$.

Ad 2. The element w_2 commutes with $\mathbb{X}_1$, and w_1 commutes with $\mathbb{X}_2$.

Ad 3. The specified inclusions hold as the corresponding subgroups of $L_{i,1}$ or $L_{i,2}$ form a BN-pair there. E.g.,

$$\mathbb{X}_1^{w_1} \subseteq B_1 \cup B_1 w_1 \mathbb{B}_1,$$

with $\mathbb{B}_1$ a Borel subgroup of L_1, and $\mathbb{X}_1^{w_1} \cap B_1 = 1$.

Ad 4. We claim that $\mathbb{Z}_1^j$ and $\mathbb{Z}_1^w$ are both equal to the unipotent subgroup of L_t opposite to $\mathbb{Z}_1$. Since jw centralizes L_t it suffices to consider $\mathbb{Z}_1^j$. But we know that j acts like the Weyl group element w_t. So the claim holds.

Now by looking at the BN-pair of L_t induced by $(\mathbb{B}, \mathbb{N})$ we find that $\mathbb{Z}_1^j$ is contained in $\{1\} \cup \mathbb{B}_t j \mathbb{B}_t$ with $\mathbb{B}_t = \mathbb{B} \cap L_t$.

Ad 5. $\mathbb{Z}_1 = \mathbb{Z}_2^{w_2}$ by definition.

Ad 6. The involution jw centralizes L_t and j centralizes L_{ti}. □

We deal now with the cases of (BN3) of type $v\mathbb{B}j$, which we break down into three groups as follows.

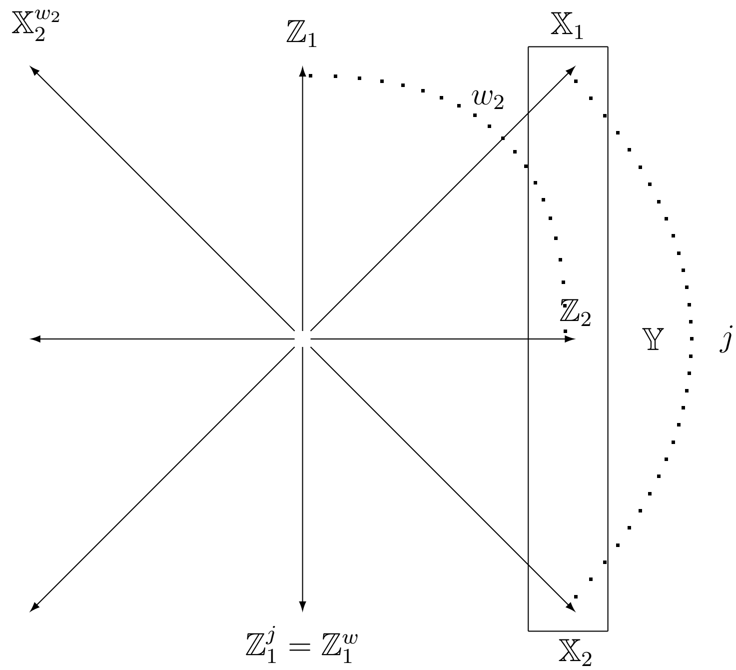

FIGURE 1. Root groups

Lemma 5.10.

(1) $jUj \subseteq U \cup \mathbb{B}j\mathbb{B}$

(2) $vUj \subseteq UvjU$ *for* $v = w_2$, jw, jw_1, *or* w_1.

(3) $vUj \subseteq UvU \cup UvjU$ *for* $v = w$ *or* jw_2.

Proof.
Ad 1.

$$jUj = j\mathbb{Z}_1\mathbb{Y}j = j\mathbb{Z}_1j\mathbb{Y} \subseteq (\{1\} \cup \mathbb{B}j\mathbb{B})\mathbb{Y} \subseteq U \cup \mathbb{B}j\mathbb{B}.$$

Ad 2.

$$vUj = v\mathbb{Z}_1\mathbb{Y}j = v\mathbb{Z}_1j\mathbb{Y},$$

so it will suffice to show

$$v\mathbb{Z}_1 j \subseteq Uvj$$

in these cases.

$$\begin{aligned}
w_2\mathbb{Z}_1 j &= \mathbb{Z}_2 w_2 j \subseteq U w_2 j;\\
jw\mathbb{Z}_1 j &= \mathbb{Z}_1 jwj \subseteq U(jw)j;\\
jw_1\mathbb{Z}_1 j &= \mathbb{Z}_1^{jw_1} jw_1 j = \mathbb{Z}_1^{(jw)w_2} jw_1 j = \mathbb{Z}_1^{w_2} jw_1 j = \mathbb{Z}_2 jw_1 j \subseteq U(jw_1)j;\\
w_1\mathbb{Z}_1 j &= j(jw_1\mathbb{Z}_1 j) = j(\mathbb{Z}_2 jw_1 j) = \mathbb{Z}_2 j(jw_1 j) = \mathbb{Z}_2 w_1 j.
\end{aligned}$$

Ad 3.

$$\begin{aligned}
wUj &= (wj)jUj \subseteq (wj)(U \cup \mathbb{B}j\mathbb{B}) = wjU \cup (wj\mathbb{B}j)\mathbb{B}\\
&\subseteq UwjU \cup (\mathbb{B}wj \cdot j\mathbb{B})U = UwjU \cup \mathbb{B}w\mathbb{B};\\
jw_1Uj &= w_2(jUj) \subseteq w_2(U \cup \mathbb{B}j\mathbb{B}) = w_2U \cup (w_2\mathbb{B}j)\mathbb{B} \subseteq Uw_2U \cup \mathbb{B}w_2j\mathbb{B} \cdot U\\
&= Uw_2U \cup \mathbb{B}w_2j\mathbb{B} = Ujw_1jU \cup \mathbb{B}jw_1\mathbb{B}. \qquad \square
\end{aligned}$$

Lemma 5.11. *Let $\mathbb{Y}^* = \langle \mathbb{X}_1, \mathbb{Z}_1, \mathbb{Z}_2 \rangle$. Then $\mathbb{X}_1\mathbb{Z}_1$ is an abelian normal subgroup of $\mathbb{Y}^*$, $\mathbb{Y}^* = (\mathbb{X}_1\mathbb{Z}_1)\mathbb{Z}_2$, $\mathbb{Y}^*$ is normal in U, and $U = \mathbb{Y}^*\mathbb{X}_2$.*

Furthermore $\mathbb{Y}^*$ is w_2-invariant.

Proof. We begin with a different definition of $\mathbb{Y}^*$, which we will prove agrees with the above.

$$\mathbb{Y}^* = U_\pi(U \cap U^{w_2}).$$

With this definition, we find that $\mathbb{Y}^*$ is a w_2-invariant subgroup of U containing $\langle \mathbb{X}_1, \mathbb{Z}_1, \mathbb{Z}_2 \rangle$ and disjoint from $\mathbb{X}_2$. In particular $\mathbb{X}_1\mathbb{Z}_1$ is an abelian subgroup of $\mathbb{Y}^*$.

Taking $U_1 = U_\pi(N_U(\mathbb{Y}^*))$ we find $U_1 > \mathbb{Y}^*$ and

$$U_1 = \left\langle C^\circ_{U_1}(v) \,:\, v \in I(\langle i, j \rangle) \right\rangle = \left\langle C^\circ_{U_1}(i), \mathbb{Z}_1, \mathbb{Z}_2 \right\rangle$$

and $\mathbb{X}_1 \le C^\circ_{U_1}(i)$. As $U_1 > \mathbb{Y}^*$ we find $C^\circ_{U_1}(i) = \mathbb{X}$ and $U_1 = U$, that is

$$\mathbb{Y}^* \lhd U$$

and thus $U = \mathbb{X}_2\mathbb{Y}^*$.

Similarly $N_{\mathbb{Y}^*}(\mathbb{X}_1\mathbb{Z}_1) = \mathbb{Y}^*$ and thus $\mathbb{Y}^* = (\mathbb{X}_1\mathbb{Z}_1)\mathbb{Z}_2$. $\square$

Recall that we have taken j and w_2 as our distinguished generators in this section. So the remaining part of condition (BN3) may be stated as follows.

Lemma 5.12. *$vUw_2 \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw_2\mathbb{B}$ for all $v \in \mathbb{N}$.*

Proof. We have

$$vUw_2 = v\mathbb{X}_2\mathbb{Y}^*w_2 = v\mathbb{X}_2w_2\mathbb{Y}^*,$$

So what we require is

$$v\mathbb{X}_2w_2 \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw_2\mathbb{B}$$

for representatives v of all nontrivial elements of N.

For $v = w_2$ this follows using the induced BN-pair in L_2.

Since $\mathbb{X}_2^{w_1} = \mathbb{X}_2$ and $\mathbb{X}_2^j = \mathbb{X}_1$, for $v = w_1$, j, or jw_1 we get $\mathbb{X}_2^v \subseteq U$ and

$$v\mathbb{X}_2w_2 \subseteq Uvw_2 \subseteq Uvw_2U$$

This leaves the cases $v = jw_2$, w, and jw.

$$\begin{aligned}
w\mathbb{X}_2w_2 &= w_1w_2\mathbb{X}_2w_2 \subseteq w_1(\mathbb{B} \cup \mathbb{B}w_2\mathbb{B}) \subseteq \mathbb{B}w_1\mathbb{B} \cup (\mathbb{B}w_1w_2\mathbb{B})\mathbb{B} \\
&= \mathbb{B}ww_2\mathbb{B} \cup \mathbb{B}w\mathbb{B}; \\
jw\mathbb{X}_2w_2 &= w_2jw_2\mathbb{X}_2w_2 \subseteq w_2j(\mathbb{B} \cup \mathbb{B}w_2\mathbb{B}) \subseteq \mathbb{B}w_2j\mathbb{B} \cup (jw_1\mathbb{B}w_2)\mathbb{B} \\
&\subseteq \mathbb{B}jw_1\mathbb{B} \cup \mathbb{B}jw_1w_2\mathbb{B} \cdot \mathbb{B} = \mathbb{B}(jw)w_2\mathbb{B} \cup \mathbb{B}jw\mathbb{B}; \\
jw_2\mathbb{X}_2w_2 &\subseteq j(\mathbb{B} \cup \mathbb{B}w_2\mathbb{B}) \subseteq \mathbb{B}j\mathbb{B} \cup (\mathbb{B}jw_2\mathbb{B})\mathbb{B} \\
&= \mathbb{B}(jw_2)w_2\mathbb{B} \cup \mathbb{B}jw_2\mathbb{B}. \qquad \square
\end{aligned}$$

So finally (BN3) holds by Lemmas 5.10 and 5.12, and our general remarks earlier.

Proof of Proposition 5.1. We have an irreducible split BN-pair of Tits rank two. Hence G is algebraic (Remark 2.7). $\square$

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