METRICALLY HOMOGENEOUS GRAPHS OF DIAMETER FOUR

GREGORY CHERLIN

Abstract. We classify metrically homogeneous graphs of diameter 4 (work in progress).

Contents

Date: September, 2013.

1991 *Mathematics Subject Classification.* Primary 03C10; Secondary 03C13, 03C15, 05B, 20B22, 20B27.

Key words and phrases. homogeneity, permutation group, model theory, amalgamation, metric space.

Supported by NSF Grant DMS 1101597 and HIM, Bonn (Fall 2013).

Introduction

Work in progress—written with considerable optimism, but still under development.

A connected graph is *metrically homogeneous* if it is homogeneous when considered as a metric space in the graph metric. A catalog of the known metrically homogeneous graphs is given in [Che13]. There is some evidence to support the view that this catalog should give a complete classification, or nearly so ([Che13, ACM13]).

In diameter at most 2, the metrically homogeneous graphs are simply the homogeneous graphs, classified by Lachlan and Woodrow [LW80] by a subtle inductive argument. A full classification of the metrically homogeneous graphs of diameter 3 is given in [ACM13].

We are not ready to tackle the general case, though we believe the plan of attack used in diameter 3 may be fundamentally sound, as far as it goes, in general. The difficulty is that the implementation of every step of that plan depends at present on concrete considerations.

Our goal here is to extend the analysis to diameter 4. In the process the general structure of the argument should become more visible, and this produces a kind of general template that one could take as a starting point in an attempt to devise a more general strategy.

Since the present article is a direct continuation of [ACM13], we continue the numbering of the sections from that paper. But we insert a review of some necessary facts from [ACM13].

General Theory. Our presentation makes use of the theory developed in [Che13], with some further developments introduced in [ACM13].

Definition 1. *Let* Γ *be a metrically homogeneous graph. Then* Γ *is of exceptional type if*

(1) Γ_1 *is imprimitive; or*
(2) *For some pair of vertices at distance* 2 *in* Γ*, their set of common neighbors contains no infinite independent set.*

Otherwise, Γ *is said to be of generic type.*

The main point of this definition is that we are able to give an explicit classification of the exceptional graphs, and that our analysis of the generic case depends on completely different methods from the exceptional case.

The Aim. We have a general conjecture about the structure of metrically homogeneous graphs. More precisely, we have a full classification of the exceptional metrically homogeneous graphs, and we conjecture a uniform description of the ones of generic type.

It is not necessary to review the classification of exceptional type here. This was given originally in [Che11] and repeated in [Che13, ACM13].

But we require a detailed description of the known generic type metrically homogeneous graphs. These depend on five numerical parameters $(\delta, K_1, K_2, C_0, C_1)$ and a finite set $\mathcal{S}$ of so-called *δ-Henson constraints.* In addition the five numerical parameters satisfy various numerical constraints, and for some particular values of the parameters the set $\mathcal{S}$ must be empty. We call combinations of numerical parameters and δ-Henson constraints for which the associated metrically homogeneous graph exists "admissible."

Furthermore, in one special case (namely, when $C_1 = 2\delta + 1$ and $C_0 = C_1 + 1$) there is a variation on the notion of δ-Henson constraint. This case is called the "antipodal" case for reasons which will be explained later.

Once we have all this notation, we introduce the notations $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ and $\Gamma^\delta_{K_1,K_2,C_0,C_1}$ for the amalgamation class or the metrically homogeneous graph associated to a specific admissible choice of parameters, with the notation $\mathcal{A}^\delta_{a,n}$ or $\Gamma^\delta_{a,n}$ for the antipodal variation alluded to.

Then our goal is to prove that a metrically homogeneous graph of diameter 4 and generic type is one of the graphs $\Gamma^4_{a,n}$ or $\Gamma^4_{K_0,K_1,C_0,C_1,\mathcal{S}}$ with admissible parameters. The elaborate conditions which define admissibility in general can be substantially simplified in the case of small diameter, and the resulting conjecture can be neatly displayed in tabular form.

In order to make sense of this, we must first explain how to define

$$\delta,\ K_1,\ K_2,\ C_0,\ C_1, \text{ and } \mathcal{S}$$

for any metrically homogeneous graph Γ, then what the "canonical" graph $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ is for a given set of parameters (bearing in mind that it only exists under certain supplementary conditions on the parameters).

We need the following terminology: the *type* of a metric triangle (a,b,c) is the triple of distances $(d(a,b),d(b,c),d(a,c))$, taken in any order; a $(1,\delta)$-space is a metric space in which all distances equal 1 or δ.

Definition 2. *Let Γ be a metrically homogeneous graph (or more generally, an integer-valued metric space).*

(1) *δ is the diameter of Γ.*
(2) *K_0 is the least k such that Γ contains a triangle of type $(1,k,k)$.*
(3) *K_1 is the largest k such that Γ contains a triangle of type $(1,k,k)$.*
(4) *C_0 is the smallest even number such that Γ contains no triangle of perimeter C_0.*
(5) *C_1 is the smallest odd number greater than 2δ such that Γ contains no triangle of perimeter C_1.*
(6) *$\mathcal{S}$ is the set of $(1,\delta)$ spaces S with the following properties;*
   (a) *S does not embed isometrically into Γ;*
   (b) *Every proper subspace of S does embed isometrically into Γ;*
   (c) *S embeds isometrically in the graph $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ which will be defined below.*

Now we turn to the reverse procedure, where we are given the parameters and we look for the corresponding graph. Here we use the Fraïssé theory. That is, we define a class of finite metric structures $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$, and in the cases in which this class of structures has the amalgamation property, we take $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ to be Fraïssé limit of this class; in other words, the unique countable homogeneous metric structure Γ for which the set of finite metric spaces isometrically embeddable in Γ coincides with the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$. This will define $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ as a metric space; to see it as a graph, take the edge relation "$d(x,y)=2$."

So we now give an explicit definition of the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$.

Definition 3. *Let δ,K_1,K_2,C_0,C_1 be positive integers, and $\mathcal{S}$ a set of $(1,\delta)$-subspaces.*

1. $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ is the class of all finite integer-valued metric spaces satisfying the following conditions.

(1) *All distances are bounded by δ;*
(2) *Any triangle of odd perimeter p satisfies $p \geq 2K_1+1$.*
(3) *Any triangle of type (i,j,k) and odd perimeter p satisfies $2p \leq 2K_2 + 2\min(i,j,k)$*

(4) *Any triangle of perimeter p satisfies $p < C_\epsilon$, where $\epsilon = 0$ or 1 represents $p \pmod 2$.*

2. $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ is the subset of $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ consisting of metric spaces with no $(1,\delta)$-subspaces isometric to an element of $\mathcal{S}$.

This is only of interest if $1 \le K_1 \le K_2 \le \delta$ (or $K_1 = \infty$ and $K_2 = 0$, when there are no triangles of odd perimeter), and furthermore $C_0, C_1 > 2\delta$. But there are other necessary conditions of a more subtle kind: we call a set of parameters *admissible* if the class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ is an amalgamation class, and has associated parameters $K_1, K_2, C_0, C_1, \mathcal{S}$.

We will need to give the precise conditions on the parameters eventually, at least in the case $\delta = 4$, in order to have a definite conjecture to aim at. At this point, we note only that there are some extreme cases, and a non-obvious dividing line.

The extreme cases are as follows.

- $K_1 = \infty$, $K_2 = 0$: there are no triangles of odd perimeter, and in particular $C_1 = 2\delta + 1$. This is the *bipartite* case.
- $C_1 = 2\delta + 1$, $C_0 = 2\delta + 2$. Then for every vertex there is a unique "antipodal" vertex v^* at distance δ, and we have the law

$$d(u, v^*) = \delta - d(u, v)$$

  The involution $v \leftrightarrow v^*$ defines an automorphism of Γ. This is called the *antipodal* case.

A variant of Smith's Theorem (originally for distance transitive graphs) says that in the metrically homogeneous case, in diameter at least 3, and vertex degree at least 3, the only imprimitive graphs are the bipartite and antipodal ones [Che13]. Note that these graphs do not fall on the exceptional side of our classification. However we have previously classified the bipartite metrically homogeneous graphs of finite diameter under the inductive hypothesis that all graphs of smaller diameter are of known type.

So the extreme values of the K_i or C_i correspond respectively to the two types of imprimitive graphs. Furthermore, since the bipartite graphs are adequately classified under appropriate inductive assumptions, we restrict attention in the future to the cases in which

$$K_1 < \infty$$

and we really have ordinary numerical parameters (the others are bounded by $3\delta + 2$).

A close study of the conditions for admissibility reveals a dividing line corresponding to the inequality

$$\min(C_0, C_1) \le 2\delta + K_1$$

The antipodal graphs satisfy this condition in its most extreme form, and one may view this class as consisting of graphs with very limited perimeters.

The other class is simply the complement

$$\min(C_0, C_1) > 2\delta + K_1$$

and these seem to be more free, more like the random graph.

We will refine this picture a little as we go along.

Plan of the analysis. The form of our conjecture suggests a natural sequence of steps in any proof, as follows.

One fixes a metrically homogeneous graph Γ of diameter δ and generic type, under an appropriate inductive assumption.

- **Step 1:** Show the parameters of Γ are admissible; let Γ^* be the "canonical" graph $\Gamma^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ with the same parameters (or $\Gamma^\delta_{a,n}$ in the antipodal case);
- **Step 2:** Show that the triangles and $(1,\delta)$-spaces embedding in Γ are those which embed in the target graph Γ^*;
- **Step 3:** Show that any finite metric space embedding isometrically into Γ^* embeds into Γ.

It is helpful in practice, at the end of Step 1, to have the objective Γ^* clearly in view, and to know something about the amalgamation procedure associated with it.

The following terminology is useful.

Definition 4. *Let Γ be a metrically homogeneous graph and A a finite metric space. Then A is* Γ-constrained *if every triangle in A and every $(1,\delta)$-subspace of A embeds isometrically into Γ.*

Overall, our strategy may be described as follows: show that the Γ-constrained metric spaces embed into Γ. We have previously characterized the amalgamation classes with this property in [Che13] and shown that they are the ones of generic type. In fact, the proof of that result is essentially the source of the original catalog. From that point of view it would not be strictly necessary to know in advance that the parameters of Γ are admissible—that would follow. But practically speaking, one needs first to show that the parameters are admissible, so that there is a definite target graph Γ^* in view. Then Step 2 says that the Γ-constrained graphs and the subgraphs of Γ are the same. In particular, this class is closed under amalgamation with respect to a known amalgamation procedure, and one may use that information in Step 3.

We delay a precise description of the admissibility conditions to the beginning of our analysis (§2).

Generally speaking, it is important to keep track of the relative sizes of the parameters C_0 and C_1, so we introduce the notation

$$C = \min(C_0, C_1) \text{ and } C' = \max(C_0, C_1)$$

The case in which

$$C_1 = C_0 + 1$$

is common, perhaps even typical, and easier to handle, as the issue of parity falls away and we have simply a bound on perimeter.

More general background material is included in [ACM13, §1]. The present article is in essence a continuation of that one, aiming to apply the same techniques in diameter 4 as were applied there in diameter 3. In the next section we will simply list the information obtained in [ACM13] which will be needed in the continuation, then take up our problem from the beginning, in diameter 4.

Draft — to see how far the analysis goes, look at the table of contents.

1. Useful Facts

We review useful general principles found either in the general discussion of metrically homogeneous graphs of known type [Che13] or in the more specific discussion relating to the case of diameter 2 [ACM13].

The following is often used as a standard ingredient in amalgamation arguments, as a way of verifying the embeddability of some basic configurations.

Fact 1.1 (Common Neighbors: [Che13, Lemma 6.8]). *Let Γ be a connected metrically homogeneous graph, not a tree. Let $v_1, v_2 \in \Gamma$ with $d(v_1, v_2) = 2$. Then*

$$\Gamma_1(v_1) \cap \Gamma_2(v_2) \cong \Gamma_1$$

The first result is a portion of Step 2 of the proof strategy outlined above. Here we impose some mild conditions on the parameters. In practice they should be known to be admissible by the time this result is wanted, which is a sharper condition.

Fact 1.2 (Forbidden Triangles: [ACM13, §3]). *Let Γ be a primitive metrically homogeneous graph of generic type with associated parameters*

$$\delta,\ K_1,\ K_2,\ C,\ C',\ \mathcal{S}$$

Suppose also

$$\text{If } C' > C + 1, \text{ then } C \geq 2\delta + K_2$$

If a triangle embeds isometrically in Γ, then it belongs to the class

$$\mathcal{A}^{\delta}_{K_1,K_2,C,C'}$$

1.1. **Local Structure.** Given a metrically homogeneous graph Γ and an arbitrarily selected base point v_0 in Γ, we denote by Γ_i the metric space induced by the metric on Γ restricted to the points at distance i from the base point. This is again a homogeneous metric space and is frequently a metrically homogeneous graph with respect to the usual edge relation "$d(x, y) = 1$."

Fact 1.3 (Γ_i Connected).

1. [Che13, Proposition 6.1and Lemma 6.15]

Let Γ be a metrically homogeneous graph of diameter δ of generic type and let $1 \leq i \leq \delta$. If Γ_i contains an edge, then Γ_i is connected.

2. [Che13, Lemmas 6.2, 6.14, 6.19]

Let Γ be a connected metrically homogeneous graph of generic type, with

$$K_1 \le 2$$

Then for all $i < \delta$, Γ_i contains an edge.

Corollary 1.4 (Admissibility, $K_1 \le 2$: [ACM13, §2]). *Let Γ be a metrically homogeneous graph of generic type, diameter δ, which is neither bipartite nor antipodal. If $K_1 \le 2$ then $K_2 \ge \delta - 1$.*

Fact 1.5 ([Che13]). *Let Γ be a connected metrically homogeneous graph, not a tree. Suppose that Γ_i contains an edge, and Γ_i is not primitive. Then Γ is antipodal, δ is even, and $i = \delta/2$.*

A more technical result which plays a role in the proofs of the above and is sometimes useful in its own right is the following.

Fact 1.6 ([Che13, Lemma 6.5]). *Let Γ be a connected metrically homogeneous graph with $\Gamma_1 \cong I_\infty$, and not a tree. Then for all i, Γ_i is connected with respect to the relation*

$$d(x, y) = 2$$

1.2. **Small values of C.**

Fact 1.7 ($C = 2\delta+1$: [ACM13, §2]). *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ. Then $C \ge 2\delta + 2$.*

Fact 1.8 ([ACM13, §2]). *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ containing no triangle of type $(2, \delta, \delta)$. Then Γ_δ is an infinite complete graph. In particular*

$$K_1 = 1$$
$$K_2 = \delta$$

and Γ contains no triangle of perimeter greater than $2\delta + 1$. Furthermore, any $(1, \delta)$-space which does not contain a forbidden triangle is realized in Γ.

We also have the following general reduction of Step 1 of our program.

Fact 1.9 ([ACM13, §2]). *Let Γ be an infinite primitive metrically homogeneous graph of generic type with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. If the numerical parameters $(\delta, K_1, K_2, C, C')$ are admissible, then the full parameter sequence*

$$\delta,\ K_1,\ K_2,\ C,\ C',\ \mathcal{S}$$

is admissible.

Finally, we note the following.

Lemma 1.10. *In an antipodal metrically homogeneous graph, we have*

$$K_1 + K_2 = \delta$$

Indeed, triangles of type $(1, i, i)$ and $(1, \delta - i, \delta - i)$ correspond under the antipodal pairing.

1.3. More special cases.

Fact 1.11 ($K_1 = \delta$: [ACM13, §2]). *Let Γ be an infinite primitive metrically homogeneous graph of generic type, with diameter $\delta \geq 3$. If $K_1 = \delta$, then Γ_δ is a primitive metrically homogeneous graph of diameter δ for which the corresponding parameter $K_{\delta,1}$ is also equal to δ. Furthermore we have*

$$\begin{aligned} K_2 &= \delta \\ C &= 3\delta + 1 \\ C' &= 3\delta + 2 \\ \mathcal{S} &= \emptyset \end{aligned}$$

and in particular the parameters $(K_1, K_2, C, C', \mathcal{S})$ are admissible.

Fact 1.12 ($K_2 = \delta$: [ACM13, §2]). *Let Γ be an infinite primitive metrically homogeneous graph of diameter δ, for which $K_2 = \delta$. Then Γ contains triangles of type (i, δ, δ) for all $i \leq K_1$. Thus*

$$C > 2\delta + K_1$$

1.4. **Admissibility.** While we need only the very simplified form of admissibility that applies when the diameter is 4, we give the general notion to supply some context.

Definition 1.13. *Let $(\delta, K_1, K_2, C_0, C_1)$ be a sequence of natural numbers, and let $\mathcal{S}$ be a set of finite $(1, \delta)$-spaces. Write $K = (K_1, K_2)$ and $C = (C_0, C_1)$ for brevity.*

1. The sequence of parameters $\delta, K, C, \mathcal{S}$ is acceptable *if the following conditions are satisfied.*

- $\delta \geq 2$;
- $1 \leq K_1 \leq K_2 \leq \delta$, *or else* $K_1 = \infty$ *and* $K_2 = 0$;
- C_0 *is even and* C_1 *is odd;*
- $2\delta + 1 \leq C_0, C_1 \leq 3\delta + 2$;
- $\mathcal{S}$ *is* irredundant *(see below).*

In particular if $\delta = \infty$ then $C_0, C_1 = \infty$ and $\mathcal{S}$ consists of a set of cliques (in fact, of just one clique).

2. An acceptable sequence of parameters is admissible *if one of the following sets of conditions is satisfied.*

I $K_1 = \infty$:
- $K_2 = 0$, $C_1 = 2\delta + 1$; *this is the* bipartite case

II $K_1 < \infty$ *and* $C \leq 2\delta + K_1$:
- $\delta \geq 3$;
- $C = 2K_1 + 2K_2 + 1$;
- $K_1 + K_2 \geq \delta$;
- $K_1 + 2K_2 \leq 2\delta - 1$

IIA $C' = C + 1$ *or*

IIB $C' > C + 1$, $K_1 = K_2$, *and* $3K_2 = 2\delta - 1$

III $K_1 < \infty$ *and* $C > 2\delta + K_1$*:*
- *If* $\delta = 2$ *then* $K_2 = 2$ *and* $\mathcal{S}$ *consists of a single clique or anti-clique;*
- $K_1 + 2K_2 \geq 2\delta - 1$ *and* $3K_2 \geq 2\delta$*;*
- *If* $K_1 + 2K_2 = 2\delta - 1$ *then* $C \geq 2\delta + K_1 + 2$*;*
- *If* $C' > C + 1$ *then* $C \geq 2\delta + K_2$*.*
- *If* $K_1 = \delta$ *or* $C = 2\delta + 2$*, then* $\mathcal{S}$ *is empty;*

We need still to define irredundance of $\mathcal{S}$, a point we have actually seen earlier without the accompanying terminology. The set $\mathcal{S}$ is said to be *irredundant* if no space in $\mathcal{S}$ contains an isometric copy of a forbidden triangle, or of another space in $\mathcal{S}$. In other words, $\mathcal{S}$ consists of minimal forbidden $(1, \delta)$-spaces, with the proviso that any forbidden triangles will be provided by the numerical parameters.

2. Overview

2.1. **Expectations.** The exceptional metrically homogeneous graphs of finite diameter $\delta \geq 4$ are simply the n-cycles C_n with $n = 2\delta$ or $n = 2\delta + 1$ [Che13].

Within the generic type metrically homogeneous graphs, we have mentioned the case division according as $C \leq 2\delta + K_1$ or $C > 2\delta + K_1$. We will refer to the first case as Atypical Generic, and the second as Typical Generic. Since by definition C is always finite, the bipartite case ($K_1 = \infty$) falls on the atypical side according to this definition; the antipodal case does as well, and is the archetypal example for this class.

In the atypical generic case, leaving aside the bipartite case, we have $K_1 + 2K_2 \leq 2\delta - 1$ and in particular $3K_2 \leq 2\delta - 1$, so with $\delta = 4$ this means $K_2 \leq 2$. On the other hand $K_1 + K_2 \geq \delta$ so we arrive at

$$K_1 = K_2 = 2,\ C = 9 = 2\delta + 1$$

and here there is only the antipodal case.

So we come to the typical generic case with $C > 2\delta + K_1$, or more concretely

$$C \geq 9 + K_1$$

Then $3K_2 \geq 2\delta$ so

$$K_2 \geq 3$$

We have a special constraint when $K_1 = 1$, $K_2 = 3$; since $K_1 + 2K_2 = 2\delta - 1$ we have $C > 2\delta + K_1$, i.e.

$$\text{If } K_1 = 1 \text{ and } K_2 = 3 \text{ then } C \geq 11$$

By definition we have the range of values $2\delta + 1 \leq C < C' \leq 3\delta + 2$, that is

$$9 \leq C < C' \leq 14$$

where C, C' have opposite parity.

We have one more condition in typical generic type when $C' > C+1$, namely $C \geq 2\delta + K_2$. Now when $K_2 \leq K_1 + 1$ this is vacuous, and when $K_1 = 1$, $K_2 = 3$ it follows from the condition already mentioned that $C \geq 11$. So this condition is relevant only when $K_2 = 4$, in which case we are requiring $C \geq 12 = 3\delta$. But then necessarily $C' = 3\delta + 1$. So this condition becomes

$$\text{If } K_2 = 4 \text{ then } C' = C + 1$$

Now we may tabulate the possibilities as follows. While there are a number of special cases, we will see that it is not hard to derive the same restrictions for the parameters associated to an arbitrary metrically homogeneous graph of generic type, thereby dealing with Step 1 of our general plan.

		Exceptional	
	n-cycles	C_8, C_9	
		Atypical Generic Type: $C \leq 2\delta + K_1$	
K_1	Description	Notation	
∞	Bipartite	$\Gamma^4_{\infty,0,9,C_0,\mathcal{S}}$	
1	Antipodal	$\Gamma^4_{1,3,9}$ or $\Gamma^4_{a,n}$ $(4 \leq n < \infty)$	
2	Antipodal	$\Gamma^4_{2,2,9} = \Gamma^4_{a,3}$	
		Typical Generic Type: $K_1 < \infty$, $C > 2\delta + K_1$	
K_2	K_1	C, C'	$\mathcal{S}$
3	1	$C \geq 11$, $C' = C+1$ or $C+3$	
"	2	$C \geq 9 + K_1$, $C' = C + 1$ or $C + 3$	
"	3	$C \geq 9 + K_1$, $C' = C + 1$	
4	1–4	$C \geq 9 + K_1$, $C' = C + 1$	If $K_1 = 4$ or $C = 10$: $\mathcal{S} = \emptyset$

The conditions on $\mathcal{S}$ include irredundancy, which is not worth incorporating into the table. This means that when $K_1 > 1$ none of the constraints in $\mathcal{S}$ contains a clique of order 3; when $K_2 < \delta$ then $\mathcal{S}$ consists of at most one clique and one anticlique; and when $C_\epsilon < 3\delta$ (with ϵ the parity of δ) then none of these constraints contains an anticlique of order 3.

In the Typical Generic case everything shown is of type $\Gamma^4_{K_1,K_2,C,C',\mathcal{S}}$, so we are just keeping track of the admissibility conditions here—and later we will need to refer to this table to check that we have derived all appropriate restrictions on these parameters for an example which is not necessarily of known type.

2.2. **Ambiguities.** From a more abstract point of view, some of the cases shown as distinct above are actually equivalent. The most useful notion for our purposes is not isomorphism per se, but isomorphism up to a permutation of the language. Typically, if one permutes the distances in a metric space one no longer has a metric space, but there are exceptions

Remark 2.1.

1. (14)*: Metrically homogeneous graphs of diameter* 4 *with parameters*

$$K_1 = 1,\ K_2 = 3,\ C = 11,\ C' = 12$$

are equivalent to metrically homogeneous graphs of diameter 4 with

$$K_1 = 2,\ K_2 = 3,\ C = 11,\ C' = 14$$

by interchanging distance 1 *and* 4*. This transformation may be applied to the set* $\mathcal{S}$ *as well.*

More generally, the class of metrically homogeneous graphs with

$$K_1 \leq 2,\ K_2 = 3,\ C_1 = 11,\ \text{and } C_0 \geq 12$$

is closed under this operation. The minimal and maximal known amalgamation classes of this type have parameters $(2, 3, 11, 14)$ *and* $(1, 3, 11, 12)$ *and are invariant under this operation.*

2. $(1, 2, 4)$*: Metrically homogeneous graphs of diameter* 4 *with* $K_1 = 1$*,* $K_2 = 4$*,* $C = 10$*,* $C' = 11$ *are equivalent to metrically homogeneous graphs of diameter* 4 *with* $K_1 = K_2 = 4$*,* $C = 13$*,* $C' = 14$ *under the permutation of distances* (124)*.*

Here $\mathcal{S} = \emptyset$*.*

3. (13)*: Antipodal graphs with* $K_1 = K_2 = 2$ *correspond to themselves if we interchange distances* 1 *and* 3*.*

Proof. In all or our classes we have certain permitted triangles—the geodesics and the triangles of even perimeter up to $2\delta = 8$. We also have the forbidden triples which do not satisfy the triangle inequality. This gives us the following initial list of contraints.

Positive	$(1, 1, 2)$, $(1, 2, 3)$ $(1, 3, 4)$, $(2, 2, 4)$, $(2, 2, 2)$, $(2, 3, 3)$
Negative	$(1, 1, 3)$, $(1, 1, 4)$, $(1, 2, 4)$

So we are only interested in permutations which do not carry constraints of one type into constraints of the other type.

If we consider the preimage of the distance 1 under our permutation then this analysis leads to the following possibilities:

$1 \mapsto 1$: identity

$2 \mapsto 1$: (142)

$3 \mapsto 1$: (13)

$4 \mapsto 1$: (14), (124)

If distance 1 is fixed: Consider the positive constraints $(1, 1, 2)$, $(1, 2, 3)$: first distance 2 must be fixed, then distance 3.

If $2 \mapsto 1$: The positive constraint $(2, 2, 4)$ must go to $(1, 1, 2)$, so $4 \mapsto 2$. The negative constraint $(1, 2, 4)$ cannot go to $(1, 2, 3)$, so $1 \mapsto 4$.

If $3 \mapsto 1$: Considering the positive constraints $(1, 2, 3)$ and $(1, 3, 4)$ we find $1 \mapsto 3$. Then considering the negative constraint $(1, 1, 4)$ shows that distances $2, 4$ are fixed.

If $4 \mapsto 1$: Considering the negative constraint $(1, 2, 4)$ shows that $\{1, 2\}$ must correspond to $\{2, 4\}$, leaving two possibilities.

For the rest, it suffices to consider the possible examples which correspond under one of the permutations (142), (13), or (14), starting with the known constraints on triangles on both sides, and referring to the known constraints on the parameters.

□

2.3. **Admissibility.** As we have mentioned, the metrically homogeneous graphs not of generic type have been classified in general, and the bipartite ones have been classified modulo the full classification in diameter $\lfloor \delta/2 \rfloor$, which certainly applies when $\delta = 4$.

Since the imprimitive ones are necessarily bipartite or antipodal, this leaves us with the following cases.

— Antipodal of generic type;
— Primitive of generic type.

In particular the parameter K_1 must be finite.

The following lemma contains an extraneous case that will need to be eliminated afterward.

Lemma 2.2. *If Γ is a primitive metrically homogeneous graph of diameter $\delta = 4$, then one of the following holds.*

- $C > 2\delta + K_1$;
- $K_1 = K_2 = 3$, $C = 2\delta + 3$

Proof. We suppose

$$C \leq 2\delta + K_1$$

As Γ is primitive, $C \geq 2\delta + 2$. By Lemma 1.8, we have $C \geq 2\delta + 3$. Therefore $K_1 \geq 3$.

If $K_2 = 4$ then we contradict Lemma 1.12. So $K_2 \leq 3$.

At this point we have $K_1 = K_2 = 3$ and $C \geq 2\delta + 3$, so $C = 2\delta + 3$. □

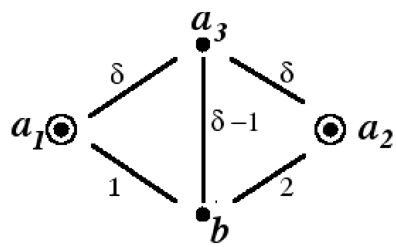
Now we eliminate the last possibility.

Lemma 2.3. *Let Γ be a primitive metrically homogeneous graph of diameter δ. Suppose*

$$K_1 \geq 3$$
$$K_2 = \delta - 1$$

Then there is a triangle of type $(3, \delta, \delta)$ in Γ.

Proof. We consider the following amalgamation.



As $K_1 > 2$ and $K_2 < \delta$, the distance $d(a_1, a_2)$ must be 3. So it suffices to show the factors embed isometrically in Γ. This means that we require a triangle of type $(2, \delta - 1, \delta)$.

For this we consider the configuration a_2a_3bc where a_2a_3b is the desired triangle (as shown) and

$$d(c, a_2) = d(c, b) = 1$$
$$d(c, a_3) = \delta - 1$$

We view this as a 2-point amalgamation problem with the distance $d(a_2, b)$ to be determined. As $K_1 > 1$ the point c ensures that $d(a_2, b) = 2$. The factors of this amalgamation are triangles of types $(1, \delta - 1, \delta - 1)$ and $(1, \delta - 1, \delta)$. As $K_2 = \delta - 1$ the former embeds isometrically in Γ, and the latter is a geodesic.

This concludes the construction.

□

Lemma 2.4. *If Γ is a metrically homogeneous graph of diameter $\delta = 4$ and generic type, then the associated parameters $(K_1, K_2, C, C', \mathcal{S})$ are admissible. In particular, either Γ is imprimitive or we have the following.*

1. $K_2 \geq 3$;
2. $C > 2\delta + K_1$;
3. *If* $K_1 = 1$ *and* $K_2 = 3$ *then* $C \geq 11$;
4. *If* $C = 2\delta + 2$ *then* $C' = C + 1$.

Proof. In the bipartite case we have the full classification already, so we suppose K_1 is finite.

We next consider the case in which $C = 2\delta + 1$. By Lemma 1.7 we have Γ imprimitive in this case, hence antipodal, so $C' = C + 1$ and $K_1 + K_2 = \delta$. It is easy to check the conditions for admissibility of (K_1, K_2, C, C') in this case.

So going forward we assume

$$K_1 < \infty \text{ and } C \geq 2\delta + 2$$

Point (1) follows by Lemma 1.4.

For point (2), by Lemma 2.2 the only possible exception has $K_1 = K_2 = 3$ and $C = 2\delta + 3$. But then Lemma 2.3 gives a triangle of perimeter $2\delta + 3$, for a contradiction

For point (3), we know $C \geq 2\delta + 2 = 10$, and in case $C = 10$ Lemma 1.8 gives $K_2 = \delta$.

Point (4) is general (Lemma 1.8).

For admissibility, we know that it suffices to check admissibility of (K_1, K_2, C, C'), and point (2) puts us in the typical case, for which the constraints are as follows.

- $K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$;
- If $C' > C + 1$ then $C \geq 2\delta + K_2$

Now points $(1, 3)$ above cover the first two conditions.

We must check that when $C' > C + 1$, we have $C \geq 2\delta + K_2$. We have previously seen that $C' = C + 1$ when $C \leq 2\delta + 2$ (Lemmas 1.7 and 1.8). So we may suppose $C \geq 2\delta + 3$, and we need only consider the case $K_2 = 4$, $C = 2\delta + 3 = 11$. The claim is that there is no triangle of type $(4, 4, 4)$.

As $K_2 = 4$ there is an edge in Γ_4, and therefore Γ_4 is connected by Lemma 1.3. If there is a triangle of type $(4, 4, 4)$ then the diameter of Γ_4 is 4, and there is a triangle of type $(4, 4, 3)$, giving a contradiction. Thus $C' = C + 1 = 12$ in this case.

□

2.4. **Realization of Triangles.** The next stage of analysis is the following.

Proposition 2.5. *Let Γ be a metrically homogeneous graph of generic type, with associated parameters K_1, K_2, C, C'. Then a triangle embeds isometrically into Γ if and only if it belongs to*

$$\mathcal{A}^{\delta}_{K_1,K_2,C,C'}$$

We recall by Proposition 1.2 that any triangle which embeds isometrically in Γ must belong to $\mathcal{A}^{\delta}_{K_1,K_2,C,C'}$. So only the converse is at issue, and as the bipartite case is fully classified we may set that aside. So we assume without further comment that K_1 is finite.

Lemma 2.6. *Let Γ be a metrically homogeneous graph of generic type of diameter δ. Let $i \leq \delta$, with $i < \delta$ if Γ is antipodal. Then there is a triangle of type $(2, i, i)$ in Γ.*

Proof. Lemma 1.6 gives a pair at distance 2 in Γ_i unless Γ_i reduces to a single vertex, in which case $i = \delta$ and Γ is antipodal. □

Lemma 2.7. *Let Γ be a metrically homogeneous graph of generic type of diameter 4, with associated parameters K_1, K_2, C_0, C_1. Then any triangle of even perimeter $p < C_0$ embeds isometrically into Γ.*

Proof. Leaving aside geodesics, the minimum edge length is at least 2, and if the type is $(2, j, k)$ then $j = k$, so Lemma 2.6 applies.

So this leaves types $(3, 3, 4)$ and $(4, 4, 4)$ to be considered.

For type $(4, 4, 4)$, we are supposing $C > p = 12$ and so some triangle of perimeter 12 occurs in Γ. As the type of such a triangle must be $(4, 4, 4)$, this case is complete.

So we are left with the case of type

$$(3, 3, 4)$$

Suppose there is no triangle of this type in Γ. Then there is no pair of vertices u, v with $u \in \Gamma_3$, $v \in \Gamma_4$, and $d(u, v) = 3$.

By Lemma 2.6 there is a pair of vertices v_1, v_2 in Γ_4 at distance 2. Let u be a neighbor of v_2 with $d(v_1, u) = 3$. Then we find $u \in \Gamma_4$. In particular Γ_4 contains pairs at distances $1, 2, 3$.

By Lemma 1.3, Γ_4 is connected. For $u \in \Gamma_3$, let $I_u = \{v \in \Gamma_4 \mid d(u, v) \leq 2\}$.

If $I_u \neq \Gamma_4$ then take $v_1 \in I_u$, $v_2 \in \Gamma_4 \setminus I_u$ with $d(v_1, v_2) = 1$. Then clearly $d(u, v_2) = 3$ and we are done.

So we may suppose

$$d(u, v) \leq 2 \text{ for } u \in \Gamma_3,\ v \in \Gamma_4$$

Take v_1, v_2 in Γ_4 with $d(v_1, v_2) = 3$ and u adjacent to v_2 with $d(v_1, v_2) = 4$. Then $u \in \Gamma_4$. Take $u' \in \Gamma_3$ adjacent to u. Then $d(u', v_1) \geq 3$, a contradiction.

□

Lemma 2.8. *Let Γ be a primitive metrically homogeneous graph of generic type and diameter 4 with associated parameters $(\delta, K_1, K_2, C, C', \mathcal{S})$. Then any triangle of type (i, j, k) with odd perimeter $p < C_1$ which satisfies the following conditions embeds isometrically in Γ.*

- $p \geq 2K_1 + 1$;
- $p \leq 2K_2 + 2\min(i, j, k)$.

Proof. Let the triangle have type (i, j, k) with $i \leq j \leq k$.
Case 1. $i = 1$:

Leaving aside geodesics, if $i = 1$ then the triangle has type $(1, j, j)$ with $K_1 \leq j \leq K_2$, and if $j = K_1$ or K_2 then the type is realized in Γ by definition. So we suppose

$$K_1 < j < K_2$$

In particular $K_1 \leq 2$, so by Lemma 1.4 we have triangles of type $(1, i, i)$ for all $i < \delta$, and the claim follows.

Case 2. $i = 2$:

Then the triangle type is $(2, j, j + 1)$ with $K_1 - 1 \leq j \leq K_2$.

In particular there is an edge in either Γ_j (if $j \geq K_1$) or in Γ_{j+1} (if $j < K_2$). Let us write the pair $\{j, j+1\}$ as $\{j_1, j_2\}$, where there is an edge in Γ_{j_2}. Then Γ_{j_2} is connected, by Lemma 1.3.

For $u \in \Gamma_{j_1}$ let I_u be $\{v \in \Gamma_{j_2} \mid d(u,v) = 1\}$. If $I_u \neq \Gamma_{j_2}$ then we may take an adjacent pair of points v_1, v_2 in Γ_{j_2} with $v_1 \in I_u$, $v_2 \notin I_u$, and then $d(u, v_2) = 2$ and we have a triangle of type $(2, j_1, j_2)$.

So now suppose

$$I_u = \Gamma_{j_2}$$

Then all distances between Γ_{j_1} and Γ_{j_2} are equal to 1, and hence Γ_{j_1} and Γ_{j_2} have diameter at most 2. In particular neither j nor $j+1$ is equal to 2, so $j = 3$ and $j + 1 = 4$.

Take v_1, v_2 in Γ_4 at maximal distance, and v_3 adjacent to v_2 with $d(v_1, v_3) = d(v_1, v_2) + 1$. Then $v_3 \in \Gamma_j$, $v_1 \in \Gamma_{j+1}$, and $d(v_1, v_3) > 1$ a contradiction.

Case 3. $i > 2$.

Then $i = 3$ and the triangle type is $(3, 3, 3)$ or $(3, 4, 4)$.

In particular our assumption $p < C_1$ (p odd) implies that there is a triangle in Γ with the same perimeter p, and then the type is unique. □

2.5. **The Antipodal Case with $K_1 > 1$.** We aim at an identification theorem.

Proposition 2.9. *Let Γ be an antipodal graph of generic type and diameter 4, with $K_1 > 1$. Then $\Gamma \cong \Gamma^4_{2,2,9,10,\emptyset}$.*

Proof. We have

$$K_1 = K_2 = 2$$

In particular Γ_2 contains an edge. By Lemma 1.3, Γ_2 is connected. Also Γ_2 has diameter 4 and is antipodal, with $K_1 > 1$. Furthermore Γ_2 is infinite since Γ contains an infinite set of points at mutual distance 2.

It follows that Γ_2 is also an antipodal graph and diameter 4 with$K_1 > 1$.

It suffices to show that any finite configuration A which embeds isometrically in $\Gamma^4_{2,2,9,10,\emptyset}$ embeds isometrically into Γ.

We suppose A os a counterexample with $|A|$ minimal (where we allow Γ to vary as well, within the class of antipodal graphs of generic type, diameter 4, with $K_1 > 1$). We view A as a graph with edge relation "$d(x, y) = 1$ or 3" and we refer to the connected components of this graph as the $(1, 3)$-components of A.

Claim 1. *The distance 4 does not occur.*

If $d(u, u') = 4$ then u' is the antipodal vertex to u, and it suffices to embed the configuration with u' omitted.

Claim 2. *All $(1, 3)$-components of A are nontrivial.*

Otherwise, we have a vertex $a \in A$ with $d(a, x) = 2$ for $x \in A \setminus \{a\}$. Then by minimality of $|A|$ we have $A \setminus \{a\}$ embedding isometrically in Γ_2, and hence A embeds isometrically in Γ.

Claim 3. *In each $(1, 3)$-component only one of the two distance 1 or 3 occurs.*

Otherwise, there is are points u, v_1, v_2 in A with $d(u, v_1) = 1$, $d(u, v_2) = 3$. Then view A as a 2-point amalgamation problem with $d(v_1, v_2)$ to be determined. As $d(u, v_1) \neq d(u, v_2)$ the two points cannot be identified, and the only available distance is 2. By minimality the factors embed isometrically in Γ, and therefore A does as well, which is a contradiction.

Claim 4. *Each* $(1, 3)$*-component has order* 2.

Otherwise there are vertices u, v_1, v_2 with $d(u, v_1) = d(u, v_2) = 1$ or 3. Then replacing v_2 by its antipodal pair we get $d(u, v_2) \neq d(u, v_1)$ and we contradict the previous claim.

So now the structure of A is clear: it is a union of pairs of points $A_i = \{a_i, b_i\}$ with $d(a_i, b_i) = 1$ or 3 and with all other distances equal to 2. There are at least two such pairs.

Adjoin a point c adjacent to all a_i and with $d(c, b_i) = 2$ for all i. Write $B = \{b_i \mid \text{all } i\}$ and view $A \cup \{c\}$ as the result of amalgamating all configurations

$$B_i = B \cup \{c, a_i\}$$

over the base $B \cup \{c\}$, to determine the distances $d(a_i, a_j)$ for all i, j. The vertex c and the condition $K_1 > 1$ guarantees that in the amalgam we have $d(a_i, a_j) = 2$. Thus it suffices to check that the factors B_i all embed into Γ. If not, then as $|B_i| \leq |A|$ (with equality only if $|A| = 4$), we may apply Claim 2 to B_i and arrive at a contradiction.

□

2.6. **The Antipodal Case with** $K_1 = 1$**.** We aim at the following identification theorem.

Proposition 2.10. *Let* Γ *be an antipodal graph of generic type and diameter* 4*, with* $K_1 = 1$*. Then* Γ *is isomorphic to one of the following.*

- *An antipodal graph of Henson type* $\Gamma^4_{a,n}$*, with* $n \geq 4$*; or*
- *The generic antipodal graph of diameter* 4*,* $\Gamma^4_{1,3,9,10,\emptyset}$*.*

We make a formal definition of the parameter n.

Notation. *For* Γ *a metrically homogeneous graph, let* $n = n(\Gamma)$ *be the maximal clique size, or* ∞*.*

Note that the condition $K_1 > 1$ is the same as $n(\Gamma) = 2$.

Lemma 2.11. *Let* Γ *be an antipodal graph of generic type and diameter* 4*, with* $n = n(\Gamma)$*. Then* Γ_2 *is an antipodal graph of generic type and diameter* 4*, with* $n(\Gamma_2) = n$*.*

Proof. All that needs to be shown here is that a clique of order n embeds isometrically into Γ_2.

We distinguish two cases.

Case 1. $n = \infty$

Then Γ_1 is a random graph, so for $a \in \Gamma_1$, $\Gamma_2(a)$ contains a random graph. This proves the claim in this case.

Case 2. $n < \infty$

We perform an amalgamation construction.

Let $A = A_1 \cup A_2$ be the union of two cliques of order $n-1$, together with edges forming a perfect matching between A_1 and A_2. Let a_1, a_2 be additional vertices with a_1 adjacent to all vertices of A, a_2 adjacent to the vertices of A_2, and unspecified distances equal to 2.

View $A \cup \{a_1, a_2\}$ as a 2-point amalgamation problem determining the distance $d(a_1, a_2)$. As a_1, a_2 have common neighbors, this distance is at most 2. If the distance were 1, then $A_2 \cup \{a_1, a_2\}$ would be a clique of order $n+1$, a contradiction.

So $d(a_1, a_2) = 2$ and $A_1 \cup \{a_1\}$ is contained in $\Gamma_2(a_2)$. So the claim follows in this case, once we embed the factors $A_1 \cup A_2 \cup \{a_i\}$ of the amalgamation in Γ.

In the factor $A_1 \cup A_2 \cup a_1$, we have $A_1 \cup A_2$ contained in $\Gamma_1(a_1)$. So it suffices to embed $A_1 \cup A_2$ isometrically in Γ_1. But Γ_1 is the Henson graph G_{n-1}, so this is possible.

In the factor $A_1 \cup A_2 \cup \{a_2\}$ we have cliques $A_1, A_2 \cup \{a_2\}$ of orders $n-1$ and n, and a perfect matching between A_1 and A_2.

Let $e_1, \ldots, e_{n-1}$ denote the edges of the perfect matching, and write $e_i = (u_i, v_i)$ with $u_i \in A_1$. Adjoint points $c_1, \ldots, c_{n-1}$ with c_i adjacent to u_i, v_i, and with unspecified distances equal to 2.

View the resulting configuration as a 2-point amalgamation with the distance $d(u_1, v_1)$ to be determined. The point c_1 forces this to be at most 2, and distance 2 would put $A_2 \cup \{a_2\}$ into $\Gamma_2(u_1)$, while distance 1 gives the desired configuration.

So we examine the factors of this amalgamation, omitting u_1 or v_1. If we omit v_1 then there is no clique of order n and this embeds into Γ_1. If we omit u_1 then we consider the configuration as a 2-point amalgamation determining $d(u_2, v_2)$, and continue in this vein until we come down finally to the configuration

$$A_2 \cup \{c_1, \ldots, c_{n-1}, a_2\}$$

which is a clique of order n with $n-1$ points c_i having distinct neighbors in A_2.

Again we argue inductively that the configuration $A_2 \cup \{c_1, \ldots, c_k\}$ embeds isometrically into Γ for $k \leq n-1$. We consider the configuration $A_2 \cup \{c_1, \ldots, c_k\}$ as a 2-point amalgamation problem determining the distance $d(c_k, v_k)$, with the factor omitting c_k given by induction, and the factor omitting v_k embedding in Γ_1.

We are aiming at $d(c_k, v_k) = 1$. if $d(c_k, v_k) = 2$ then A_2 is contained in $\Gamma_2(c_k)$. If $d(c_k, v_k) = 3$ then replace c_k by the antipodal vertex c'_k. We cannot have $d(c_k, v_k) = 4$ since $d(v_k, a_2) = 1$ and $d(c_k, a_2) = 2$.

This completes the construction.

□

Lemma 2.12. *Let Γ be an antipodal graph of generic type and diameter 4, with $n = n(\Gamma)$. Let v_1, v_2 be a pair of adjacent vertices, and write $\Gamma_2(v_1, v_2)$ for $\Gamma_2(v_1) \cap \Gamma_2(v_2)$. Then $\Gamma_2(v_1, v_2)$ is an antipodal graph of generic type and diameter 4, with $n(\Gamma_2(v_1, v_2)) = n$.*

Proof.

Claim 1. *$\Gamma_2(v_1, v_2)$ is nonempty.*
This holds as $K_1 \leq 2$ by Lemma 1.4.

Claim 2. *$\Gamma_2(v_1, v_2)$ contains a geodesic triangle of type $(1, 1, 2)$.*

We make an amalgamation argument.
Let $u_1, u_2, , u_3, v_1, v_2$ be a configuration with

$$d(u_1, u_2) = d(u_2, u_3) = d(v_1, v_2) = 1$$

and other distances equal to 2. We need to embed this configuration isometrically into Γ.

We adjoin vertices a_1, a_2 satisfying the following.

$$d(a_1, u_2) = 1 \qquad d(a_2, u_2) = 1$$
$$d(a_1, v_1) = 1 \qquad d(a_2, v_1) = 3$$

and unspecified distances equal to 2, i.e., the standard witnessing pair for the condition "$d(u_2, v_1) = 2$."

We view the configuration $(u_1, u_2, u_3, v_1, v_2, a_1, a_2)$ as a 2-point amalgamation problem with $d(u_2, v_1)$ to be determined. Since the distance $d(u_2, v_1) = 2$ is forced, it suffices to embed the factors

$$(u_1, u_2, u_3, v_2, a_1, a_2) \text{ and } (u_1, u_3, v_1, v_2, a_1, a_2)$$

isometrically into Γ.

By Lemma 2.11, Γ_2 satisfies the same hypotheses as Γ, and we can eliminate vertices lying at distance 2 from the others, in either factor. This reduces the problem to the configurations

$$(u_1 u_2 u_3 a_1 a_2) \text{ and } (v_1 v_2 a_1 a_2)$$

In each case the structure is a star, with center u_2 or v_1 respectively, and with distances 1 or 3 on the edges of the star, with distance 2 elsewhere.

We may also replace a_2 by its antipodal point whenever convenient, and in this way replace the condition $d(a_2, v_1) = 3$ by $d(a_2, v_1) = 1$. In other words, we may take these stars to be true stars, with respect to the edge relation "$d(x, y) = 1$."

So we then require an embedding of $(u_1 u_3 a_1 a_2)$ or $(v_2 a_1 a_2)$ into Γ_1, and this is possible.

Claim 3. *$\Gamma_2(u_1, v_1)$ contains pairs at distances 1, 2, 3, and 4.*

The claims so far cover distances 0, 1, and 2, and $\Gamma_2(u_1, v_1)$ is closed under the antipodal pairing, so that gives 3 and 4 as well.

Claim 4. $\Gamma_2(v_1, v_2)$ *contains geodesic triangles of types* $(1, 2, 3)$ *and* $(1, 3, 4)$.

Under the antipodal pairing this corresponds to types $(1, 2, 1)$ and $(1, 1, 0)$, both of which we have.

Claim 5. $\Gamma_2(v_1, v_2)$ *is a connected antipodal metrically homogeneous graph of diameter 4.*

As we have the geodesics of type $(1, 1, 2)$, $(1, 2, 3)$, and $(1, 3, 4)$, and $\Gamma_2(v_1, v_2)$ is metrically homogeneous, it is connected as a graph under the edge relation "$d(x, y) = 1$," and the metric is the graph metric. We have already seen that the diameter is 4 and the antipodality is inherited from Γ.

Claim 6. $\Gamma_2(v_1, v_2)$ *is of generic type.*

In the contrary case $\Gamma_2(v_1, v_2)$ would be a cycle of girth 9.

It suffices to embed the configuration $(v_1, v_2, u, u_1, u_2, u_3)$ isometrically into Γ, where (v_1, v_2) is an edge, (u, u_1, u_2, u_3) is a star with center u and distances $d(u, u_i) = 1$, and unspecified distances are equal to 2.

Add the usual witnessing pair a_1, a_2 to ensure $d(u, v_1) = 2$, taking unspecified distances again equal to 2. View the resulting configuration as a 2-point amalgamation problem determining the distance $d(u, v_1)$. In the factors, after removal of isolated points (at distance 2 from the remainder) and after replacing a_2 when necessary by its antipodal point, we come down to true stars with distance 1 on each edge and distance 2 elsewhere. These embed into Γ by considering Γ_1.

Claim 7. $n(\Gamma_2(v_1, v_2)) = n$

If $n = \infty$ the desired configuration is realized in Γ_1, so we suppose n is finite.

The configuration we require is $A \cup \{v_1, v_2\}$ where A is a clique of order n and (v_1, v_2) is an edge, other distances being equal to 2.

We take $u \in A$ and adjoin a witnessing pair a_1, a_2 to ensure $d(u, v_1) = 2$, again taking unspecified distances to be 2. It remains to check that the factors embed isometrically into Γ.

The factor $(A \setminus \{u\}) \cup \{v_1, v_2, a_1, a_2\}$ embeds in Γ_1 as it contains no n-clique.

So this leaves the factor

$$A \cup \{v_2, a_1, a_2\}$$

Here by Lemma 2.11 we may delete the vertex v_2, leaving $A \cup \{a_1, a_2\}$. After replacing a_2 by its antipodal point we have a configuration in $\Gamma_1(a)$ which embeds into Γ_1, so we conclude. □

Proof of Proposition 2.10. We let n be minimal such that Γ omits a clique of order n, if there is one, and $n = \infty$ otherwise. We write $\Gamma^4 a, \infty$ for $\Gamma^4_{1,3,9,10,\emptyset}$, to unify notation. We set $\Gamma^* = \Gamma^4_n$, so that our claim is

$$\Gamma \cong \Gamma^*$$

It suffices to show that any finite configuration A which embeds isometrically into Γ^* embeds isometrically into Γ.

If we have a pair of points v_1, v_2 at distance 4 in A then we may omit v_2: if $A \setminus \{v_2\}$ embeds in Γ, then so does A. So we suppose all distances are 1, 2, or 3.

Let G_A be the graph on A with edge relation "$d(x, y) = 1$ or 3." We call the connected components of G_A the $(1, 3)$-components of A.

We consider the subset A' of points with at least two neighbors in G_A. We choose a hypothetical counterexample A as follows (here we vary not over A but also Γ).

- Minimize $|A'|$; then
- Minimize the number of $(1, 3)$-connected components of A; then
- Minimize $|A|$.

Claim 1. *There is no pair of points in A at distance* 4.

By antipodality if $d(v, v') = 4$ and $A \setminus \{v\}$ embeds isometrically in Γ, then so does A.

Claim 2. *Every $(1, 3)$-component contains a point of A'.*

Otherwise, there is a $(1, 3)$-component A_0 of order at most 2, and then by minimality of A and Lemma 2.11 or 2.12, $A \setminus A_0$ embeds isometrically into $\Gamma_2(A_0)$, and hence A embeds into Γ.

Claim 3. *A' is a $(1, 3)$-complete graph, that is, a clique in G_A.*

Suppose on the contrary that $v_1, v_2 \in A'$ with $d(v_1, v_2) = 2$.

Adjoin vertices a_1, a_2 with

$$\begin{aligned} d(a_1, v_1) &= 1 \qquad & d(a_2, v_1) &= 1 \\ d(a_1, v_2) &= 1 \qquad & d(a_2, v_2) &= 3 \end{aligned}$$

and unspecified distances equal to 2.

View the resulting configuration as a 2-point amalgamation problem determining the distance $d(v_1, v_2)$. The points a_1, a_2 ensure that this distance is 2. So it suffices to embed the factors of $A \cup \{a_1, a_2\}$ omitting v_1 or v_2 into Γ. Here the size of A' decreases in each factor and we conclude by the minimality of A.

Claim 4. *Without loss of generality, A' is a* 1*-clique (an ordinary clique with mutual distance* 1*).*

As A' is a Γ-constrained $(1, 3)$-space, A' consists of 1-cliques lying at mutual distance 3, and (by the bound $C = 9$) there are at most two such cliques. If there are two cliques, then by replacing vertices by antipodal vertices we arrive at an equivalent configuration in which A' is a single clique.

Going forward, therefore, we will make the assumption

$$A' \text{ is a clique for the edge relation "} d(x, y) = 1. \text{"}$$

Claim 5. *A consists of a single $(1,3)$-connected component.*

Each $(1,3)$-component meets A', and A' is connected.

Claim 6. *Without loss of generality, all distances in A are equal to 1 or 2, and the graph with the edge relation "$d(x,y) = 1$" is a clique with some attached edges.*

Suppose $u \in A \setminus A'$. Then there is a unique vertex u^* at distance 1 or 3 from u, and after replacing u if necessary by its antipodal point, that distance is 1. Furthermore, as A is $(1,3)$-connected, u^* must belong to A'. This gives the desired structure.

Now to conclude the proof, we will show by induction on the parameter

$$k = |\{u \in A' \mid \text{There is a neighbor of } u \text{ outside } A'\}|$$

that A embeds isometrically in Γ.

If $k \leq 1$ then there is a vertex $u \in A$ at distance 1 from all vertices of A, so it suffices to embed $A \setminus \{u\}$ in Γ_1. As this contains no n-clique, this is possible.

So suppose

$$k \geq 2$$

Now proceed by induction on the minimal degree of a vertex in A' with neighbors outside A'. Take a vertex u minimizing this parameter, a neighbor v_1 of u outside A', and another vertex $u_1$in A' with a neighbor outside A'.

Adjoin the usual witnessing pair a_1, a_2 ensuring $d(u_1, v_1) = 2$, taking unspecified distances equal to 2. It suffices to embed the factors omitting u_1 or v_1 isometrically into Γ.

The factor omitting u_1 contains no n-clique so embeds into Γ_1.

The factor omitting v_1 embeds isometrically into Γ by induction: either u no longer has neighbors outside A', and the number of such vertices is decreased, or else the minimal degree of such a vertex has been lowered.

This completes the proof.

□

3. Diameter 4: $C = 10$ or $K_1 = 4$

Now we work toward the following.

Proposition 3.1. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then $\Gamma \cong \Gamma^4_{1,4,10,11}$.*

As noted earlier, after a permutation of the language this is equivalent to the following.

Proposition 3.2. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $K_1 = 4$. Then $\Gamma \cong \Gamma^4_{4,4,13,14}$.*

3.1. **Structure of Γ_2 and Γ_3.** Our first goal is the following.

Lemma 3.3. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then*

- *Γ_2 is an infinite primitive metrically homogeneous graph of diameter 4 with associated parameters $\tilde{K}_1 = 1$, $\tilde{K}_2 = 4$, and $\tilde{C} = 10$.*
- *$\Gamma_3 \cong \Gamma^3_{1,3,10,11}$*

We work toward this in stages.

Lemma 3.4. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then Γ_3 is an infinite primitive metrically homogeneous graph of diameter 3 with associated parameters $\tilde{K}_1 = 1$ and $\tilde{K}_2 = 3$.*

Proof. By Lemma 1.8 we have

$$K_1 = 1,\ K_2 = 4,\ C' = C + 1,\ \mathcal{S} = \emptyset$$

In particular Γ contains an infinite clique and so Γ_1 is a random graph.

Claim 1. *Γ_3 is an infinite primitiive metrically homogeneous graph of diameter 3, of generic type, with $\tilde{K}_1 = 1$.*

Γ_3 has an edge and is therefore connected by Lemma 1.3. Thus Γ_3 is a metrically homogeneous graph.

Triangles of type $(3, 3, 3)$, but not $(3, 3, 4)$, occur in Γ, so the diameter of Γ_3 is 3.

If we take two vertices at distance 2 in Γ_2, Γ_4, then their common neighbors form a copy of Γ_1, a random graph, contained in Γ_3, by Lemma 1.1. In particular Γ_3 is infinite with $\tilde{K}_1 = 1$, and of generic type.

By Lemma **??**, Γ_3 is primitive.

Claim 2. *For $u \in \Gamma_1$, there are at least two points of Γ_4 at distance 4 from u.*

Otherwise, u determines a unique point $u' \in \Gamma_4$ at distance 4. This gives a function from Γ_1 onto Γ_4, and as Γ_1 is primitive this function is either 1-1 or constant. As Γ_4 is nontrivial, the function is $1-1$. Since Γ_4 is complete, it follows that the automorphism group of Γ_1 acts 2-transitively on Γ_1, a contradiction.

Claim 3. $\tilde{K}_2 = 3$

We need to find a triangle of type $(3, 3, 1)$ in Γ_3.

Begin with $u \in \Gamma_1$ and $v_1, v_2 \in \Gamma_4$ with

$$d(u, v_1) = d(u, v_2) = 4$$

Extend v_1, v_2 to a geodesic (v_0, v_1, v_2, v_3) with $d(v_0, v_3) = 3$.

As Γ_4 is complete we find $v_0, v_3 \in \Gamma_3$. As v_0, v_3 are adjacent to v_1, v_2 respectively, we find

$$d(u, v_0), d(u, v_3) \geq 3$$

As $C = 10$ and $C' = 11$ we find that (u, v_0, v_3) has type $(3, 3, 3)$. Thus $\Gamma_3(v_0)$ contains a triangle of type $(3, 3, 1)$ consisting of u, v_3, and the base point.

□

Lemma 3.5. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then Γ_3 is an infinite primitive metrically homogeneous graph of diameter 3 with associated parameter $\tilde{C} = 10$*

Proof. By Lemma 3.4, Γ_3 is infinite primitive metrically homogeneous of diameter 3 with associated parameters

$$\tilde{K}_1 = K_1 = 1 \text{ and } \tilde{K}_2 = 3$$

As Γ_3 has diameter 3 we also have $\tilde{C}' = \tilde{C} + 1$. In principle it would be sufficient to show that there is a triangle of type $(3, 3, 3)$ in Γ_3, but from a practical point of view it is convenient to deal separately—and first—with type $(3, 3, 2)$.

Claim. *There is a triangle of type $(3, 3, 2)$ in Γ_3.*

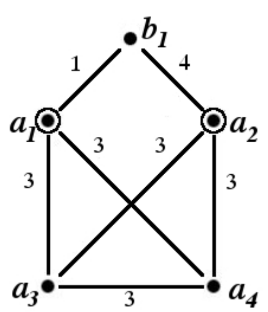
Take $u_1.u_2 \in \Gamma_3$ with $d(u_1, u_2) = 3$. Take u adjacent to u_2 with $d(u_1, u) = 4$. As $C = 10$ we have $u \in \Gamma_2$. Now u has a neighbor u_3 in Γ_3 at distance 2 from u_2. Since $d(u_1.u) = 4$ we have $d(u_1, u_3) = 3$. Thus (u_1, u_2, u_3) is a triangle of type $(3, 3, 2)$ in Γ_3. This proves the claim.,

Now we take up the problem of finding a triangle of type $(3, 3, 3)$ in Γ_3. We use an explicit amalgamation argument. We are aiming at the configuration (a_1, a_2, a_3, a_4) with $d(a_i, a_j) = 3$.

Adjoin b_1 with

$$d(b_1, a_1) = 1 \qquad d(b_1, a_2) = 4$$
$$d(b_1, a_i) = 2 \ (i = 3, 4)$$

View this as a 2-point amalgamation problem in which the distance $d(a_1, a_2)$ is to be determined.



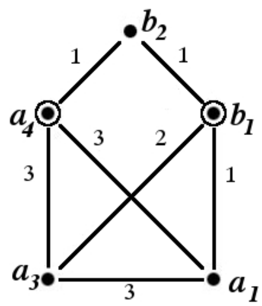
$d(b_1, a_3) = d(b_1, a_4) = 2$

The bound $C = 10$ gives $d(a_1, a_2) \leq 3$ and the point b_1 gives $d(a_1, a_2) \geq 3$. So it suffices to embed the factors of this configuration omitting a_1 or a_2 isometrically into Γ.

I. The factor $(a_1a_3a_4b_1)$*:*
We adjoin a point b_2 with

$$\begin{aligned} d(b_2,a_4) = d(b_2,b_1) &= 1 \\ d(b_2,a_1) &= 2 \\ d(b_2,a_3) &= 3 \end{aligned}$$

We view this as a 2-point amalgamation in which the distance $d(a_4,b_1)$ is to be determined.



$$\begin{aligned} d(b_2,a_1) &= 2 \\ d(b_2,a_3) &= 3 \end{aligned}$$

The points a_1 and b_2 force $d(a_4,b_1) = 2$. So it suffices to embed the factors omitting a_4 or b_1 isometrically into Γ.

The factor $(a_1a_3a_4b_2)$ has the triangle (a_1,a_4,b_2) of type $(1,2,3)$ in $\Gamma_3(a_3)$. As Γ_3 is connected of diameter 3, this factor embeds isometrically into Γ.

Now we consider the factor $(a_1a_3b_1b_2)$. Taking b_1 as base point, we require a point a_3 in Γ_2 at distance 3 from two nonadjacent points of Γ_1.

Fix a_3 in Γ_2 and take a in Γ_2 at distance 4 from a_3. Then there is a 4-cycle (acb_1c') embedding isometrically in Γ. In particular $c,c' \in \Gamma_1$ are at distance 2.

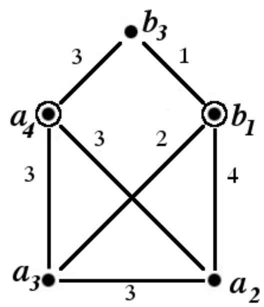
As c,c' are adjacent to a we have $d(a_3,c), d(a_3,c') \geq 3$. As $a_3 \in \Gamma_2$ and $c,c' \in \Gamma_1$, we have $d(a_3,c), d(a_3,c') \leq 3$. Thus we have the desired configuration in the form (b_1a_3cc').

The factor $(a_2a_3a_4b_1)$*:*
We adjoin a vertex b_3 with

$$\begin{aligned} d(b_3,a_2) &= 3 \\ d(b_3,a_3) &\leq 2 \\ d(b_3,a_4) &= 3 \\ d(b_3,b_1) &= 1 \end{aligned}$$

where the choice of $d(b_3,a_3)$ will be settled later.

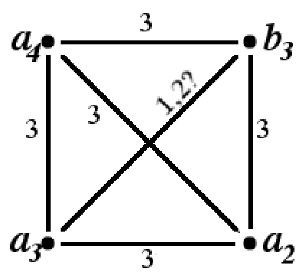
We treat the resulting configuration as a 2-point amalgamation problem in which the distance $d(b_1,a_4)$ is to be determined.



$d(b_3, a_3) = 1$ or 2
$d(b_3, a_2) = 3$

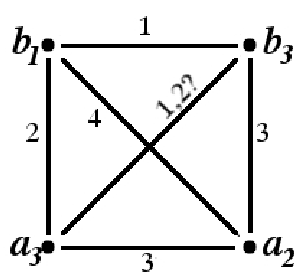
So it suffices to check that for some choice of the distance $d(b_3, a_3)$, the factors of this amalgamation embed isometrically into Γ.

The configuration $(a_2a_3a_4b_3)$*:*



This consists of a triangle of type $(1,3,3)$ or $(2,3,3)$ in $\Gamma_3(a_4)$. Since triangles of both types embed in Γ_3, this configuration embeds isometrically in Γ, for either value of $d(a_3, b_3)$.

The configuration $(a_2a_3b_1b_3)$*:*



We adjoin a point b_4 with

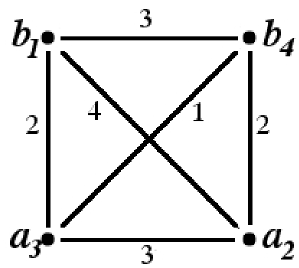
$$\begin{aligned} d(b_4, a_3) = d(b_4, b_3) &= 1 \\ d(b_4, a_2) &= 2 \\ d(b_4, b_1) &= 3 \end{aligned}$$

and view the resulting configuration as a 2-point amalgamation problem with the distance $d(a_3, b_3)$ to be determined. Here we have the possibilities $d(a_3, b_3) = 1$ or 2 but as we have already seen, either one suffices.

So it suffices to embed the factors $(a_2a_3b_1b_4)$ and $(a_2b_1b_3b_4)$ of this amalgamation isometrically in Γ.

In the factor $(a_2b_1b_4b_4)$ the distance $d(b_1, b_4) = 2$ is determined uniquely by the other two points, so this may be obtained by amalgamating two triangles which embed into Γ.

This leaves only the factor $(a_2a_3b_1b_4)$ to be dealt with.



If we think of this as a 2-point amalgamation problem with the distance $d(a_2, b_4)$ to be determined, then as $C = 10$ the value $d(a_2, b_4) = 2$ is forced. Thus this results from the amalgamation of two triangles which embed in Γ. □

Lemma 3.6. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then Γ_2 is an infinite primitive metrically homogeneous graph of diameter 4 with the same parameters $\tilde{K}_1 = K_1 = 1$, $\tilde{K}_2 = K_2 = 4$, and $\tilde{C} = C = 10$.*

Proof. Clearly Γ_2 contains an edge and has diameter 4. By Lemma 1.3, Γ_2 is connected, and is therefore a metrically homogeneous graph of diameter 4. By Lemma **??**, Γ_2 is primitive. As Γ_1 contains an infinite independent set, Γ_2 is infinite.

Thus Γ_2 is an infinite primitive metrically homogeneous graph of diameter 4. We claim that the associated parameter $\tilde{C}$ is 10, and then Lemma 1.8 does the rest.

It will suffice to show that Γ_2 contains a triangle of type $(3, 3, 3)$. We consider the confiburation (a_1, a_2, a_3, b) with a_1, a_2, a_3 the desired triangle in $\Gamma_2(b)$. Observe that this configuration is Γ_3-constrained and therefore embeds in Γ_3, by the classification in diameter 3. Thus this configuration embeds in Γ. □

We return to consideration of Γ_3.

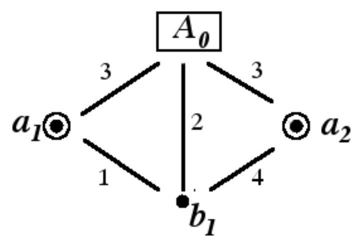
Lemma 3.7. *Let Γ be an infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. Then Γ_3 contains $I_\infty^{(3)}$, an infinite anticlique with mutual distance 3.*

Proof. We show by induction on n that any infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$ contains an isometric copy of $I_n^{(3)}$ for each n. This is known already for $n = 4$, so take $n > 4$.

Let $A = \{a_1, \ldots, a_n\}$ be the desired configuration. Adjoin a point b with

$$d(b, a_i) = \begin{cases} 1 & i = 1 \\ 4 & i = 2 \\ 2 & \text{otherwise} \end{cases}$$

View the resulting configuration as a 2-point amalgamation problem in which the distance $d(a_1, a_2)$ is to be determined. We will set $A_0 = \{a_i \mid i > 2\}$.



As $C = 10$ and $n > 2$, the distance $d(a_1, a_2)$ is at most 4, hence exactly 3. Thus it suffices to show that the factors of this amalgamation embed isometrically in Γ.

The factor $(A_0 a_1 b_1)$*:*

We adjoin a point b_2 with

$$\begin{aligned} d(b_2, a_1) &= 1 \\ d(b_2, a_3) &= 4 \\ d(b_2, a_i) &= 2 \ (i > 3) \\ d(b_2, b_1) &= 2 \end{aligned}$$

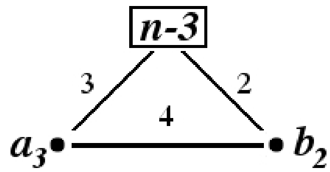
View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_3)$ to be determined. The points b_1, b_2 force this distance to be 3. Thus it suffices to embed the factors of this amalgamation isometrically in Γ. Write $A_1 = \{a_i \mid i > 3\}$.

By induction hypothesis, $I_{n-2}^{(3)}$ embeds in Γ_3. Therefore the factor $(A_1 a_1 b_1 b_2)$ is Γ_3-constrained, and hence embeds isometrically in Γ_3, and therefore also in Γ.

In the factor $(A_1 a_3 b_1 b_2)$, $A_1 a_3 b_2$ lies in $\Gamma_2(b_1)$, which is another infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$. So it will suffice now to show that the configuration

$$(A_1 a_3 b_2)$$

embeds isometrically into Γ.



We adjoin a point b_3 with

$$d(b_3, a_i) = 3 \text{ all } i$$
$$d(b_3, b_2) = 1$$

View the resulting configuration as an amalgamation in which the distances between b_2 and A_1 are to be determined. The point a_3 ensures these distances are at most 2, and the point b_3 ensures these distances are at least 2. So it suffices to show that the factors of this amalgamation embed isometrically into Γ.

These factors are a copy of $I^{(3)}_{n-1}$, which embeds by induction hypothesis, and a geodesic triangle.

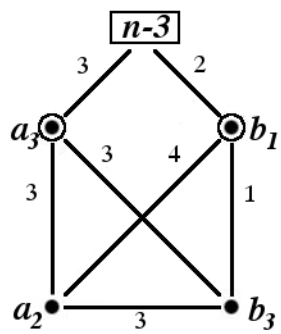
This completes the discussion of the factor $(A_0 a_1 b_1)$.

The factor $(A_0 a_2 b_1)$*:*

We adjoin a point b_3 with

$$d(b_3, a_2) = d(b_3, a_3) = 3$$
$$d(b_3, a_i) = 2 \ (i > 3)$$
$$d(b_3, b_1) = 1$$

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(a_3, b_1)$ to be determined.



$$d(a_2, a_i) = 3$$
$$d(b_3, a_i) = 2 \ (i > 3)$$

The points a_2 and b_3 force $d(a_3, b_1) = 2$. Thus it suffices to embed the factors omitting a_3 or b_1 isometrically in Γ.

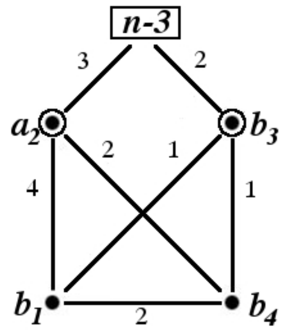
The factor omitting b_1 consists of the configuration $(A_1 a_3 b_3)$ inside $\Gamma_3(a_2)$. By induction $(A_1 a_3 b_3)$ is Γ_3-constrained and hence embeds isometrically into Γ_3. Thus this factor embeds isometrically into Γ.

This leaves the factor $(A_1a_2b_1b_3)$ to be considered.

We adjoin a point b_4 with

$$d(b_4, a_i) = 2 \text{ (all } i)$$
$$d(b_4, b_2) = 2$$
$$d(b_4, b_3) = 1$$

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, b_3)$ to be determined.



$d(b_1, a_i) = 2$ $(i > 3)$
$d(b_4, a_i) = 2$ $(i > 3)$

The points b_1, b_4 ensure that $d(a_2, b_3) = 2$. So it suffices to show that the factors $(A_1a_2b_1b_4)$ and $(A_1b_1b_3b_4)$ embed isometrically into Γ.

The factor $(A_1a_2b_1b_4)$ consists of $(A_1a_2b_1)$ inside $\Gamma_2(b_4)$. Since Γ_2 satisfies the same conditions as Γ, it suffices to show that the configuration $(A_1a_2b_1)$ embeds into Γ. But this is isometric to the configuration $(A_1a_3b_2)$ treated earlier.

The factor $(A_1b_1b_3b_4)$ is Γ_3-constrained, hence embeds isometrically in Γ_3, hence also in Γ.

This completes the proof. □

Lemma 3.8. *Let* Γ *be an infinite primitive metrically homogeneous graph of diameter* 4 *with* $C = 10$*. Let* A *be a finite* $(1, 3)$*-space with at most one nontrivial connected component. Then* A *embeds isometrically into* Γ*.*

Proof. We proceed by induction on the number n of components, and then on the size of a maximal component. The components of a $(1, 3)$-space are cliques, separated by distance 3. Our assumption is that at least $n - 1$ of these cliques consist of an isolated point.

If all components are trivial the result is covered by Lemma 3.7. So we will assume that there is a nontrivial component A_1 in A. If $A = A_1$ then the claim follows since Γ_4 is an infinite clique. So we assume $n \geq 2$ and pick a point $a_2 \in A \setminus A_1$.

Let us also treat separately the case of 2 components, that is $A = A_1 \cup \{a_2\}$. Recall that Γ_4 is a clique. Take $u \in \Gamma_4$. Then $\Gamma_1(u)$ is a random graph contained in $\Gamma_3 \cup \Gamma_4$. Therefore $\Gamma_1(u) \setminus \Gamma_4$ contains an infinite clique in Γ_3.

Thus we have an embedding of $A_1 \cup \{a_2\}$ into Γ with a_2 corresponding to our chosen base point.

So now suppose

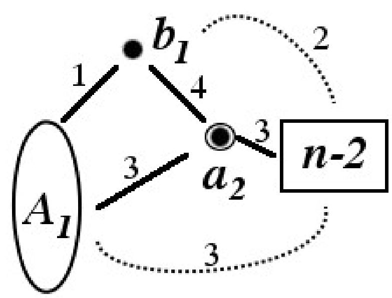
$$n \geq 3$$

Fix another point a_3 outside A_1.

Now adjoin a point b_1 with

$$d(b_1, a) = 1 \ (a \in A_1)$$
$$d(b_1, a_2) = 4$$
$$d(b_1, x) = 2 \text{ otherwise}$$

We treat the resulting configuration as an amalgamation problem in which the distances between A_1 and a_2 are to be determined.



The point b_1 ensures that these distances are at least 3, and as $C = 10$ the point a_3 ensures that these distances are exactly 3. So it suffices to show that the corresponding factors embed isometrically in Γ in each of the two cases. Set $A^* = A \setminus (A_1 \cup \{a_2\})$.

The factor $(A_1 A^* b_1)$*:*

We adjoin a vertex b_2 with

$$d(b_2, a_3) = d(b_2, b_1) = 1$$
$$d(b_2, x) = 2 \text{ otherwise}$$

We view the resulting configuration as a 2-point amalgamation problem in which the distance $d(a_3, b_1)$ is to be determined. The point b_2 forces this distance to be at most 2 and the points of A_1 force it to be at least 2. So it suffices to show that the factors of this amalgamation embed isometrically into Γ.

We claim that the factor $(A \setminus \{a_2, a_3\}, b_1 b_2)$ is Γ_3-constrained and hence embeds isometrically even into Γ_3.

The maximal $(1, 3)$-spaces in the factor $(A \setminus \{a_2, a_3\}, b_1 b_2)$ are on the one hand some cliques which contain b_1 or b_2 and on the other hand the space $A \setminus \{a_2, a_3\}$. To embed these into Γ_3 we adjoin and additional isolated point a_{n+1} and apply induction.

This leaves the factor $(A \setminus \{a_2\}, b_2)$ for consideration.

We adjoin a further point b_3 with

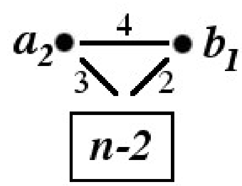
$$\begin{aligned} d(b_3, a) &= 1 \ (a \in A_1) \\ d(b_3, a_3) &= 2 \\ d(b_3, b_2) &= 1 \\ d(b_3, x) &= 3 \text{ otherwise} \end{aligned}$$

We view the resulting configuration as an amalgamation problem in which the distances between b_2 and A_1 are to be determined, and are forced to be equal to 2. So it suffices now to embed the factors of this amalgamation into Γ.

The factor omitting b_1 is a $(1,3)$-space with a unique nontrivial component, and a total of $n-1$ components, so this embeds in Γ by induction.

The factor omitting A_1 is is Γ_3-constrained since its maximal $(1,3)$-subspaces are the clique $\{b_1, b_3\}$ and some anticliques $I^{(3)}_{n-1}$.

The factor $(a_2 A^* b_1)$*:*



Adjoin a vertex b_3 with

$$\begin{aligned} d(b_3, b_1) &= 1 \\ d(b_3, a) &= 3 \text{ otherwise} \end{aligned}$$

View the resulting configuration as an amalgamation in which the distances between b_1 and a_i for $i \geq 3$ are to be determined. The point a_2 ensures that these distances are at most 2, and the point b_3 ensures that they are at least 2. So it suffices to show that the factors of this amalgamation embed isometrically into Γ.

The factor $(a_2 b_1 b_3)$ is a geodesic triangle.

The factor $\{a_i \mid i \geq 2\} \cup \{b_3\}$ is an anticlique $I^{(3)}_n$.

This concludes the analysis.

□

Proof of Lemma 3.3. We dealt with Γ_2 in Lemma 3.6.

And we have shown so far that Γ_3 has diameter 3, $K_1 = 1$, $K_2 = 3$, $C = 10$, so by the classification in diameter 3 we have

$$\Gamma_3 \cong \Gamma^3_{1,3,10,\tilde{\mathcal{S}}}$$

for some set of $(1, 3)$-spaces $\tilde{\mathcal{S}}$. It remains to be proved that

$$\tilde{\mathcal{S}} = \emptyset$$

In other words, we must embed an arbitrary $(1, 3)$-space A isometrically into Γ.

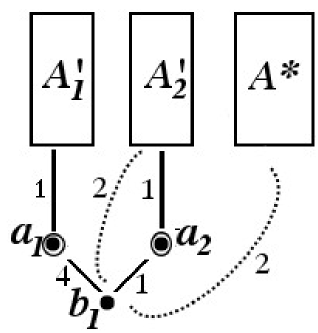
Now the connected components of a $(1, 3)$-space are cliques. We will proceed by induction on the sum of the orders of the nontrivial components of A. The case in which there is at most one nontrivial component in A was treated in Lemma 3.8, so we suppose that there are at least two such.

Let A_1, A_2 be two nontrivial components of A. Fix $a_1 \in A_1$ and $a_2 \in A_2$. We may suppose that A has at least 3 components, as we may add trivial components without altering our inductive parameters.

Adjoin a vertex b_1 with

$$\begin{aligned} d(b_1, a_1) &= 4 \\ d(b_1, a) &= 3 \ (a \in A_1 \setminus \{a_1\}) \\ d(b_1, a_2) &= 1 \\ d(b_1, a) &= 2 \ (a \notin A_1 \cup \{a_2\}) \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_2)$ to be determined.



As we assume A has at least 3 components, and $C = 10$, we have the upper bound $d(a_1, a_2) \leq 3$. The point b_1 then forces $d(a_1, a_2) = 3$. So it suffices to show that the factors of this amalgamation embed isometrically into Γ.

The factor $(A \setminus \{a_1\}, b_1)$*:*

As we omit a_1, all distances are among $1, 2, 3$. It will suffice to check that this configuration is Γ_3-constrained, as it then embeds isometrically into Γ_3 and hence into Γ.

Write $A_1' = A_1 \setminus \{a_1\}$, $A' = A \setminus \{a_1\}$. The maximal $(1, 3)$-subspaces of this configuration are

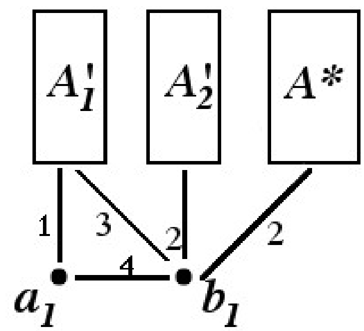
$$A_1' \cup \{a_2, b_1\} \text{ and } A'$$

Let B' be the extension of A' by one more trivial component. Then our induction hypothesis applies to B', so B' embeds isometrically into Γ. This means that A' embeds isometrically into Γ_3.

If $|A_1| > 2$ then induction also applies to $A'_1 \cup \{a_2, b_1\}$ and thus we conclude that $(A'b_1)$ is Γ_3-constrained, as required.

Otherwise, both A_1 and A_2 have order 2. Then $A'_1 \cup \{a_2, b_1\}$ has only one nontrivial component.

The factor $(A \setminus \{a_2\}, b_1)$*:*



Here $A'_1 = A_1 \setminus \{a_1\}$, $A'_2 = A_2 \setminus \{a_2\}$, $A^* = A \setminus (A_1 \cup A_2)$.

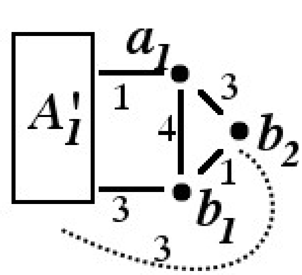
Adjoin a point b_2 with

$$d(b_2, b_1) = 1$$
$$d(b_2, a) = 3 \ (a \in A \setminus \{a_2\})$$

View the resulting configuration as an amalgamation problem with the distances between b_1 and $A \setminus A_1$ to be determined. As $C = 10$, the point a_1 ensures that these distances are at most 2, while the point b_2 ensures that they are at least 2. So it suffices to show that the factors $(A_1 b_1 b_2)$ and $(A_1 A'_2 A^* b_2)$ embed isometrically into Γ.

The factor $(A_1 A'_2 A^* b_2)$ is a $(1,3)$-space such that the sum of the orders of the nontrivial components is smaller than the sum for A. So this embeds isometrically in Γ by induction.

This leaves the factor $(A_1 b_1 b_2)$ for consideration.



We adjoin a point b_3 with

$$d(b_3, a_1) = 1$$
$$d(b_3, x) = 2 \text{ otherwise}$$

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, b_2)$ to be determined. The points b_1, b_3 force this distance to be 3. So it suffices to show that the factors embed isometrically into Γ.

The factor omitting a_1 has no distance greater than 3, so it suffices to check that it is Γ_3-constrained. The maximal $(1,3)$-subspaces are $(A_1 b_2)$, $(A_1' b_1 b_2)$, and $b_1 b_3$. Induction applies to these spaces.

In the factor omitting b_2, we have $(A_1 b_1)$ contained in $\Gamma_2(b_3)$. Since Γ_2 satisfies the same conditions as Γ, it suffices to check that $(A_1 b_2)$ embeds in Γ. Again, this follows by induction.

This completes the proof. □

3.2. **Proof of Proposition 3.1.** Our goal is to prove that any finite space which embeds into $\Gamma^4_{1,4,10,11}$ embeds into every infinite primitive metrically homogeneous graph of diameter 4 with $C = 10$.

We first make some reductions.

Lemma 3.9. *Suppose that there is an infinite primitive metrically homogeneous graph Γ of diameter 4 with $C = 10$, and a finite metric subspace A of $\Gamma^4_{1,4,10,11}$, such that A does not embed isometrically into Γ. Let A be chosen to minimize the number of pairs u, v with $d(u,v) = 4$. Then A contains a unique pair at distance 4.*

Proof. If the distance 4 does not occur then by Lemma 3.3, A embeds isometrically into Γ_3, and hence into Γ.

So it remains to reduce all configurations involving at least two such pairs to configurations involving fewer such pairs.

We note that the amalgamation strategy for the class $\mathcal{A}^4_{1,4,10,11}$ given in [Che13] never introduces new pairs at distance 4 (in the notation of that article, 2-point amalgamation problems are completed using either r^- or K_1).

Claim 1. *For each $u \in A$ there is at most one $v \in A$ with $d(u,v)=4$.*

Suppose we have v_1, v_2 in A distinct with $d(u, v_1) = d(u, v_2) = 4$. Then $d(v_1, v_2) = 1$.

Adjoin a vertex b with

$$d(b, v_1) = 1$$
$$d(b, v_2) = 2$$

Complete the configuration Ab to a Γ-constrained metric space without introducing additional pairs at distance 4.

Now view Ab as a 2-point amalgamation problem with the distance $d(v_1, v_2)$ to be determined. The vertex b prevents these vertices from being identified, and then the vertex u forces their distance to be 1. Each factor of Ab has fewer pairs at distance 4.

This proves the claim.

Claim 2. *There are no pairs* (u_1, v_1), (u_2, v_2) *at distance* 4 *with*

$$d(u_1, u_2) = 2$$

Supposing the contrary, we adjoin points b_1, b_2 with

$$\begin{aligned} d(b_1, u_1) &= 1 & \qquad d(b_2, u_1) &= 1 \\ d(b_1, u_2) &= 1 & \qquad d(b_2, u_2) &= 3 \end{aligned}$$

Complete to a Γ-constrained configuration Ab_1b_2 introducing no additional pairs at distance 4. View this as a 2-point amalgamation problem with the distance $d(u_1, u_2)$ to be determined. The points b_1, b_2 ensure $d(u_1, u_2) = 2$.

The factors of this amalgamation have fewer pairs at distance 4.

Claim 3. *There are no pairs* (u_1, v_1), (u_2, v_2) *at distance* 4 *with* $d(u_1, u_2) = 3$.

Supposing the contrary, adjoin a vertex b with

$$\begin{aligned} d(b, u_1) &= 1 \\ d(b, u_2) &= 2 \end{aligned}$$

Complete Ab toa Γ-constrained configuration with no additional pairs at distance 4 and view the result as a 2-point amalgamation problem determining $d(u_1, u_2)$, with b and the v_i ensuring $d(u_1, u_2) = 3$. The factors involve fewer pairs at distance 4.

Now to conclude, if there are two distinct pairs (u_1, v_1) and (u_2, v_2) at distance 4, then they are disjoint, and $d(u_1, u_2) = d(u_1, v_2) = 1$, a contradiction.

Lemma 3.10. *Suppose that there is an infinite primitive metrically homogeneous graph* Γ *of diameter* 4 *with* $C = 10$, *and a finite metric subspace* A *of* $\Gamma^4_{1,4,10,11}$, *such that* A *does not embed isometrically into* Γ. *Then* A *may be chosen as follows.*

- *A contains a unique pair (u, v) with $d(u, v) = 4$;*
- *For every $x \neq u, v$ we have $d(u, x) = 1$, $d(v, x) = 3$.*

Proof. By Lemma 3.9 we may suppose that A contains a unique pair (u, v) with $d(u, v) = 4$. Let us take such an A so as to minimize the number of vertices x for which $d(u, x) = 2$ or $d(v, x) = 2$.

Claim 1. *For* $x \neq u, v$ *we have* $d(u, x) = 1$, $d(v, x) = 3$, *or vice versa.*

Suppose first that $d(u,w)=2$ for some $w\neq u,v$. Then we adjoin points b_1,b_2 with

$$\begin{aligned} d(b_1,u)&=1 & d(b_2,u)&=1\\ d(b_1,w)&=1 & d(b_2,w)&=3\\ d(b_1,v)&=3 & d(b_2,v)&=3 \end{aligned}$$

We complete the configuration Ab_1b_2 to a Γ-constrained configuration without adding any more pairs at distance 4. We view the resulting configuration as a 2-point amalgamation problem with the distance $d(u,w)$ to be determined, with the value $d(u,w)=2$ forced by the points b_1,b_2. The factor of this amalgamation omitting u has no pairs at distance 4 and the factor omitting w has fewer vertices x violating the conditions $d(u,x)=1$, $d(v,x)=3$. So both factors embed isometrically in Γ and then so does A, for a contradiction.

Similarly $d(v,x)\neq 2$ for $x\neq u,v$.

Thus $d(u,x)$ and $d(v,x)$ must be 1 or 3 for $x\neq u,v$, and the claim follows, recalling $C=10$.

Now write $A=\{u,v\}\cup A_1\cup A_3$ where $d(u,x)=i$ for $x\in A_i$ ($i=1$ or 3). Take A so as to minimize $|A_3|$.

Claim 2. $A_3=\emptyset$

Suppose A_3 is nonempty. We have $|A_3|\leq|A_1|$ since otherwise by a simple change of notation we can reduce $|A_3|$. In particular $A_1\neq\emptyset$. Fix $w_1\in A_1$ and $w_3\in A_3$.

Note that $2\leq d(w_1,w_3)\leq 3$.

Adjoin a point b with

$$\begin{aligned} d(b,u)&=1\\ d(b,v)&=3\\ d(b,w_1)&=2\\ d(b,w_3)&=2 \end{aligned}$$

□

To check that this is a metric space it suffices to inspect the triples the point b together and involving either a pair at distance 4 (i.e., (u,v,b)) or two pairs at distance 1 (i.e., (u,w_1,b)).

Now complete the configuration Ab to a Γ-constrained configuration involving no additional pairs at distance 4, and view the result as a 2-point amalgamation problem with the distance $d(u,w_3)$ to be determined. The points v and b force this distance to be 3. The factor omitting u contains no pair at distance 4, and the factor omitting w_3 has smaller $|A_3|$. So we conclude.

□

Proof of Proposition 3.1. We have Γ with the parameters of $\Gamma^4_{1,4,10,11}$ and we claim that any Γ-constrained finite metric space A embeds into Γ. By Lemma 3.10, it suffices to treat the case in which

- A contains a unique pair (u, v) with $d(u, v) = 4$;
- For every $x \neq u, v$ we have $d(u, x) = 1$, $d(v, x) = 3$.

Let $A^* = A \setminus \{u, v\}$. This is a metric space with distances among $1, 2$.

Taking v as a base point in Γ, fix $u \in \Gamma_4$. Then $\Gamma_1(u)$ is a random graph contained in $\Gamma_3 \cup \Gamma_4$, while $\Gamma_1(u) \cap \Gamma_4$ is a clique. It follows that every finite graph embeds in $\Gamma_1(u) \cap \Gamma_3$. So embedding A^* isometrically into $\Gamma_1(u) \cap \Gamma_3$, we arrive at the required isometric embedding of A. □

4. Embedding Lemmas; Γ_3

We have dealt with the cases $C_0 = 10$ or $K_1 = 4$ in the previous section. In this section we begin the treatment of all remaining cases. While we have not managed to avoid further consideration of the precise values of the parameters (particularly K_1), certain uniformities begin to appear. We will see that a natural focus of attention is the structure of Γ_3.

For the treatment of all other cases, the following point will be fundamental.

Fact 4.1 ([Che13])**.** *Let Γ be a primitive metrically homogeneous graph of generic type, and of known type, whose associated parameters satisfy the following conditions.*

$$C > 2\delta + 2$$
$$K_1 < \delta$$

Then any associated amalgamation problem can be completed without introducing new pairs at distance 1 or δ.

This follows the proof of amalgamation given in [Che13] as Part I of the Main Theorem. Note that two slightly different amalgamation procedures were described there, one which sometimes uses the value K_1 (which may be equal to 1) in the absence of Henson constraints, and a more refined version which varies at one point to avoid that extreme case.

We take note of the following consequence.

Lemma 4.2. *Let Γ be a metrically homogeneous graph of diameter 4 which is not of known type, and $\mathcal{A}$ the class of finite Γ-constrained metric spaces. Then any amalgamation diagram in $\mathcal{A}$ can be completed without adding new pairs at distance 1 or 4.*

Proof. We know that the $\mathcal{A}$ coincides with $\mathcal{A}^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$ for some admissible set of parameters, so that this is an amalgamation class covered by the procedure given in [Che13]. As we have already identified the metrically homogeneous graphs of diameter 4 with $C = 2\delta + 2$ or $K_1 = \delta$, Fact 4.1 applies. □

More generally, in a systematic approach to identification, the first three steps will be the following.

- Admissibility of parameters;
- Determination of forbidden triangles;
- Treatment of the cases $C = 2\delta + 2$ or $K_1 = \delta$;
- Further analysis based on Fact 4.1.

We give the next part of the analysis in a general setting.

4.1. **The Embedding Principle: Reductions.** The Embedding Principle for Γ states that any finite Γ-constrained metric space embeds isometrically in Γ. This is equivalent to the conjecture that Γ is isomorphic to the graph of known type with the same parameters.

Now we work toward an analysis of the structure of a suitably minimized counterexample to the Embedding Principle.

Definition 4.3. *Let Γ be a metrically homogeneous graph of diameter δ and generic type, and A a finite Γ-constrained metric space. We denote by $A(1,\delta)$ the graph on the vertex set A with edge relation*

$$d(x,y) \in \{1,\delta\}$$

In this context we speak of the $(1,\delta)$-connected components of A and adapt other graph theoretic terminology similarly.

Lemma 4.4. *Let Γ be a metrically homogeneous graph of diameter $\delta \geq 4$ with admissible parameters $(K_1, K_2, C, C', \mathcal{S})$, and suppose that Γ realizes the same triangles as the corresponding space $\Gamma^\delta_{K_1,K_2,C,C',\mathcal{S}}$. Suppose that $C \geq 2\delta + 3$ and that $K_1 < \delta$. Suppose that there is a finite Γ-constrained metric space A which does not embed isometrically into Γ. let $A' = A'(1,\delta)$ be the subgraph of $A(1,\delta)$ induced on vertices of degree at least 2. Then*

- *If $|A'|$ is minimized, then $|A'|$ is a $(1,\delta)$-space (equivalently, a $(1, delta)$-clique).*
- *If the minimal A' is nonempty and the number of nontrivial connected components of $A(1,\delta)$ is also minimized (subject to the preceding), then $A(1,\delta)$ has a unique nontrivial connected component.*

Proof. This is a matter of checking that when the desired conditions are not met, the configuration can be reduced to an amalgam of simpler Γ-constrained configurations with a unique solution.

We suppose first that $|A'|$ is minimized.

Claim 1. *A' is a $(1,\delta)$-space.*

Suppose on the contrary $u, v \in A'$ and $d(u,v) = k \neq 1,\ \delta$. We adjoin the usual witnesses b_1, b_2 with

$$d(b_1,u) = 1 \qquad d(b_2,u) = 1$$
$$d(b_1,v) = k-1 \qquad d(b_2,v) = k+1$$
$$d(b_1,b_2) = 2$$

To complete the configuration Ab_1b_2 we amalgamate A and uvb_1b_2 to get a Γ-constrained configuration, without introducing any new pairs at distance 1 or δ.

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(u,v)$ to be determined. The witnesses ensure $d(u,v)=k$, and it suffices to embed the factors F of this amalgamation isometrically into Γ. As these factors omit u or v, it and as neither b_1 nor b_2 will be in the corresponding subset F', this decreases the parameter $|A'|$ and we may conclude by minimality.

Now with $|A'|$ minimized, and supposing A' is nonempty, we take the number of nontrivial connected $(1,\delta)$-components of A to be minimized.

Claim 2. *A contains a unique nontrivial $(1,\delta)$-component.*

The $(1,\delta)$-space A' is contained in a unique nontrivial $(1,\delta)$-component A_1 of A.

Suppose that there is another nontrivial $(1,\delta)$-component A_2. Then A_2 consists of a pairs of points at distance 1 or δ.

Fix $u\in A_1$ and $v\in A_2$, let $k=d(u,v)$, and as usual adjoin witnessing points b_1,b_2 with

$$\begin{aligned} d(b_1,u)&=1 & d(b_2,u)&=1\\ d(b_1,v)&=k-1 & d(b_2,v)&=k+1\\ d(b_1,b_2)&=2 \end{aligned}$$

forcing $d(u,v)=k$. Complete to a Γ-constrained configuration without adjoining additional pairs at distance 1 or δ. View the resulting configuration as a 2-point amalgamation problem with the distance $d(u,v)$ to be determined.

The factor omitting v has fewer $(1,\delta)$-connected components, and the same value of $|A'|$, so embeds isometrically in Γ by assumption.

In the factor omitting u, we either have a smaller value of $|A'|$, or we have the same value, with the point v replacing the point u. In the first case the factor embeds isometrically in Γ by assumption.

In the second case, we have a factor F with associated subset $F'\subseteq(A'\setminus\{u\})\cup\{v\}$ and therefore if $|F'|=|A'|$ we have

$$F'=(A'\setminus\{u\})\cup\{v\}$$

But if $|F'|=|A'|$ is minimal, then F' is a $(1,\delta)$-space. As $v\in F'$ this forces

$$F'=\{v\}$$

Thus $A'=\{u\}$. It follows that the $(1,\delta)$-connected component of A containing u is a star and that F has fewer nontrivial $(1,\delta)$-connected components than A. Thus we find again that F embeds isometrically into Γ. □

Lemma 4.5. *Let Γ be a metrically homogeneous graph of diameter $\delta\geq 4$ with admissible parameters $(K_1,K_2,C,C',\mathcal{S})$ satisfying*

$$C\geq 2\delta+3 \text{ and } K_1<\delta$$

Suppose that Γ *realizes the same triangles as the corresponding graph*

$$\Gamma^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$$

Suppose that there is a finite Γ*-constrained metric space* A *which does not embed isometrically into* Γ*. Let* A *be taken with* $|A'|$ *is minimized, and suppose that* A' *is nonempty. Denote by* A'' *be the nontrivial* $(1,\delta)$*-connected component of* A*. Suppose further that the number of vertices* $v \in A'$ *joined to a point of* $A'' \setminus A'$ *is minimized. Then this number is at most* 1.

Proof. For the present, we use the term "adjacent" in the sense of "$(1,\delta)$-adjacent," that is, at distance 1 or δ.

Fix a point $u \in A'$ and take A to minimize the number of vertices of A'' not adjacent to u. If all vertices of A'' are adjacent to u, we are done.

Suppose therefore that $v \in A''$ is not adjacent to u. Then $v \in A'' \setminus A'$. Let $k = d(u,v)$. Adjoin witnesses b_1, b_2 to the relation $d(u,v) = k$ as usual, with $d(b_1,u) = d(b_2,u) = 1$, extending to a Γ-constrained configuration without adding pairs at distance 1 or δ.

View the result as a 2-point amalgamation problem with $d(u,v)$ to be determined. It suffices to show that the factors embed isometrically in Γ.

The factor omitting v reduces the number of vertices non-adjacent to u.

It remains to consider the factor F omitting u. Then $F' \subseteq (A' \setminus \{u\}) \cup \{v\}$. If $|F'| < |A'|$ we conclude by minimality so we may suppose $F' = (A' \setminus \{u\}) \cup \{v\}$. But then $|F'| = |A'|$ is minimal so if F does not embed isometrically into Γ, it follows that F' is a $(1,\delta)$-space containing v. However v has at most one neighbor in F'. Thus $|A'| = |F'| = 2$.

Let $A' = \{u, v'\}$ where v' must be adjacent to v. If u has m neighbors in $A'' \setminus A'$ and v has n neighbors in $A'' \setminus A'$, then in the factor F of Ab_1b_2 omitting u, $F' = \{v, v'\}$ where v' has $n-1$ neighbors in $F'' \setminus F'$ and v has only one. Taking (F, v) in place of (A, u) we reduce the number of points in $F'' \setminus F'$ not adjacent to v to $n-1$, and conclude by minimality. □

Lemma 4.6. *Let* Γ *be a metrically homogeneous graph of diameter* $\delta \geq 4$ *with admissible parameters* $(K_1, K_2, C, C', \mathcal{S})$ *satisfying*

$$C \geq 2\delta + 3 \text{ and } K_1 < \delta$$

Suppose that Γ *realizes the same triangles as the corresponding graph*

$$\Gamma^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$$

Suppose that there is a finite Γ*-constrained metric space* A *which does not embed isometrically into* Γ*, and for which* $|A'| \leq 1$*. If we take such an* A *which minimizes the number of nontrivial* $(1,\delta)$*-components, that that number is* 1.

Proof. Suppose there are at least two nontrivial $(1,\delta)$-components, A_1, A_2. Take $u \in A_1$, $v \in A_2$, set $k = d(u,v)$, and add the usual witnesses b_1, b_2 with

$$\begin{aligned} d(b_1,u) &= 1 & d(b_2,u) &= 1 \\ d(b_1,v) &= k-1 & d(b_2,v) &= k+1 \\ d(b_1,b_2) &= 2 \end{aligned}$$

Furthermore, when A' is nonempty, take $u \in A'$. Extend to a Γ-constrained configuration without adding additional pairs at distance 1 or δ.

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(u,v)$ to be determined. We claim that the factors F of this amalgamation embed isometrically in Γ, by minimality: that is, we have $|F'| \le 1$ and the number of nontrivial components is reduced.

The only noteworthy point is that in case $A' = \emptyset$ we may have $|F'| = 1$. But this does not affect the argument.

□

We summarize the discussion as follows.

Lemma 4.7. *Let Γ be a metrically homogeneous graph of diameter $\delta \ge 4$ with admissible parameters $(K_1, K_2, C, C', \mathcal{S})$ satisfying*

$$C \ge 2\delta + 3 \text{ and } K_1 < \delta$$

Suppose that Γ realizes the same triangles as the corresponding graph

$$\Gamma^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$$

Suppose that there is a finite Γ-constrained metric space A which does not embed isometrically into Γ. Then A may be taken to have a unique nontrivial $(1,\delta)$-connected component A_0, satisfying one of the following.

- *$|A'| \le 1$; so A_0 is a single pair, or a star with center $u \in A'$; or*
- *$|A'| \ge 2$ and there is a point of A' which is $(1,\delta)$-adjacent to all points of A_0.*

In the second case, we may also suppose that any Γ-constrained configuration B with $|B'| < |A'|$ embeds isometrically into Γ.

Proof. We first take A to minimize A'.

If $|A'| \le 1$ we may apply Lemma 4.6 to replace A by a configuration of the first kind. (Here we may possibly pass from a case with A' empty to a case with $|A'| = 1$.)

If $|A'| \ge 2$ then we apply Lemma 4.5 to ensure that there is at most one point of A' joined to a point of $A_0 \setminus A'$. Every point of $A_0 \setminus A'$ has a unique $(1,\delta)$-neighbor, and that neighbor is in A', since otherwise there would be a second nontrivial $(1,\delta)$-connected component, So u is $(1,\delta)$-adjacent to every point of A_0 (and is the unique such point if $A_0 \ne A'$).

□

Lemma 4.8. *Let Γ be a metrically homogeneous graph of diameter $\delta \geq 4$ with admissible parameters $(K_1, K_2, C, C', \mathcal{S})$ satisfying*

$$C \geq 2\delta + 3 \text{ and } K_1 < \delta$$

Suppose that Γ realizes the same triangles as the corresponding graph

$$\Gamma^{\delta}_{K_1,K_2,C,C',\mathcal{S}}$$

Suppose that there is a Γ-constrained configuration A which does not embed isometrically into Γ, and which satisfies the following conditions.

- *$|A'|$ is minimal for all such configurations.*
- *There is a point u of A' which is $(1, \delta)$-adjacent to all points in the $(1, \delta)$-connected component A_0 containing A'; in particular, A' is nonempty.*

If A is chosen to minimize $|A \setminus A_0|$, then $A = A_0$.

Proof. Otherwise take $v \in A \setminus A_0$ and set $k = d(u, v)$. Adjoin points b_1, b_2 with

$$\begin{aligned} d(b_1, u) &= 1 & d(b_2, u) &= 1 \\ d(b_1, v) &= k - 1 & d(b_2, v) &= k + 1 \\ d(b_1, b_2) &= 2 \end{aligned}$$

Extend to a Γ-constrained configuration with no new pairs at distance 1 or δ. It suffices to show that the factors embed isometrically in Γ.

The factor F_1 omitting u has $|F_1'| < |A_1'|$, so embeds by hypothesis.

The factor F_2 omitting v has fewer points outside the nontrivial $(1, \delta)$-connected component. □

One can reduce these configurations further, but it seems one will need to consider the various subcases before long, in particular the cases $K_1 > 1$, $K_2 < \delta$, $C \leq 3\delta$.

4.2. **Direct Sums.** Now we return to the case of diameter 4. Our first major goal is the following.

Lemma 4.9 ((2, 3)-Embedding Principle)**.** *Let Γ be a primitive metrically homogeneous graph of diameter 4 and generic type with $C > 10$ and $K_1 < 4$. Then any Γ-constrained finite $(2, 3)$-space embeds isometrically into Γ.*

Of course, the excluded cases in this statement were covered previously.

The following operation is very useful, and will occupy us for some time.

Definition 4.10. *In the category of metric spaces of diameter δ, for $r \geq \delta/2$ the r-direct sum of two metric spaces A, B, denoted $A \perp^{(r)} B$, is the disjoint union of A and B with $d(a, b) = r$ for $a \in A$, $b \in B$.*

As we work with integer valued metric spaces the "default" value of r is $\lfloor \delta/2 \rfloor$: we write $A \perp B$ in this case.

Lemma 4.11. *Let Γ be a metrically homogeneous graph of diameter δ, of generic type and of known type. Suppose $r \geq \delta/2$ is an integer. Then the following are equivalent.*

- *Γ is closed under r-direct sum*
- *$K_1 \leq \delta/2 \leq K_2$ and $C > 2r + \delta$*

In particular for $r = \lfloor \delta/2 \rfloor$ this reduces to

- *$K_1 \leq \delta/2$*

Proof. Clearly closure under r-direct sum is equivalent to the condition

All triangles of type (r, r, k) embed isometrically into Γ

With $k = 1$ this gives $K_1 \leq r \leq K_2$, and with $k = \delta - 1$ and δ this gives $C > 2r + \delta$.

Now we suppose $K_1 \leq r \leq K_2$, $C > 2r + \delta$, and we check that no triangle of type (r, r, k) is forbidden.

The following three conditions on the perimeter $p = 2r + k$ suffice.

- $2r + k \geq 2K_1 + 1$—true by hypothesis;
- $2r + k \leq 2K_2 + 2\min(r, k)$—since $K_2 \geq r \geq \delta/2$;
- $2r + k < C$: by hypothesis.

For the final point, we always have $K_2 \geq \delta/2$ and $C > 2\delta$, by admissibility. □

We apply this lemma to metrically homogeneous graphs having the same triangles as a known metrically homogeneous graph of generic type, in which case the conclusion is that the Γ-constrained graphs are closed under direct sum if and only if the parameter K_1 is at most $\delta/2$.

Since $K_2 \geq 3$ when $\delta = 4$, we have the following conclusions in diameter 4:

- If $K_1 \leq 2$ then the Γ-constrained graphs are closed under the direct sum with distance 2;
- If $K_1 \leq 3$ and $C > 10$ then the the Γ-constrained graphs are closed under the direct sum with distance 3.

Of course we are already assuming $K_1 \leq 3$ and $C > 10$ so we will have closure under $\perp^{(3)}$ in all remaining cases of interest. However this relates only to Γ-constrained configurations, so we need to turn this analysis into something more concrete.

We begin with direct sum at distance 3.

Lemma 4.12. *Let $\delta, K_1, K_2, C, C', \mathcal{S}$ be an admissible choice of parameters with $\delta = 4$, $K_1 \leq 3$, $C > 10$. Let A and B be finite metric spaces which embed isometrically into every metrically homogenous graph with the given parameters. Then $A \perp^{(3)} B$ embeds into every metrically homogeneous graph with the given parameters.*

This easily reduces to the "base case" in which the metric spaces A and B are the "generators" for the given class of metrically homogeneous graphs. Namely, let $G(K_1, K_2, C, C')$ be the following set of triangle types.

- $(K_1, K_1, 1)$, $(K_2, K_2, 1)$
- $(4, 4, C-10)$ and $(4, 4, C'-10)$

Let $G(K_1, K_2, C, C', \mathcal{S})$ be the union of $G(K_1, K_2, C, C')$ together with all $(1,4)$-spaces which are not forbidden by $\mathcal{S}$.

The base case of the lemma is then as follows.

Lemma 4.13. *Let $\delta, K_1, K_2, C, C', \mathcal{S}$ be an admissible choice of parameters with $\delta = 4$, $K_1 \leq 3$, $C > 10$. Let A and B be finite metric spaces in $G^*(K_1, K_2, C, C', \mathcal{S})$. Then $A \perp^{(3)} B$ embeds into every metrically homogeneous graph with the given parameters.*

We must work toward this gradually.

4.3. **An Inductive Framework.**

Definition 4.14. *Let Γ be a metrically homogeneous graph of finite diameter δ.*

1. $\mathcal{E}(\Gamma)$ be the family of all triangles and finite $(1,\delta)$-spaces which embed isometrically in Γ.

2. $\Gamma \preceq \Gamma'$ if $\mathcal{E}(\Gamma) \subseteq \mathcal{E}(\Gamma')$.

The relation $\preceq$ is a quasiorder on the set of metrically homogeneous graphs of specified diameter.

Lemma 4.15. *For fixed δ, the relation $\prec$ is a well quasiorder (wqo)—there are no strictly descending chains, and any infinite collection of such graphs Γ contains a comparable pair.*

Proof. Associate to each such graph Γ the a set $M(\Gamma)$ of representatives for the minimal forbidden triangles and $(1,\delta)$-spaces for Γ. Then the quasiorder $\preceq$ is equivalent to the following quasiorder on these sets:

$$M_1 \prec M_2 \text{ iff every element of } M_2 \text{ contains an isometric copy of some element of } M_1$$

We encode each $(1,\delta)$ space by a sequence of integers representing the sizes of the components. If two such sequences σ, σ' are comparable in the sense that there is a subsequence of σ' of the same length as σ, whose terms dominate the corresponding terms of σ, then there is an embedding of the corresponding structures. By Higman's Lemma, it follows that the set of possible constraints ($(1,\delta)$-spaces and triangles) is well quasiordered under the isometric embedding relation. In particular the sets $M(\Gamma)$ are always fiinite.

By another application of Higman's Lemma, if we view the sets $M(\Gamma)$ as sequences and strengthen the relation $\prec$ correspondingly, the sets $M(\Gamma)$ are wqo. This applies a fortiori to the definition as gi9ven. □

The point is that we argue by induction over the order $\prec$. Since any nonempty set of metrically homogeneous graphs of fixed diameter has minimal elements with respect to this quasiorder, we have the following.

Lemma 4.16. *Suppose there is a metrically homogeneous graph Γ of diameter δ which is not of known type. Then there is such a graph with the property that whenever Γ' is another metrically homogeneous graph with $\mathcal{E}(\Gamma')$ strictly conteined in $\mathcal{E}(\Gamma)$, then Γ' is of known type.*

We apply this in particular to the graphs Γ_i when Γ_i contains an edge. Either Γ_i is again a graph with the same parameters as Γ, or Γ_i is a known graph.

Definition 4.17. *A metrically homogeneous graph Γ of diameter δ is of* K^*-type *if*

- Γ *is of generic type.*
- *Any metrically homogeneous Γ' strictly below Γ in the quasiorder $\preceq$ is of known type.*

In order to show that every metrically homogeneous graph is of known type, it suffices to prove that metrically homogeneous graphs of K^*-type are of known type.

4.4. **The Structure of Γ_3.** The base case for a treatment of 3-direct sums is the analysis of Γ_3. We undertake that here.

Let Γ be a primitive metrically homogeneous graph of diameter 4 of generic type. For $K_1 \leq i \leq K_2$, Γ_i contains an edge and is therefore a primitive metrically homogeneous graph by Lemma 1.5.

Lemma 4.18. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$C > 10$$

Then every point in Γ_3 has a pair of neighbors v_1, v_2 in Γ_4 with $d(v_1, v_2) = 2$.

Proof. Let u be a point of Γ_3.

We show first that u has at least two neighbors in Γ_4. Otherwise, we define a function $u \mapsto u'$ from Γ_3 to Γ_4 by $d(u, u') = 1$. If $v \in \Gamma_4$, then v has neighbors u_1, u_2 at distance 2 in Γ_3 by Lemma 1.1. Then $u_1' = u_2' = v$ and as Γ_3 is connected with respect to the relation $d(x, y) = 2$, it follows that $u' \in \Gamma_4$ is independent of u. Then $|\Gamma_4| = 1$, a contradiction.

So if the lemma fails, then for $u \in \Gamma_3$ the set $I_u = \{v \in \Gamma_4 \mid d(u, v) = 1\}$ is a nontrivial complete graph. Hence for v_1, v_2 in Γ_4 adjacent, there is some $u \in \Gamma_3$ adjacent to both.

Since $C > 10$ we may take $v_1, v_2 \in \Gamma_4$ at distance 2, and then v at distance 1 from both. If $v \in \Gamma_3$ then our claim follows, so suppose $v \in \Gamma_4$. We may fix $u_1, u_2 \in \Gamma_3$ with u_i adjacent to v, v_i for $i = 1, 2$. Then $u_1 \neq u_2$ and $d(u_1, u_2) \leq 2$.

By Lemma 1.1 applied to vertices in Γ_2 and Γ_4, there is $u \in \Gamma_2$ adjacent to u_1, u_2. Then $d(u, v_1) = d(u, v_2) = 2$. Since u, v_1, v_2 are at mutual distance 2, there is a point w adjacent to all three. This forces $w \in \Gamma_3$ and then $v_1, v_2 \in I_w$, a contradiction. □

Lemma 4.19. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \leq 3 \text{ and } C > 10$$

Then Γ_3 is a primitive metrically homogeneous graph with the same parameters.

Proof. As $C > 10$, Γ_3 has diameter 4. As $K_1 \leq 3$, Γ_3 contains an edge, thus is a primitive metrically homogeneous graph. Also Γ_3 is infinite and thus is of generic type by our previous analysis.

Let us write as usual $\tilde{K}_1$, $\tilde{K}_2$, $\tilde{C}$, $\tilde{C}'$, $\tilde{\mathcal{S}}$ for the parameters associated with Γ_3.

Claim 1. *If Γ contains a clique of order n, then Γ_3 contains a clique of order n.*

First, if Γ contains a clique of order $n+1$, then Γ_1 contains a clique of order n. By Lemma 1.1, applied to a pair u, v at distance 2 in Γ_2 and Γ_4, the graph Γ_3 contains a copy of Γ_1, and hence contains a clique of order n.

So now suppose that Γ contains no clique of order $n+1$. We perform an amalgamation.

Let A be a clique of order $n-1$. For $i = 1, 2, 3$ let Ab_i be a clique of order n with $d(b_i, b_j) = 2$. Let c be a point satisfying

$$\begin{aligned} d(c, a) &= 3 \ (a \in Ab_3) \\ d(c, b_1) &= 2 \\ d(c, b_2) &= 4 \end{aligned}$$

View $Ab_1b_2b_3c$ as an amalgamation problem with the distances between c and A to be determined. The points b_1, b_2 force $d(c, a) = 3$ for $a \in A$. So it suffices to check that the factors $b_1b_2b_3c$ and $Ab_1b_2b_3$ embed in Γ.

The factor $b_1b_2b_3c$:

This consists of a pair of points b_2, c at distance 4 and all other distances equal to 2. So take b_2 as the base point of Γ and c in Γ_4. Take u adjacent to c in Γ_3 and v_1, v_2 adjacent to u in Γ_2. Then v_1, v_2 are at distance 2 from b_2 and c, and at distance at most 2 from each other.

If $d(v_1, v_2) = 2$ we have the desired factor, and otherwise the configuration (u, v_1, v_2) shows that $K_1 = 1$ and thus Γ_2 contains an edge. Therefore Γ_2 is connected of diameter 4. But $b_1b_2b_3c$ can be viewed as a geodesic triangle (b_2b_3c) of type $(2, 2, 4)$ inside $\Gamma_2(b_1)$, so we have the desired embedding in either case.

The factor $Ab_1b_2b_3$:

Take $a \in A$. With a as base point, we must embed $(A \setminus \{a\})b_1b_2b_3$ in Γ_1. But Γ_1 contains a clique of order $n-1$ and $(A \setminus \{a\})b_1b_2b_3$ contains no larger clique, so this embeds in Γ_1.

This proves our first claim. For cliques of order 3 this gives

$$\text{If } K_1 = 1 \text{ then } \tilde{K}_1 = 1$$

Claim 2. *If $K_1 = 2$ then $\tilde{K}_1 = 2$.*

By assumption Γ contains triangles of type $(1, 2, 2)$, but none of type $(1, 1, 1)$.

Our goal is the configuration a_1a_2bc with

$$\begin{aligned} d(a_1, a_2) &= 1 \\ d(a_i, b) &= 2 \\ d(c, x) &= 3 \text{ (all } x) \end{aligned}$$

We adjoin a point c' with

$$\begin{aligned} d(c', a_1) = d(c', b) = 1 \\ d(c', a_2) = d(c', c) = 2 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, b)$ to be determined. The points a_2 and c' ensure first that a_1, b remain distinct, and second that $d(a_1, b) = 2$. So it suffices to embed the factors (a_1a_2cc') and (a_2bcc') isometrically into Γ.

The factor (a_1a_2cc'):

View this as a 2-point amalgamation problem with the distance $d(a_2, c)$ to be determined. The point c' ensures that a_2, c are not identified, and then a_1 ensures that the distance is 2.

The factor (a_2bcc'):

View this as a 2-point amalgamation problem with the distance $d(c, c')$ to be determined. As a_2bc' is a triangle of type $(2, 2, 1)$, if we have $d(c, c') = 3$ in the amalgam then we have a triangle of type $(2, 2, 1)$ in Γ_3 and we are done. The point a_2 ensures that $d(c, c') \geq 2$, so the alternative is $d(c, c') = 2$, in which case we have the desired configuration.

This proves the claim.

Claim 3. *If $K_1 = 3$ then $\tilde{K}_1 = 3$.*

Notice first that as $K_3 \leq \tilde{K}_3$, if $\tilde{K}_1 \neq 3$ then $\tilde{K}_1 = 4$ and hence

$$\Gamma_3 \cong \Gamma^4_{4,4,13,14}$$

so that any configuration with no triangles of odd perimeter less than 9 will embed into Γ_3 and hence into Γ.

We aim via an amalgamation argument at the configuration $(a_1a_2b_1b_2)$ with

$$\begin{aligned} d(a_1,a_2) &= 1 \\ d(x,y) &= 3 \text{ otherwise} \end{aligned}$$

Adjoin a point c_1 with

$$\begin{aligned} d(c,a_1) &= 2 \\ d(c,b_1) &= 1 \\ d(c,a_2) &= 3 \\ d(c,b_2) &= 4 \end{aligned}$$

View this configuration as a 2-point amalgamation problem with the distance $d(a_1,b_1)$ to be determined. The point c_1 provides an upper bound of 3 and also eliminates the possibility $d(a_1,b_1) = 2$ since $K_1 > 2$. Then a_2 provides the lower bound $d(a_1,b_1) \geq 2$ and thus the distance $d(a_1,b_i)$ must be 3. We must show that the factors $(a_1a_2b_2c_1)$ and $(a_2b_1b_2c_1)$ embed isometrically into Γ.

The factor $(a_1a_2b_2c_1)$*:*
Adjoin a point c_2 with

$$\begin{aligned} d(c_2,a_1) &= 2 \\ d(c_2,a_2) &= 1 \\ d(c_2,b_2) &= 3 \\ d(c_2,c_1) &= 4 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2,c_1)$ to be determined. The points a_1,c_2 determine this distance uniquely, so it suffices to show that the factors $(a_1a_2b_2c_2)$ and $(a_1b_2c_1c_2)$ embed isometrically into Γ.

The factor $(a_1a_2b_2c_2)$ has no triangles of small odd perimeter, hence embeds isometrically into Γ_3, hence into Γ.

The factor $(a_1b_1c_1c_2)$ consists of a geodesic of type $(1,1,2)$ in $\Gamma_3(b_2)$, so embeds isometrically into Γ.

The factor $(a_2b_1b_2c_1)$*:*
This is a geodesic $(b_1b_2c_1)$ of type $(1,3,4)$ in $\Gamma_3(a_2)$, so embeds isometrically in Γ.

This proves the claim, and so we may sum up as follows.

$$\tilde{K}_1 = K_1$$

in all cases.

Claim 4. $\tilde{K}_2 = K_2$

Now $3 \leq \tilde{K}_2 \leq K_2$, so we may suppose for the present that $K_2 = 4$ and our claim is that Γ_3 contains a triangle of type $(4, 4, 1)$. If this fails, we have $\tilde{K}_2 = 3$ and then as Γ is of K^*-type we find

$$\Gamma_3 \cong \Gamma^4_{K_1,3,\tilde{C},\tilde{C}',\tilde{S}}$$

We aim at the configuration (a_1, a_2, a_3, b) with

$$d(a_1, a_2) = 1$$
$$d(a_i, a_3) = 4 \ (i = 1, 2)$$
$$d(b, a_i) = 3 \ (i = 1, 2, 3)$$

We extend this configuration by points c_1, c_2 with

$$d(c_1, c_2) = 4$$
$$d(c_1, a_1) = 2 \qquad d(c_2.a_1) = 3$$
$$d(c_1, a_2) = 1 \qquad d(c_2, a_2) = 4$$
$$d(c_1, a_3) = 3 \qquad d(c_2.a_3) = 3$$
$$d(c_1, b) = 2 \qquad d(c_2, b) = 3$$

We view this configuration as a 2-point amalgamation problem with the distance $d(a_2, a_3)$ to be determined. The point a_1 ensures that this distance is 3 or 4, and then either $(a_1a_2a_3b)$ or $(a_2c_1c_2a_3)$ is the desired configuration. So it suffices to show that the factors $(a_1a_2bc_1c_2)$ and $(a_1a_3bc_1c_2)$ embed isometrically into Γ.

The factor $(a_1a_2bc_1c_2)$*:*

We view this as a 2-point amalgamation problem with the distance $d(a_1, c_2)$ to be determined. The point a_2 ensures that this distance is at least 3. If it is exactly 3 we have the desired configuration, and if it is 4 then $(a_1a_2c_2)$ is a triangle of type $(1, 4, 4)$ in $\Gamma_2(b)$ and we have our claim.

So it will suffice to embed the factors $(a_1a_2bc_1)$ and $(a_2b_1c_1c_2)$ isometrically in Γ.

The factor $(a_1a_2bc_1)$ embeds isometrically in Γ_3, hence in Γ. There are no pairs at distance 4 here, and the only triangle of odd perimeter involving distance 1 is of type $(1, 3, 3)$.

This leaves the factor $(a_2bc_1c_2)$ for consideration.

We adjoin a point c_3 with

$$d(c_3, b) = 1$$
$$d(c_3, c_i) = 2 \ (i = 1, 2)$$

We leave $d(c_3, a_2)$ to be chosen below, among the values $2, 3, 4$.

We view the resulting configuration as a 2-point amalgamation problem with distance $d(b, c_1)$ to be determined. The points a_2, c_3 ensure that this distance is 2 or 3. If it is 2 then we have the required configuration and if it is 3 then $(a_2c_1c_2)$ is a triangle of type $(1, 4, 4)$ in $\Gamma_2(b)$.

So it suffices to check that for some choice of $d(c_3, a_2)$ we have both factors of this amalgamation in Γ.

The factor $a_2c_1c_2c_3$ may be viewed as a 2-point amalgamation problem with $d(a_2, c_3)$ to be determined. The factors are triangles which embed isometrically in Γ. In view of the point c_2 the amalgam has $d(a_2, c_3) \geq 2$. We take whatever value results as the value of $d(a_2, c_3)$ in our configuration, so that the structure of the other factor $(a_2bc_2c_3)$ is now fully determined.

We claim that the factor $(a_2bc_2c_3)$ embeds isometrically into Γ_3 and hence into Γ. Let $k = d(a_2, c_3)$. Then the triangle types occurring in this factor are $(1, 2, 3)$, $(3, 3, 4)$, $(k, 2, 4)$, and $(k, 1, 3)$, where $k > 1$, by construction—fortunately, as the value of k was inserted without checking the triangle inequality.

All of the triangle types which may occur here embed into Γ_3 as $K_1 \leq 3 \leq \tilde{K}_2$. This concludes the treatment of the factor $(a_1a_2bc_1c_2)$.

The factor $(a_1a_3bc_1c_2)$*:*

We show this embeds isometrically into Γ_3 and hence into Γ.

As there are no $(1, 4)$-spaces involved other than pairs, it suffices to check that the triangles present embed isometrically. There are only two pairs (a_2, a_3) and (c_1, c_2) at distance 4, so there are no triangles of perimeter greater than 10. As $K_1 \leq 3$ and $\tilde{K}_2 \geq 3$ the only forbidden triangles of odd perimeter less than 10 for Γ_3 are (at worst) types $(1, 1, 1)$, $(1, 2, 2)$, $(1, 4, 4)$. There are no pairs at distance 1 in this factor.

So Claim 4 is proved.

The following claim depends on the hypothesis $C > 10$ (and is clearly false otherwise).

Claim 5. $\tilde{C} > 10$

We will show that a triangle of type $(4, 4, 2)$ embeds isometrically in Γ_3. So we aim at the configuration $(a_1a_2a_3b)$ with the (a_1, a_2, a_3) the specified triangle, and $d(a_1, a_2) = 2$, and $d(b, a_i) = 3$ for $i = 1, 2, 3$.

Assuming the contrary, Γ_3 must be isomorphic to $\Gamma^3_{1,4,10,11}$ by the minimality of Γ.

Adjoin points c_1, c_2 with

$$d(c_i, a_j) = 3 \ (i = 1, 2;\ j = 1, 2)$$
$$d(c_i, a_3) = 1 \ (i = 1, 2)$$
$$d(c_1, b) = 2,\ d(c_2, b) = 4$$
$$d(c_1, c_2) = 2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_3, b)$ to be determined. The points c_1, c_2 ensure that this distance is 3. So it suffices to show that the factors $(a_1a_2a_3c_1c_2)$ and $(a_1a_2bc_1c_2)$ embed isometrically into Γ.

The factor $(a_1a_2a_3c_1c_2)$*:*

Adjoin a point c_3 with

$$d(c_3, a_i) = 2\ (i = 1, 2)$$
$$d(c_3, a_3) = 2$$
$$d(c_3, c_i) = 1\ (i = 1, 2)$$

View this configuration as an amalgamation problem with the distances between a_i and c_j to be determined ($i = 1, 2$; $j = 1, 2$).

The points a_3 and c_3 ensure that all these distances equal 3. So it suffices to check that the factors $(a_1a_2a_3c_3)$ and $(a_3c_1c_2c_3)$ embed isometrically into Γ.

The factor $(a_3c_1c_2c_3)$ is a $(1, 2)$-space and embeds into Γ_3.

For the factor $(a_1a_2a_3c_3)$, take a_3 as basepoint. Then we require $c_3 \in \Gamma_2$ and $a_1, a_2 \in \Gamma_4$ with a_1, a_2, c_3 all at distance 2.

Take $u \in \Gamma_3$ and v_1, v_2, v_3 adjacent to u with $v_1 \in \Gamma_2$ and $v_2, v_3 \in \Gamma_3$, and $d(v_2, v_3) = 2$ (using the previous claim).

Then the configuration (v_1, v_2, v_3) is as required.

The factor $(a_1a_2bc_1c_2)$:

Here the largest distance occurring is 3 and hence this embeds into Γ_3, and therefore into Γ.

This proves the claim.

Claim 6. *If there is a triangle of type $(3, 4, 4)$ or $(4, 4, 4)$ in Γ, then there is a triangle of the same type in Γ_3.*

As always, we suppose this fails, and then by minimality of Γ the metrically homogeneous graph Γ_3 is of known type, with $\tilde{K}_1 = K_1 \leq 3$ and $\tilde{K}_2 = K_2 \geq 3$.

For a time we will consider both cases simultaneously. Let (a_1, a_2, a_3) be a triangle of type $(4, 4, k)$ with $k = 3$ or 4 and $d(a_i, a_3) = 4$ for $i = 1, 2$, $d(a_1, a_2) = k$. We aim at the configuration $(a_1a_2a_3b)$ where $d(a_i, b) = 3$, all i.

Adjoint two points c_1, c_2 with

$$d(c_i, a_j) = 3\ (i = 1, 2;\ j = 1, 2)$$
$$d(c_1, a_3) = 1 \qquad d(c_2, a_3) = 1$$
$$d(c_1, b) = 2 \qquad d(c_2, b) = 4$$
$$d(c_1, c_2) = 2$$

View the resulting configuration as a 2-point amalgamation with the distance $d(a_3, b)$ to be determined. The points c_1, c_2 ensure that $d(a_3, b) = 3$. So it suffices to show that the factors $a_1a_2a_3c_1c_2$ and $a_1a_2bc_1c_2$ embed isometrically in Γ. One checks first that the second factor is Γ_3-constrained and hence embeds isometrically in Γ_3. Thus we may focus our attention on the factor

$$(a_1a_2a_3c_1c_2)$$

Adjoin a point c_3 with

$$d(c_3, a_i) = 2 \ (i = 2, 3)$$
$$d(c_3, c_i) = 1 \ (i = 1, 2)$$

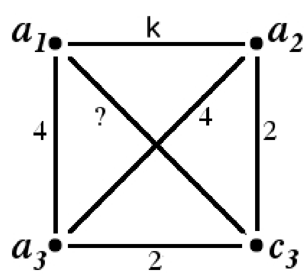
We will chose $d(c_3, a_1)$ later, subject to

$$d(c_3, a_1) = 1\text{or } 2$$

We view this configuration as an amalgamation problem with the distances $d(a_2, c_i)$ to be determined for $i = 1, 2$. The points a_3, c_3 ensure that these distances are equal to 3. So it suffices to show that the two factors $(a_1a_2a_3c_3)$ and $(a_1a_3c_1c_2c_3)$ embed isometrically in Γ.

The factor $(a_1a_3c_1c_2c_3)$ embeds isometrically in Γ_3, whether $d(c_3, a_1)$ is equal to 2 or 3. The only pair at distance 4 in this factor is (a_1, a_3), so all perimeters are bounded by 10, and it suffices to check the triangles of odd perimeter.

So we come down to the factor $(a_1a_2a_3c_3)$, where the distance $d(c_3, a_1)$ still remains to be chosen.



We adjoin a point c_4 with

$$d(c_4, a_1) = 1$$
$$d(c_4, a_2) = k - 1$$
$$d(c_4, a_3) = 3$$
$$d(c_4, a_4) = 2$$

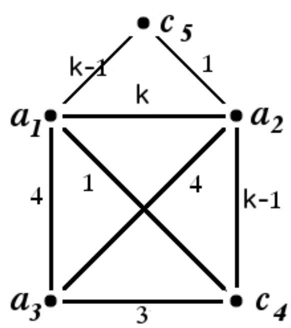
View this configuration as a 2-point amalgamation problem with the distance $d(a_1, c_3)$ to be determined. The points a_3 and c_4 ensure that this distance will be 2 or 3, as required. So it suffices to show that the factors $(a_1a_2a_3c_4)$ and $(a_2a_4c_3c_4)$ embed isometrically into Γ.

The factor $(a_2a_3c_3c_4)$ embeds into Γ_3, so we need only consider the factor

$$(a_1a_2a_3c_4)$$

We adjoin a point c_5 with

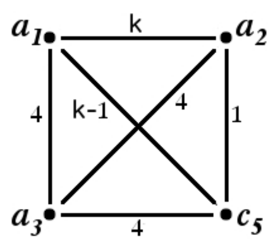
$$\begin{aligned} d(c_5, a_1) &= k - 1 \\ d(c_5, a_2) &= 1 \\ d(c_5, a_3) &= 4 \\ d(c_5, c_4) &= k - 2 \end{aligned}$$



$d(c_5, a_3) = 4$
$d(c_5, c_4) = k - 2$

We view the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, c_4)$ to be determined. The points a_1 and c_5 ensure that this distance is $k - 1$. So it suffices to show that the factors $(a_1a_2a_3c_5)$ and $(a_1a_3c_4c_5)$ embed isometrically into Γ.

The factor $(a_1a_2a_3c_5)$*:*



We view this as a 2-point amalgamation problem with the distance $d(a_3, c_5)$ to be determined. The point a_2 ensures that this distance is at least 3.

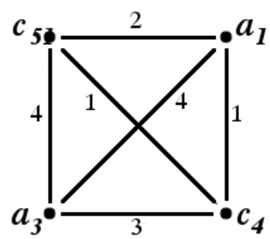
If $d(a_3, c_5) = 4$: then we have the desired configuration.

If $d(a_3, c_5) = 3$: then we have a configuration isometric to $(a_1a_2a_3c_4)$ above, and we may conclude.

We must check that the two triangles occurring as factors in this amalgamation embed into Γ. These are of types $(4, 4, k)$ and $(1, k - 1, k)$, so both occur.

That completes the treatment of this factor—but there is a minor subtlety that may be worth pointing out here. In the event that Γ contains no triangle of type $(4, 4, 3)$ but does contain a triangle of type $(4, 4, 4)$ then the configuration we are aiming at is impossible, but then the argument simply lands in the second alternative.

The factor $(a_1a_3c_4c_5)$:



We claim that this factor embeds isometrically into Γ_3 and hence into Γ. For this it suffices to check that there are no forbidden triangles.

Thus we conclude the treatment of the second factor, and the proof of the claim.

We summarize the last three claims as follows.

Claim 7.
$$\tilde{C} = C \text{ and } \tilde{C}' = C'$$

Thus we have checked all the numerical parameters, and it remains to consider the set $\mathcal{S}$.

We dealt with the case of cliques in Claim 1. It is convenient to separate off the case of anticliques of type $I_n^{(4)}$ (mutual distance 4).

Claim 8. *Suppose that* Γ *contains a* $(1,4)$*-space* A*. Then so does* Γ_3.

We proceed by induction on $|A|$. So let A be a minimal counterexample. Then by minimality of Γ, Γ_3 is of known type.

If $|A| \leq 3$ or A is a clique then this has been dealt with above. So we suppose
$$|A| \geq 4 \text{ and } A \text{ is not a clique}$$

Fix $a_1, a_2 \in A$ with $d(a_1, a_2) = 4$. Adjoin points c_1, c_2 with the following properties.

$d(c_i, a_1) = 1$ $(i = 1, 2)$

$d(c_i, a) = 2$ if $a \in A$ and $d(a_1, a) = 1$ $(i = 1, 2)$

$d(c_i, a) = 3$ if $a \in A$, $d(a_1, a) = 4$, $a \neq a_2$ $(i = 1, 2)$

$d(c_1, b) = 2$ $\qquad\qquad$ $d(c_2, b) = 4$

$d(c_1, c_2) = 2$

We will determine $d(c_i, a_2)$ in a moment. Note that the point a_1 ensures that $d(c_i, a_2) \geq 3$ for $i = 1, 2$.

To determine the structure of Ac_1c_2 completely, we treat the diagram Ac_1c_2 as an amalgamation problem with the distances $d(c_i, a_2)$ to be determined $(i = 1, 2)$. It suffices to check that the factors of this amalgamation embed isometrically into Γ. These factors are A and $A'c_1c_2$ where $A' = A \setminus \{a_2\}$.

The factor A embeds isometrically by hypothesis. We claim that the factor $A'c_1c_2$ is Γ_3-constrained, and hence embeds in Γ_3, and a fortiori in Γ. Now A' embeds in Γ, and $|A'| < |A|$, so by assumption A' embeds in Γ_3.

So it suffices to consider triangles and $(1,4)$-subspaces of $A'c_1c_2$ which contain at least one of the parameters c_1, c_2. The triangles have at most the following types.

$$(1,1,2),\ (2,2,2),\ (2,3,3),\ (1,1,2),\ (1,3,4),\ (1,2,2),\ (2,3,4),\ (3,3,4)$$

The only doubtful case is $(1,2,2)$, which arises only when A contains a clique of order 3. In this case $K_1 = 1$ and triangles of type $(1,2,2)$ are permitted.

Thus we may perform our amalgamation to determine the structure of Ac_1c_2, and our configuration is now completely determined.

We view the configuration as a 2-point amalgamation problem with the distance $d(a_1, b)$ to be determined. The points c_1, c_2 ensure that this distance will be 3. So it suffices to check that the factors Ac_1c_2 and $A'bc_1c_2$ embed isometrically into Γ.

The factor Ac_1c_2 was just constructed via an amalgamation in Γ, so that is no longer an issue. We claim that the factor $A'bc_1c_2$ is Γ_3-constrained and therefore embeds into Γ_3, hence into Γ.

The factor $A'c_1c_2$ embeds in Γ. Furthermore all its $(1,4)$-subspaces have order smaller than $|A|$, since those containing one of the points c_i have order at most 3. Therefore $A'c_1c_2$ is also Γ_3-constrained.

It remains to consider triangles and $(1,4)$-spaces containing b.

Triangles in $A'b$ containing b are of the types $(1,3,3)$ or $(3,3,4)$, both of which embed in Γ_3. Other triangles containing b are of the types

$$(2,2,4),\ (1,2,3),\ (1,3,4),\ (2,2,3),\ (2,3,4),\ (2,3,3),\ \text{or}\ (3,3,4)$$

None of these present any issues.

The only nontrivial $(1,4)$-space containing b is the pair (c_2, b).

So this factor is indeed Γ_3 constrained.

With this the proof of the claim, and also of the lemma, is complete.

□

Now the analysis of Γ_3 is nothing but the study of isometric embeddings of configurations $(a) \perp^{(3)} A$ with A either a triangle of interest, or a $(1,4)$-space.

We are concerned more generally with configurations of the form $A \perp^{(3)} B$ where A, B are triangles or $(1,4)$-spaces. From that point of view the analysis of Γ_3 is a small but essential step.

5. $\Gamma_3(A)$: First Steps

We denote by $\Gamma_3(A)$ the intersection $\bigcap_{a\in A} \Gamma_3(a)$. Similarly, if A is a triangle of type (i,j,k) we may write $\Gamma_3(i,j,k)$ for $\Gamma_3(A)$.

In the cases remaining for study, we expect $\Gamma_3(A)$ to be a connected metrically homogeneous graph with the same parameters as Γ. Our first substantial goal is connectedness, but the present section is devoted to preparatory amalgamation arguments, some dealing with $\Gamma_3(A)$ when A has two points, some simply dealing with small configurations.

5.1. **Small Direct Sums.** Our next objective is the following.

Lemma 5.1 (Edge Sums)**.** *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \le 3 \text{ and } C > 10$$

Let A, B be two pairs of points, each at distance at most 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

We begin with a simple case.

Lemma 5.2. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \le 3 \text{ and } C > 10$$

Let A, B be two pairs of points, with the points of A at distance 3 and the points of B at distance at most 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. Write $A \perp^{(3)} B$ as $u \perp^{(3)} v \perp^{(3)} B$ and apply Lemma 4.19 twice. □

Now we pull out some special cases.

Lemma 5.3. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \le 3 \text{ and } C = 11$$

Let A, B be two pairs of points, with the points of A at distance 1 and the points of B at distance 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. We let $A = \{u_1, v_2\}$ and $B = \{u_2, v_2\}$, and adjoin a point c with

$$d(c, u_1) = 4 \qquad d(c, v_1) = 4$$
$$d(c, u_2) = 3 \qquad d(c, v_2) = 1$$

View this configuration as a 2-point amalgamation problem with the distance $d(v_1, v_2)$ to be determined. The point c ensures $d(v_1, v_2) \ge 3$, and the bound $C = 11$ together with the point u_2 ensures $d(v_1, v_2) \le 3$. So it suffices to embed the factors $(u_1 u_2 v_1 c)$ and $(u_1 u_2 v_2 c)$ isometrically into Γ.

Each of these factors can be viewed as a geodesic of type $(1, 3, 4)$ in Γ_2, hence embeds isometrically in Γ. □

Lemma 5.4. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 = 3$$

Let A, B be two pairs of points, with the points of A at distance 1 and the points of B at distance $k = 2$ or 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. We write $A = \{u_1, v_1\}$ and $B = \{u_2, v_2\}$, and adjoin a point c with

$$d(c, u_1) = 2 \qquad d(c, v_1) = 1$$
$$d(c, v_2) = 2$$

We will determine $d(c, u_2)$ below, with the proviso $d(c, u_2) > 1$.

We view this configuration as a 2-point amalgamation problem with the distance $d(v_1, v_2)$ to be determined. The point u_1 ensures that this distance is at least 2. The point c together with the condition $K_1 = 3$ ensures that the distance is not 2, and is at most 3. Thus $d(v_1, v_2)$ must be 3. Therefore it suffices to embed the factors of this amalgamation isometrically into Γ, for some choice of the distance $d(c, u_2)$.

The factor $u_1u_2v_2c$:

We treat this as an amalgamation problem with the distance $d(c, u_2)$ to be determined. The point v_2 shows that this distance is not 2, via the triangle inequality if $k = 4$ and via the condition $K_1 = 3$ if $k = 2$. The factors of this amalgamation are triangles of types $(3, 3, k)$ and $(2, 2, 3)$. So this factor can be constructed with the distance $d(c_2, u_2) > 1$.

The factor $u_1v_1u_2c$:

If $d(c, u_2) = 3$ then this represents a triangle of type $(1, 1, 2)$ in $\Gamma_3(u_2)$ and there is no problem. So suppose $d(c, u_2) \neq 3$.

Then we view this configuration as a 2-point amalgamation problem with the distance $d(c, u_1)$ to be determined, and as $K_1 > 1$ the point v_1 forces the distance to be 2, with the assistance of the point u_2 which ensures that the two points are not identified in the amalgam.

So it suffices to check that the factors of this amalgamation embed isometrically into Γ. These are triangles of types $(1, 3, 3)$ and $(1, 3, d(c, u_2))$. As $d(c, u_2) \neq 1$ these factors embed isometrically into Γ.

□

Lemma 5.5. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 = 3$$

Let A, B be two pairs of points, each at distance 1. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. We take $A = \{u_1, v_1\}$ and $B = \{u_2, v_2\}$, and much as in the previous argument adjoin a point c with

$$d(c, u_1) = 3 \qquad d(c, v_1) = 2$$
$$d(c, u_2) = 2 \qquad d(c, v_2) = 1$$

We view this as a 2-point amalgamation problem with the distance $d(v_1, v_2)$ to be determined. Again using the condition $K_1 = 3$ we see that this distance is forced to be 3. So it suffices to embed the factors of this amalgamation isometrically into Γ.

The factor $(u_1u_2v_2c)$ represents a geodesic of type $(1,1,2)$ in $\Gamma_2(u_1)$, so embeds isometrically into Γ.

The factor $(u_1v_1u_2c)$ may be interpreted by taking u_2 to be the base point for Γ. Then we need adjacent points u_1, v_1 in Γ_3 and a point c in Γ_2 with $d(c,u_1)=3$, $d(c,v_1)=2$.

Fix u in Γ_2 and let $I_u = \{v \in \Gamma_3 \mid d(u,v) \geq 3\}$. Then I_u is a proper subset of Γ_3 since there are adjacent points in Γ_2, Γ_3. As Γ_3 is connected we can find u_1, v_1 in Γ_3, adjacent, with $u_1 \in I_u$ and $v_1 \notin I_u$. Then $d(c,u_1)=3$, $d(c,v_1)=2$, as required. □

Now we treat a substantial case.

Lemma 5.6. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \leq 3 \text{ and } C > 10$$

Let A, B be two pairs of points, with the points of A at distance 1 and the points of B at distance at most 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. Take $A = \{u_1, v_1\}$, $B = \{u_2, v_2\}$, and set

$$k = d(u_2, v_2)$$

We have dealt with the case $k=3$ so we assume throughout that

$$k \neq 3$$

We have dealt with all remaining cases in which $K_1 = 3$ in Lemmas 5.4 and 5.5. So we suppose

$$K_1 \leq 2$$

We also treated the case $k=4$, $C=11$ in Lemma 5.3, so we set this aside as well.

$$\text{If } k=4, \text{ assume } C > 11$$

For $u \in \Gamma_k$ let $I_u = \{v \in \Gamma_3 \mid d(u,v) = 3\}$. If there is an adjacent pair in I_u then we have the desired configuration. Suppose toward a contradiction that there are no adjacent pairs in I_u.

Claim 1. *For $v \in I_u$, there are adjacent vertices v_1, v_2 in Γ_3 with*

$$d(u,v_1) = 2 \text{ and } d(u,v_2) = 4$$

Let $I_u^+ = \{v \in \Gamma_3 \mid d(u,v) \geq 3\}$, $I_u^- = \{v \in \Gamma_3 \mid d(u,v) \leq 3\}$. Then I_u^+ and I_u^- are proper subsets of Γ_3 since Γ contains triangles of types

$$(3,k,2) \text{ and } (3,k,4)$$

When $k=4$ this uses the hypothesis $C > 11$.

As Γ_3 is connected it follows that there are adjacent v, v_1 in Γ_3 with $v \in I_u^+$ and $v_1 \notin I_u^+$. So $d(u,v)=3$ and $d(u,v_1)=2$. We get the point v_2 similarly.

This proves the claim. We may strengthen it as follows.

Claim 2. *Let $v \in I_u$ and $i = 2$ or 4. Then there are distinct neighbors v_1, v_1' of v with $d(u, v_1) = d(u, v_1') = i$.*

Suppose on the contrary that for some choice of i, each point v of I_u has a unique neighbor v_1 with $d(u, v_1) = i$. Then every neighbor v_2 of v other than v_1 satisfies $d(u, v_2) = i'$ where i, i' are $2, 4$ in some order.

We claim that every neighbor w of v_1 satisfies

$$w \in I_u \text{ and } d(v, w) = 2$$

As w is adjacent to v_1, $d(u, w) \neq i'$ and thus w is not adjacent to v. So $d(v, w) = 2$.

Now we may take a second common neighbor v_2 of v, w in Γ_3, not equal to v_1. Then $d(u, v_2) = i'$, $d(u, v_1) = i$, so $d(u, w) = 3$.

Now by Lemma 5.2 there is a pair v_1, v_2 in I_u with $d(v_1, v_2) = 3$. Then $v_1' \neq v_2'$. We now reach a contradiction by considering $d(v_1', v_2')$.

As $d(u, v_1') = d(u, v_2') = i$, v_1' and v_2' are not adjacent. Let $(v_1', w_1, \ldots, w_2, v_2')$ be a geodesic. Then $w_1' = v_1'$, $w_2' = v_2'$, so $w_1 \neq w_2$. Furthermore $w_1, w_2 \in I_u$, so w_1, w_2 are not adjacent. On the other hand $d(v_1', v_2') \leq 4$ so we arrive at $d(w_1, w_2) = 2$. But in this case, by homogeneity, $w_1' = w_2'$ and we have a contradiction.

This proves the claim.

Recall $K_1 \leq 2$. We treat the two possibilities separately.

Claim 3. *The Lemma holds if $K_1 = 1$.*

We take $v \in I_u$ and neighbors v_1, v_2 with $d(u, v_1) = 2$, $d(u, v_2) = 4$. Then $d(v_1, v_2) = 2$.

We consider the configuration $v_1 v_2 w_1 w_2$ where all vertices are adjacent except for the pair v_1, v_2, which are at distance 2. We claim this embeds in Γ_3.

This configuration can be seen as a geodesic (v_1, w_1, v_2) in $\Gamma_1(w_1)$, so it embeds in Γ_3. (Recall that $\tilde{K}_1 = K_1$.)

Now by homogeneity we may assume that under this embedding v_1, v_2 are the neighbors of v initially chosen. As w_1, w_2 are adjacent to v_1 and v_2, this forces $d(u, w_1) = d(u, w_2) = 3$, with w_1, w_2 adjacent.

Claim 4. *The lemma holds if $K_1 = 2$.*

We take $v \in I_u$ and v_1, v_1', v_2, v_2' adjacent to v with

$$d(u, v_1) = d(u, v_1') = 2 \qquad d(u, v_2) = d(u, v_2') = 4$$

As $K_1 > 1$, all distances between v_1, v_1', v_2, v_2' are equal to 2.

We claim that the configuration v_1, v_1', v_2, v_2' can be extended by adjacent points w, w' in Γ_3 with w adjacent to v_1, v_2 and w' adjacent to v_1', v_2', and all other distances equal to 2.

It suffices to construct such a configuration in Γ_3, or for that matter in Γ, as these have the same properties.

First view the configuration as an amalgamation problem with the distances between v_1 or v_2 and w' to be determined. The point w ensures that these distances are equal to 2, and the point v_1' or v_2' ensures that none of these points will be identified. So it suffices to show that the factors of this amalgamation embed in Γ.

The factor $ww'v_1'v_2'$ is simply a point with three neighbors (recalling $K_1 > 1$). So we consider the factor

$$(v_1v_2v_1'v_2'w)$$

Adjoin a point c adjacent to v_1, v_2, v_1', v_2', with $d(c, w) = 2$, and view the result as an amalgamation problem in which the distances between $\{v_1, v_2\}$ and $\{v_1', v_2'\}$ are to be determined. The point w prevents collapse and the point c ensures that the distances are all equal to 2.

The factors of this diagram are a 4-cycle (cv_1wv_2), which certainly embeds in Γ, and the factor $(wc_1v_1'v_2')$ consisting of the geodesic $(v_1'v_2'c)$ in $\Gamma_2(w)$, a configuration which also occurs in Γ.

This proves the claim, and treats the last case of the lemma. □

So at this point we have treated all cases of the Edge Sum Lemma 5.1 in which one of the two pairs is at distance 1 or 3.

Lemma 5.7. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \le 3 \text{ and } C > 10$$

Let A, B be two pairs of points, each at distance 2. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. Take $A = \{u_1, v_1\}$ and $B = \{u_2, v_2\}$.

Adjoin points c_1, c_2 with

$$\begin{aligned} &d(c_1, u_1) = d(c_1, v_1) = 1 && d(c_2, u_1) = d(c_2, v_1) = 1\\ &d(c_1, u_2) = d(c_1, v_2) = 2 && d(c_2, u_2) = d(c_2, v_2) = 4\\ &d(c_1, c_2) = 2 \end{aligned}$$

Treat the resulting configuration as an amalgamation problem in which the distances between A and B are to be determined. The points c_1, c_2 guarantee that these distances are equal to 3. So it suffices to show that the factors $(u_1v_1c_1c_2)$ and $(u_2v_2c_1c_2)$ embed isometrically in Γ.

The factor $(u_1v_1c_1c_2)$ may be viewed as a pair of points u_1, u_2 at distance 2, together with two common neighbors at distance 2. This is covered by Fact 1.1.

The factor $(u_2v_2c_1c_2)$ may be viewed as a geodesic $(v_2c_1c_2)$ of type $(2, 2, 4)$ in $\Gamma_2(u_2)$. This configuration certainly embeds in Γ. □

Lemma 5.8. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \le 3 \text{ and } C > 10$$

Let A, B be two pairs of points, with the points of A at distance 2 and the points of B at distance 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. Take $A = \{u_1, v_1\}$ and $B = \{u_2, v_2\}$.

Adjoin points c_1, c_2 with

$$\begin{array}{ll} d(c_1,u_1) = d(c_1,v_1) = 1 & d(c_2,u_1) = d(c_2,u_2) = 1 \\ d(c_1,u_2) = d(c_2,v_2) = 2 & d(c_2,u_2) = 2 \\ & d(c_2,v_2) = 4 \\ d(c_1,c_2) = 2 & \end{array}$$

Then view this as an amalgamation problem with the distances between A and v_2 to be determined. The points c_1, c_2 ensure that these distances are equal to 3. So it suffices to embed the factors isometrically in Γ.

The factor $(u_1u_2v_1c_1c_2)$:

Taking u_2 as base point, we require a pair of points in Γ_2 at distance 2, with a pair of common neighbors in Γ_3, also at distance 2.

Fix c_1, c_2 in Γ_2 with $d(c_1, c_2) = 2$. Take c_3 in Γ_3 adjacent to c_1, c_2, and c_4 in Γ_4 adjacent to c_3. Then c_1, c_2, c_3 is a triple of points mutually at distance 2.

We claim that there are (u_1, v_1) adjacent to c_1, c_2, c_4 with $d(u_1, v_1) = 2$. To see this, take v_1, v_2 first and apply Fact 1.1 to get suitable c_1, c_2, c_4. Then apply homogeneity,

Now as $c_1, c_2 \in \Gamma_2$ and $c_4 \in \Gamma_4$, we have $v_1, v_2 \in \Gamma_3$, as required.

The factor $u_1u_2v_2c_1c_2$:

With v_2 as base point, this represents a triple c_1, c_2, u_2 with two points in Γ_4 and one point in Γ_2.

Fix a point u in Γ_2 and a neighbor v in Γ_3. By Lemma 4.18 there is a pair of points v_1, v_2 in Γ_4 adjacent to v, with $d(v_1, v_2) = 2$. Then (u, v_1, v_2) is a suitable triple.z □

Lemma 5.9. *Let Γ be a primitive metrically homogeneous graph of diameter 4 of K^*-type with*

$$K_1 \leq 3 \text{ and } C > 10$$

Let A, B be two pairs of points, each at distance 4. Then the configuration $A \perp^{(3)} B$ embeds isometrically into Γ.

Proof. Take $A = \{u_1, v_1\}$ and $B = \{u_2, v_2\}$.

Adjoin a pair of points c_1, c_2 with

$$\begin{aligned}
&d(c_i, u_1) = 3 \ (i = 1, 2) \\
&d(c_i, v_1) = 1 \ (i = 1, 2) \\
&d(c_i, u_2) = 2 \ (i = 1, 2) \\
&d(c_1, v_2) = 2 \qquad\qquad d(c_2, v_2) = 4 \\
&d(c_1, c_2) = 2
\end{aligned}$$

View the resulting configuration as a 2-point amalgamation in which the distance $d(v_1, v_2)$ is to be determined. The points c_1, c_2 ensure that this distance is 3. So it suffices to check that the factors $(u_1u_2v_1c_1c_2)$ and $(u_1u_2v_2c_1c_2)$ of this amalgamation embed isometrically into Γ.

The factor $(u_1u_2v_1c_1c_2)$*:*

Taking u_1 as base point, we want u_2, c_1, c_2 to be in Γ_3, $v_1 \in \Gamma_4$, all at distance 2, with v adjacent to c_1, c_2 and at distance 3 from u_2.

As Γ contains a triangle of type $(3, 4, 2)$, we may fix points u in Γ_4, $v \in \Gamma_3$ with $d(u, v) = 2$. As Γ_3 is connected and there are points in Γ_3 at distance 1 or 3 from u, it follows that there are neighbors v_0, v_1 of v in Γ_3 with $d(u, v_0) = 2$ and $d(u, v_1) = 3$. This forces $d(v_0, v_1) = 2$.

The points v_0, v_1 have a common neighbor w in Γ_2. Then $d(w, u) = 2$. By Fact 1.1, u and w have two common neighbors v_2, v_3 with $d(v_2, v_3) = 2$. Then v_2, v_3 are in Γ_3. Considering the points u and w, we see that the distances $d(v_1, v_2)$ and $d(v_1, v_3)$ are also equal to 2. This is the required configuration.

The factor $(u_1u_2v_2c_1c_2)$*:*

Here taking u_1 as base point, we require the configuration $(u_2v_2c_1c_2)$ in $\Gamma_3(u_1)$. Since Γ_3 and Γ satisfy the same hypotheses, it suffices to find the configuration $(u_2v_2c_1c_2)$ in Γ.

With v_2 as base point, this means we require u_2, c_2 in Γ_4 and c_1 in Γ_2 with u_2, c_1, c_2 mutually at distance 2.

We first take u_2, c_2 in Γ_4, then a common neighbor u in Γ_3 (using Lemma 4.18). Then take c_1 to be a neighbor of u in Γ_2. This is the required configuration.

□

Proof of Lemma 5.1. When one of the pairs of points lie at distance 1 or 3 this is covered by Lemmas 5.2 and 5.6.

The remaining cases are covered in Lemmas 5.7, 5.8, and 5.9.

□

5.2. **The structure of** $\Gamma_3(1, 1, 2)$**: First steps.** We begin the analysis of $\Gamma_3(A)$ for A of order 2 or 3. The various cases become intertwined but we will arrive in particular at the conclusion that $\Gamma_3(1, 1, 2)$ contains geodesics of type $(1, 1, 2)$ and $(1, 2, 3)$.

We begin the study of $\Gamma_3(A)$ for various pairs and triples A.

At this point we know that $\Gamma_3(u_1, u_2)$ is a homogeneous metric space in which the distances occurring are precisely $1, 2, 3, 4$.

Lemma 5.10. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type, with $K_1 = 1$ and $C > 10$. Let u_1, u_2 be a pair of points in Γ and let B be a $(1,2)$-space whose cliques embed in Γ_1. Suppose further*

$$\textit{If } d(u_1, u_2) = 4 \textit{ then } C > 11$$

Then B embeds in $[\Gamma_3(u_1, u_2)]_1$, by which we mean the graph Γ_1 taken relative to $\Gamma_3(u_1, u_2)$. In particular, if B is a clique embedding in Γ, or a triangle of type $(1,1,2)$, then B embeds in $\Gamma_3(u_1, u_2)$.

Proof. Set $k = d(u_1, u_2)$.

For the first claim, treat u_1 as a base point for γ and for $u \in \Gamma_k$ let $I_u = \{v \in \Gamma_3 \mid d(u, v) = 3\}$.

Fix $u \in \Gamma_k$. There are points v_2, v_3, v_4 in Γ_3 with $d(u, v_i) = i$ for $i = 2, 3, 4$, since Γ contains triangles of type $(3, k, i)$ with $i = 2, 3, 4$; recall that if $k = 4$ then $C > 11$.

As Γ_3 is connected, it follows easily that for $v \in I_u$ there are v_1, v_2 adjacent to v with $d(u, v_1) = 2$, $d(u, v_2) = 4$. As the parameters of Γ and Γ_3 are the same, the configuration B embeds in $\Gamma_{3,1} = (\Gamma_3)_1$. By Fact 1.1, the configuration B embeds in the common neighbors of v_1, v_2 in Γ_3. Then the distances from u to points of B are forced to be 3.

This proves the first point.

For the second point, we view a clique of order n embedded in Γ as a clique of order $n-1$ embedded in Γ_1, and we view a geodesic of type $(1,1,2)$ as a pair of points at distance 2 in Γ. □

Lemma 5.11. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type, with $K_1 > 1$ and $C > 10$. Let u_1, u_2 be a pair of points in Γ. Assume further*

$$\textit{If } d(u_1, u_2) = 4 \textit{ then } C > 11$$

Then a triangle of type $(1,1,2)$ embeds in $\Gamma_3(u_1, u_2)$.

Proof. We take u_1 as the base point, and set $k = d(u_1, u_2)$. Fix $u \in \Gamma_k$ and set $I_u = \{v \in \Gamma_3 \mid d(u, v) = 3\}$. Suppose toward a contradiction that there is no geodesic of type $(1,1,2)$ in $\Gamma_3(u_1, u)$.

By Lemma 5.6 there is a pair v, v_1 in $\Gamma_3(u_1, u)$ with v, v_1 adjacent.

As $C > 11$ in the case $k = 4$, there are triangles of type $(3, k, 2)$ and $(3, k, 4)$ in Γ. Thus there are points in Γ_3 at distance 2 or 4 from u.

As Γ_3 is connected it follows easily that for $v \in \Gamma_3(u_1, u)$ there are v_2, v_3 adjacent to v in Γ_3 with $d(u, v_2) = 2$, $d(u, v_3) = 4$.

By Fact 1.1, for v, v' at distance 2 in Γ_3 there are three distinct common neighbors w_1, w_2, w_3, mutually at distance 2. Therefore the there points v_1, v_2, v_3, which are mutually at distance 2, have two distinct common neighbors v, v'. As $d(u, v_2) = 2$ and $d(u, v_3) = 4$ it follows that $d(u, v') = 3$. Then (v, v_1, v') is a geodesic of type $(1,1,2)$ in $\Gamma_3(u_1, u_2)$. □

Lemma 5.12. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type, with $K_1 \le 3$ and $C = 11$. Let u_1, u_2 be a pair of points in Γ with $d(u_1, u_2) = 4$. If B is a geodesic of type $(1,1,2)$ or $(1,2,3)$, then B embeds into $\Gamma_3(u_1, u_2)$.*

Proof. Extend the configuration $(u_1u_2) \perp^{(3)} B$ by a point c with

$$d(c, u_1) = 3 \qquad d(c, u_2) = 1$$
$$d(c, b) = 4 \ (b \in B)$$

View this configuration as an amalgamation problem with the distances between u_2 and B to be determined. The point c ensures that these distances are at least 3. The point u_1 and the bound $C = 11$ ensures that these distances are not 4. So the result of the amalgamation is unique and it suffices to check that the factors (u_1u_2c) and u_1Bc embed isometrically into Γ.

The factor (u_1u_2c) is a geodesic of type $(1,3,4)$.

The factor u_1Bc can be viewed as Bc inside $\Gamma_3(u_1)$. Since Γ_3 satisfies the same conditions as Γ, the problem is to embed Bc isometrically in Γ.

Now Bc represents a geodesic B of type $(1,1,2)$ or $(1,2,3)$ inside Γ_4. Since Γ_4 is connected of diameter 3, this configuration embeds isometrically into Γ. □

Lemma 5.13. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type, with $K_1 \le 3$ and $C > 10$. Let u_1, u_2 be a pair of points in Γ. Then a geodesic of type $(1,1,2)$ embeds into $\Gamma_3(u_1, u_2)$.*

Proof. If $k < 4$ or $C > 11$ then Lemma 5.10 or 5.11 applies.

If $k = 4$ and $C = 11$ then Lemma 5.12 applies. □

Lemma 5.14. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $K_1 \le 3$ and $C > 10$. Let a_1, a_2, a_3 be a triple of points in Γ with*

$$d(a_1, a_2) = 1 \qquad d(a_2, a_3) = 2 \qquad d(a_1, a_3) = 3$$

Then Γ contains points v_2, v_3, v_4 with

$$d(v_i, a_j) = 3 \ (i = 2,3,4;\ j = 1,3)$$
$$d(v_i, a_2) = i \ (i = 2,3,4)$$

Proof. The point v_3 is afforded by Lemma 4.19.
Construction of $a_1a_2a_3v_2$:

Relative to v_2 as base point, we require points a_1, a_3 in Γ_3 at distance 3 and a point a_2 in Γ_2 with $d(a_2, a_1) = 1$, $d(a_2, a_3) = 2$.

Take a point u in Γ_2. By Fact 1.1, u has two neighbors v_1, v_2 at distance 2 in Γ_3. There is a point $v_3 \in \Gamma_3$ adjacent to v_2 and at distance 3 from v_1. Then $d(u, v_3) = 2$. Thus the configuration (u, v_1, v_3) is as required.

The construction of v_4 is the same, taking $u \in \Gamma_4$ at the outset. □

Lemma 5.15. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $K_1 = 1$ and $C > 10$. Let A be a geodesic of type $(1, 2, 3)$ and let B be a geodesic of type $(1, 1, 2)$. Then $A \perp^{(3)} B$ embeds isometrically in Γ.*

Proof. By Lemma 5.14 we have points v_2, v_3, v_4 in $\Gamma_3(a_1, a_3)$ at distances 2, 3, or 4 respectively from a_1. Now $\Gamma_3(a_1, a_3)$ is connected, and in fact has the same parameters as Γ, by Lemma **??**.

Therefore it follows easily that we may take v_2, v_4 to be adjacent to v_3. It then follows that $d(v_2, v_4) = 2$. By Fact 1.1, the common neighbors of v_2, v_4 contain a an isometric copy of any $(1, 2)$-space without cliques of order 3, since $K_1 = 1$ and $\Gamma_3(a_1, a_3)$ has the same parameters as Γ.

In particular we may find a geodesic of type $(1, 1, 2)$ in $\Gamma_3(a_1, a_2, a_3)$. □

Lemma 5.16. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $1 < K_1 \leq 3$ and $C > 10$. Let A be a geodesic of type $(1, 2, 3)$ and B a pair of points at distance 2. Then $A \perp^{(3)} B$ embeds isometrically into Γ.*

Proof. Let $A = (a_1 a_2 a_3 b)$ with $d(a_1, a_2) = 1$, $d(a_2, a_3) = 2$, $d(a_1, a_3) = 3$, and $B = \{b_1, b_2\}$. Adjoin a point c with

$$d(c, a_1) = 2$$
$$d(c, a_2) = d(c, a_3) = 1$$
$$d(c, b_1) = d(c, b_2) = 2$$

View this configuration as a 2-point amalgamation problem with the distance $d(a_2, a_3)$ to be determined. The point c ensures that this distance is 2. So it suffices to show that the factors $a_1 a_2 b_1 b_2 c$ and $a_1 a_3 b_1 b_2 c$ embed isometrically into Γ.

The factor $(a_1 a_2 b_1 b_2 c)$:

We may view this as a 2-point amalgamation problem with the distance $d(a_1, c)$ to be determined. The point a_2 ensures that this distance is 2. So it suffices to show that the subfactors $(a_1 a_2 b_1 b_2)$ and $(a_2 b_1 b_2 c)$ embed isometrically into Γ.

The subfactor $(a_1 a_2 b_1 b_2)$ is given by Lemma 5.1.

Relative to the base point a_2, the subfactor $(a_2 b_1 b_2 c)$ consists of three points at mutual distance 2, with one in Γ_1 and two in Γ_3. Take a point $u \in \Gamma_2$, a neighbor in Γ_1, and two neighbors in Γ_3 at distance 2, to get this configuration.

The factor $(a_1 a_3 b_1 b_2 c)$:

Relative to the base point a_1, we require a point c in Γ_2 and a triangle (v_1, v_2, v_3) in Γ_3 with

$$d(v_1, v_2) = 2 \qquad d(v_1, v_3) = d(v_2, v_3) = 3$$
$$d(c, v_3) = 1 \qquad d(c, v_1) = d(c, v_2) = 2$$

We first show that Γ_3 contains a configuration v, v_1, v_2, v_3 with v_1, v_2, v_3 as above, v adjacent to v_1 and v_2, and $d(v, v_3) = 2$.

As Γ_3 satisfies the same conditions as Γ, it suffices to show that this configuration embeds in Γ. Relative to v_3 as base point, the configuration $(v_1 v_2 v)$ consists of a point in Γ_2 with two neighbors at distance 2 in Γ_3. This is afforded by Fact 1.1.

Now take the configuration $(v v_1 v_2 v_3)$ to lie in Γ_3 and take a common neighbor c of v, v_3 in Γ_2. Then $(c v_1 v_2 v_3)$ is the desired configuration.

□

Lemma 5.17. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \le 3$ and $C > 10$. Let A be a geodesic of type $(1,2,3)$ and let B be a geodesic of type $(1,1,2)$. Then $A \perp^{(3)} B$ embeds isometrically in Γ.*

Proof. We deal with the case $K_1 = 1$ in Lemma 5.15, so we will suppose

$$K_1 > 1$$

Write $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ with

$$d(a_1, a_2) = 1 \qquad d(a_2, a_3) = 2 \qquad d(a_1, a_3) = 3$$
$$d(b_1, b_2) = 1 \qquad d(b_2, b_3) = 1 \qquad d(b_1, b_3) = 2$$

Adjoin a point c_1 with

$$d(c_1, a_1) = 2 \qquad d(c_1, a_2) = d(c_1, a_3) = 1$$
$$d(c_1, b_2) = 2 \qquad d(c_1, b_1) = d(c_1, b_3) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, a_3)$ to be determined. As $K_1 > 1$ the point c_1 ensures that this distance is 2. So it suffices to show that the factors $(a_1 a_2 B c_1)$ and $(a_1 a_3 B c_1)$ embed isometrically into Γ.

The factor $(a_1 a_2 B c_1)$:

View this as a 2-point amalgamation problem with the distance $d(a_1, c_1)$ to be determined. The point a_2 ensures that this distance is 2. So we may reduce to the separate factors $(a_1 a_2 B)$ and $(a_2 B c_1)$.

The factor $(a_1 a_2 B)$ is $(a_1 a_2) \perp^{(3)} B$, which is covered by Lemma 5.13.

For the factor $(a_2 B c_1)$, adjoin a point c_2 with

$$d(c_2, b_2) = d(c_2, c_1) = 1$$
$$d(c_2, a_2) = d(c_2, b_1) = d(c_2, b_3) = 2$$

View this as a 2-point amalgamation problem with the distance between c_1 and b_2 to be determined. The points a_2 and c_2 ensure that this distance is 2. So it suffices to embed the factors $(a_2 B c_2)$ and $(a_2 b_1 b_3 c_1 c_2)$ isometrically into Γ.

View the factor $(a_2 B c_2)$ as an amalgamation problem with the distances between c_2 and b_1, b_3 to be determined. The point b_2 ensures that these distances are equal to 2. So it suffices to embed the subfactors $a_2 B$ and

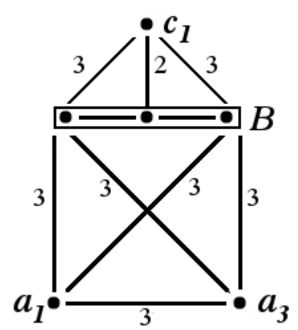
$a_2b_2c_2$ isometrically into Γ. The former embeds by Lemma 4.19 and the latter is a triangle of type $(1, 2, 3)$.

View the factor $(a_2b_1b_3c_1c_2)$ as a 2-point amalgamation problem with the distance $d(a_2, c_2)$ to be determined. The point c_1 ensures that this distance is 2. So it suffices to embed the subfactors $a_2b_1b_3c_1$ and $b_1b_2c_1c_2$ isometrically into Γ.

The configuration $a_2b_1b_3c_1$ is $(a_2c_1) \perp^{(3)} (b_1b_3)$, so embeds by Lemma 5.1.

Relative to the base point c_1, the configuration $b_1b_3c_1c_2$ consists of three points at mutual distance 2, with one in Γ_1 and two in Γ_3. Take a point u in Γ_2, a neighbor of u in Γ_1, and two neighbors of u in Γ_3, to obtain the desired configuration.

The factor $(a_1a_3Bc_1)$*:*



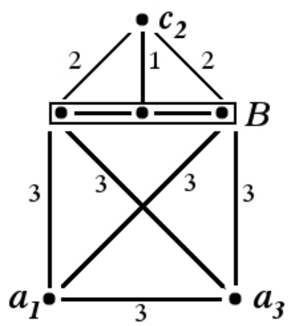
$d(c_1, a_1) = 2$
$d(c_1, a_3) = 1$

Adjoin a point c_2 with

$$d(c_2, b_2) = d(c_2, c_1) = 1$$
$$d(c_2, a_3) = d(c_2, b_1) = d(c_2, b_3) = 2$$
$$d(c_2, a_1) = 3$$

View this configuration as a 2-point amalgamation problem with the distance between c_1 and b_2 to be determined. The point c_2 ensures that this distance is 2. So it suffices to show that the subfactors $(a_1a_3Bc_2)$ and $(a_1a_3b_1b_3c_1c_2)$ embed isometrically in Γ.

The subfactor $(a_1a_3Bc_2)$ may be viewed as an amalgamation problem in which the distances between c_2 and b_1, b_3 are to be determined.



$d(c_2, a_1) = 2$
$d(c_2, a_3) = 2$

The point b_2 ensures that these distances are equal to 2. So it suffices to show that the configurations $(a_1 a_3 B)$ and $a_1 a_3 b_2 c_2$ embed isometrically into Γ.

The configuration $(a_1 a_3 B) = (a_1 a_3) \perp^{(3)} B$ is covered by two applications of Lemma 4.19.

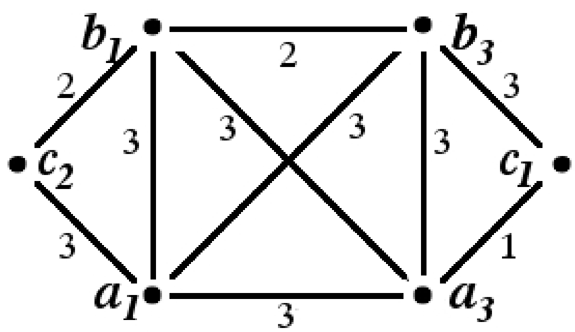
The configuration $(a_1 a_3 b_1 b_3 c_1 c_2)$ may be viewed as a 2-point amalgamation problem with the distance $d(a_3, c_2)$ to be determined. The point c_1 ensures that this distance is 2.

So we reduce to the configurations

$$(a_1 a_3 b_1 b_3 c_1) \text{ and } (a_1 b_1 b_3 c_1 c_2)$$

The factor $(a_1 a_3 b_2 c_1 c_2)$ is afforded by Lemma 5.16.

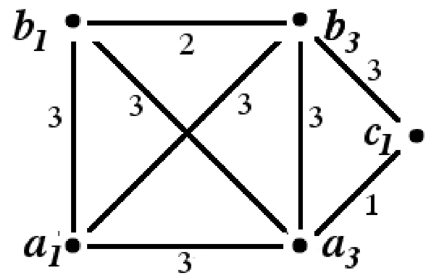
The factor $(a_1 b_1 b_3 c_1 c_2)$ may be viewed as a 2-point amalgamation problem with the distance $d(a_3, c_2)$ to be determined.



$d(c_1, a_1) = 2$
$d(c_1, b_1) = 3$
$d(c_2, a_3) = 2$
$d(c_2, b_3) = 2$

We may view this as a 2-point amalgamation problem with the distance $d(a_3, c_2)$ to be determined. The point c_1 ensures that this distance is 2. So it suffices to embed the factors $(a_1 a_3 b_1 b_3 c_1)$ and $(a_1 b_1 b_3 c_1 c_2)$ isometrically into Γ.

The factor $(a_1 a_3 b_1 b_3 c_1)$ is afforded by Lemma 5.16.



For the factor $(a_1b_1b_3c_1c_2)$ we adjoin a point c_3 with

$$d(c_3, a_1) = d(c_3, c_1) = 1$$
$$d(c_3, b_1) = d(c_3, b_3) = d(c_3, c_2) = 2$$

We view this configuration as a 2-point amalgamation problem with the distance $d(a_1, c_1)$ to be determined. The point c_3 ensures that this distance is 2. So it suffices to show that the factors $(a_1b_1b_3c_2c_3)$ and $(b_1b_3c_1c_2c_3)$ embed isometrically into Γ.

Relative to the base point a_1, the factor $(a_1b_1b_3c_2c_3)$ consists of four points at mutual distance 2, with three of them in Γ_3 and one in Γ_1. Take a point u in Γ_2, a neighbor of u in Γ_1, and three neighbors of u in Γ_3 at mutual distance 2, to produce the desired configuration.

Relative to the base point c_1, the factor $(b_1b_3c_1c_2c_3)$ consists of four points at mutual distance 2, with two of them in Γ_1 and the other two in Γ_3. We take a point u in Γ_2, two neighbors in Γ_1 at distance 2, and two neighbors in Γ_3 at distance 2, to produce the desired configuration. □

Lemma 5.18. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $K_1 = 1$, $K_2 = 4$, and $C > 10$. Then there is a geodesic A of type $(1, 1, 2)$ in Γ_4 and a point of Γ_3 adjacent to all points of A.*

Proof. As $K_2 = 4$ and $C > 10$, Γ_4 is connected of diameter at least 2.

Take $v_1, v_2 \in \Gamma_4$ at distance 2. By Fact 1.1 the common neighbors of v_1, v_2 form a connected graph $\Gamma_2(v_1, v_2)$.

As Γ_4 is connected, $\Gamma_2(v_1, v_2)$ meets Γ_4. By Lemma 4.18, $\Gamma_2(v_1, v_2)$ meets Γ_3.

As $\Gamma_2(v_1, v_2)$ is connected, there is a pair of adjacent edges u, v in $\Gamma_2(v_1, v_2)$ with $u \in \Gamma_3$, $v \in \Gamma_4$. Then (v_1, v, v_2, u) is the desired configuration. □

Lemma 5.19. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $K_1 = 1$, $K_2 = 4$, and $C > 10$. Let A be a geodesic of type $(1, 1, 2)$. Then Γ contains an isometric copy of $A \perp^{(3)} A$.*

Proof. Let $A = \{a_1, a_2, a_3\}$ with a_2 the midpoint, and consider the extension Au_1u_2 in which

$$d(u_1, u_2) = 2$$
$$d(u_1, a) = 2 \ (a \in A)$$
$$d(u_2, a) = 4 \ (a \in A)$$

If we can embed this configuration into Γ then it suffices to take a second copy of A among the points adjacent to both u_1 and u_2.

Adjoin a point c_1 with

$$d(c_1, u_1) = d(c_1, u_2) = 1$$
$$d(c_1, a) = 3 \ (a \in A)$$

Consider the resulting configuration as a 2-point amalgamation problem with the distance $d(u_1, u_2)$ to be determined. The point c_1 together with any point of A ensures that this distance is 2. So it suffices to show that the factors Au_1c_1 and Au_2c_1 embed isometrically into Γ.

The factor Au_1c_1:

Relative to the base point c_1, this consists of a point in Γ_1 at distance 2 from a copy of A in Γ_3.

Start with a point $u \in \Gamma_2$. Using Fact 1.1 we may find a copy A' of A in the neighbors of u in Γ_3. Take a neighbor u_1 of u in Γ_1. Then u_1 is at distance 2 from the points of A'.

The factor Au_2c_1:

Relative to the base point u_2, this consists of a point in Γ_1 at distance 3 from a copy of A in Γ_4.

Begin with a point $u \in \Gamma_3$. By Lemma 5.18 there is a copy A' of A in the neighbors of u in Γ_4. Take a point $u_2 \in \Gamma_1$ at distance 2 from u. Then $A'u_2$ is the required configuration, over the base point. □

Lemma 5.20. *Let Γ be a primitive metrically homogeneous graph of diameter 4 with $K_2 = 3$ and $C > 10$. Let A be a geodesic of type $(1, 1, 2)$. Then Γ contains an isometric copy of $A \perp^{(3)} A$.*

Proof. Take two copies $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ of A, with a_2 and b_2 the midpoints.

Adjoin a point c with

$$d(c, a_1) = d(c, a_3) = 1 \qquad d(c, a_2) = 2$$
$$d(c, b_1) = d(c, b_3) = 4 \qquad d(c, b_2) = 3$$

View the resulting configuration as an amalgamation problem in which the distances between a_1, a_3 and b_1, b_3 are to be determined. As $K_2 < 4$, the point c ensures that all these distances are equal to 3. So it suffices to show that the factors Ab_2c and Ba_2c embed isometrically into Γ.

The factor Ab_2c:

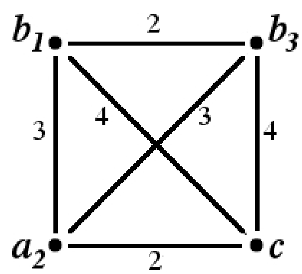
Relative to the base point b_2, this is a 4-cycle Ac embedding in Γ_3. This is straightforward.

The factor Ba_2c:

We view this as a 2-point amalgamation problem with the distance $d(c, b_2)$ to be determined. As $K_2 < 4$, the point b_1 ensures that this distance is 3. So we may reduce to the subfactors

$$Ba_2 \text{ and } (a_2b_1b_3c)$$

Now Ba_2 is simply a copy of B in $\Gamma_3(a_2)$, so this is known. We consider the remaining configuration $(a_2b_1b_3c)$.



Relative to the base point c, this is a point in Γ_2 at distance 3 from a pair of points in Γ_4 which are at distance 2.

As Γ contains triangles of type $(2, 4, 2)$, $(2, 4, 3)$, and $(2, 4, 4)$, we may find pairs of points in Γ_2 and Γ_4 at distance 2, 3, or 4.

Take a pair of points u, v at distance 3, with $u \in \Gamma_2$ and $v \in \Gamma_4$. As Γ_4 is connected we may easily find neighbors v_1, v_2 of v with

$$d(u, v_1) = 2 \qquad\qquad d(u, v_2) = 4$$

By Lemma 1.3, Γ_4 is connected. By Fact 1.1, the common neighbors of v_1, v_2 in Γ_4 contain a pair of points at distance 2. Therefore we may find a common neighbor v' of v_1, v_2 in Γ_4 at distance 2 from v. It follows that $d(u, v') = 3$ and we have the desired configuration. □

Lemma 5.21. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $1 < K_1 \leq 3$ and $C > 10$. Let u_1, u_2 be a pair of points at distance 1 in Γ. Then $\Gamma_3(u_1, u_2)$ is a connected metrically homogeneous graph of diameter 4.*

Proof. By Lemma 5.17, $\Gamma_3(u_1, u_2)$ contains geodesics of type $(1, 1, 2)$ and $(1, 2, 3)$. So it remains to show that $\Gamma_3(u_1, u_2)$ contains a geodesic of type $(1, 3, 4)$.

Let u_1, u_2 be a pair of points at distance 1, and $A = \{a_1, a_2, a_3\}$ a geodesic with

$$d(a_1, a_2) = 1 \qquad d(a_2, a_3) = 3 \qquad d(a_1, a_3) = 4$$

Adjoin points c_1, c_2 with

$$\begin{aligned} &d(c_1,u_1)=2 && d(c_2,u_1)=4\\ &d(c_1,u_2)=3 && d(c_2,u_2)=3\\ &d(c_1,a_i)=d(c_2,a_i)=i && (i=1,2,3)\\ &d(c_1,c_2)=2 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(u_1, a_1)$ to be determined. The points c_1 and c_2 ensure that this distance is 3. So it suffices to show that the factors $(u_1u_2a_2a_3c_1c_2)$ and $(u_2Ac_1c_2)$ embed isometrically in Γ.

The factor $(u_1u_2a_2a_3c_1c_2)$*:*
This can be viewed as $(u_1u_2a_2c_1c_2)$ inside $\Gamma_3(a_3)$, and since Γ_3 satisfies the same conditions as Γ, we may restrict our attention to $(u_1, u_2a_2c_1c_2)$.

We adjoin a point c_3 with

$$\begin{aligned} &d(c_3,a_2)=d(c_3,c_1)=d(c_3,c_2)=1\\ &d(c_3,u_1)=d(c_3,u_2)=3 \end{aligned}$$

The result can be viewed as an amalgamation of 3 factors in which all distances among u_1, u_2, a_2 are to be determined. The point c_3, and the fact that these three points are at different distances from u_1, ensures that the distances between them are equal to 2.

So it suffices to show that the factors $(u_1u_2a_2c_3)$, $(u_1u_2c_1c_3)$, and $(u_1u_2c_2c_3)$ all embed isometrically into Γ.

Relative to the base point c_3 these factors consist of the two adjacent points u_1, u_2 in Γ_3, and a point in Γ_1 with the pair of distances $(2,3)$, $(3,3)$, or $(3,4)$ over them.

The usual argument using connectedness of Γ_3 takes care of the points with distances $(2,3)$ or $(3,4)$. This leaves $(u_1u_2b_2c_3) = (u_1u_2) \perp^{(3)} (b_2c_3)$, covered by Lemma 5.1.

The factor $(u_2Ac_1c_2)$*:*
This is $(u_2) \perp^{(3)} (Ac_1c_2)$ so we may restrict attention to

$$(Ac_1c_2)$$

Relative to the base point a_1, this becomes three points at mutual distance 2 in Γ_1, at distance 3 from a point in Γ_4.

Beginning with a point u in Γ_2, take three neighbors at mutual distance 2 in Γ_1, using Fact 1.1, and one neighbor in Γ_4 at distance 2 from u. This gives the desired configuration. □

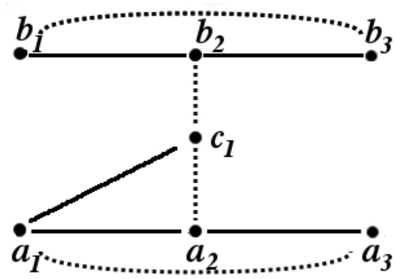
Lemma 5.22. *Let* Γ *be a primitive metrically homogeneous graph of diameter* 4 *with* $K_1 \leq 3$ *and* $C > 10$. *Let* A *be a geodesic of type* $(1,1,2)$. *Then* Γ *contains an isometric copy of* $A \perp^{(3)} A$.

Proof. All cases in which $K = 1$ are covered by Lemmas 5.20 and 5.19. So we will suppose

$$K_1 > 1$$

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be geodesics of type $(1, 1, 2)$ with midpoint a_2, b_2 respectively. Adjoin a point c_1 with

$$d(c_1, a_1) = 1 \qquad d(c_1, a_2) = 2 \qquad d(c_1, a_3) = 3$$
$$d(c_1, b_1) = 3 \qquad d(c_1, b_2) = 2 \qquad d(c_1, b_3) = 3$$



View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_3)$ to be determined. The points a_2 and c_1 ensure that this distance is 3. So it suffices to show that the factors $(a_1a_2Bc_1)$ and $(a_2a_3Bc_1)$ embed isometrically into Γ.

The factor $(a_1a_2Bc_1)$*:*

View this as a 2-point amalgamation problem with the distance $d(a_2, c_1)$ to be determined. The point a_1 ensures that this distance is 2. So it suffices to show that the subfactors (a_1a_2B) and (a_1Bc_1) embed isometrically into Γ.

The subfactor (a_1a_2B) is afforded by Lemma 5.13.

For the subfactor (a_1Bc_1), adjoin a point c_2 with

$$d(c_2, a_1) = d(c_2, b_1) = d(c_2, b_3) = 2$$
$$d(c_2, c_1) = d(c_2, b_2) = 1$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(c_1, b_2)$ to be determined. The point c_2 ensures that this distance is 2. So it suffices to show that the subfactors (a_1Bc_2) and $(a_1b_1b_3c_1c_2)$ embed isometrically into Γ.

Relative to the base point a_1, the configuration (a_1Bc_2) represents a geodesic of type $(1, 1, 2)$ in Γ_3 and a point in Γ_2 adjacent to its midpoint. This is easily obtained.

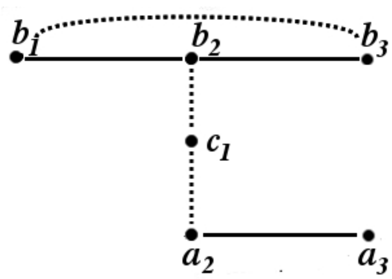
The configuration $(a_1b_1b_3c_1c_2)$ may be viewed as a 2-point amalgamation problem with the distance $d(a_1, c_2)$ to be determined. The point c_1 ensures that this distance is 2. So it suffices to show that the configurations $(a_1b_1b_3c_1)$ and $(b_1b_3c_1c_2)$ embed isometrically into Γ.

The configuration $(a_1b_1b_3c_1) = (a_1c_1) \perp^{(3)} (b_1b_3)$ is afforded by Lemma 5.1.

Relative to he base point c_1, the configuration $(b_1b_3c_1c_2)$ represents a triple of points a mutual distance 2, with two in Γ_3 and one in Γ_1. For this, begin with a point u in Γ_2, and take a neighbor in Γ_1, and two neighbors at distance 2 in Γ_3.

This completes the discussion of the first factor.

The factor $(a_2a_3Bc_1)$*:*



Claim 1. *With* c_1, a_2, a_3 *as specified, there are points* v_2, v_3, v_4 *satisfying*

$$d(v_i, a_j) = 3 \ (i = 2, 3, 4;\ j = 2, 3)$$
$$d(v_i, c_1) = i$$

The configuration $a_2a_3c_1v_3$ is $(v_3) \perp^{(3)} (a_2a_3c_1)$ and is covered by Lemma 4.19.

For $i = 2$ or 4, relative to the base point v_i the configuration $(a_2a_3c_1v_i)$ is a point c_1 in Γ_2 or Γ_4, and a pair of adjacent points a_2, a_3 in $Gamma_3$, with $d(c_1, a_2) = 2$, $d(c_1, a_3) = 3$. Since Γ_3 is connected it suffices to check that the triangles $(a_2c_1v_i)$ and $(a_3c_1v_i)$ of types $(2, 3, i)$ and $(3, 3, i)$ embed in Γ, which is clear.

This proves the claim.

Now we work relative to the points a_2, a_3. We fix u satisfying $d(u, a_2) = 2$ and $d(u, a_3) = 3$. There are points in $\Gamma_3(a_2a_3)$ at distance 2, 3, or 4 from u. By Lemma 5.21 the graph $\Gamma_3(a_2a_3)$ is connected. It follows easily that if $v_2 \in \Gamma_3(a_2, a_3)$ lies at distance 2 from u, we can find v_3, v_4 in $\Gamma_3(a_2, a_3)$ with v_3 adjacent to v_2 and v_4, satisfying $d(u, v_3) = 3$ and $d(u, v_4) = 4$. Now take v_3' in $\Gamma_3(a_2, a_3)$ adjacent to v_2, v_4 and distinct from v_3. Then v_3, v_2, v_3' is a geodesic of type $(1, 1, 2)$ with midpoint v_2, and $(a_2a_3uv_3v_2v_3')$ is the desired configuration.

□

We give an overview of the results proved in this subsection in tabular form.

Summary

Lemma	Hyp.	Conclusion
5.10	$K_1 = 1$; If $d = 4$ then $C > 11$	Constrained (1,2) in $\Gamma_3(u_1, v_2)_1$
"	"	$(1, 1, 1)$, $(1, 1, 2)$ in $\Gamma_3(u_1, u_2)$
5.11	$K_1 > 1$; if $d = 4$ then $C > 11$	$(1, 1, 2)$ in $\Gamma_3(u_1, u_2)$
5.12	$K_1 \leq 3$, $C = 11$, $d = 4$	$(1, 1, 2)$, $(1, 2, 3)$ in $\Gamma_3(u_1, u_2)$
5.13	$K_1 \leq 3$, K^*-type	$(1, 1, 2)$ in $\Gamma_3(u_1, u_2)$
5.14	$K_1 \leq 3$	[3i3]over (1,2,3)
5.15	$K_1 = 1$	(1,1,2)+(123)
5.16	$1 < K_1 \leq 3$	(d=2)+(123)
5.17	$K_1 \leq 3$, K^*-type	(1,1,2)+(123)
5.18	$K_1 = 1$, $K_2 = 4$	(1,1,2) in Γ_4 with $v \in \Gamma_3$ adjacent
5.19	$K_1 = 1$, $K_2 = 4$	$(1, 1, 2) + (112)$
5.20	$K_2 = 3$	$(1, 1, 2) + (112)$
5.21	$d = 1$, K^*-type	$\Gamma_3(u_1, u_2)$ connected
5.22	$K_1 \leq 3$, K^*-type	$(1, 1, 2) + (112)$

6. $\Gamma_3(A)$: Connectedness

6.1. $\Gamma_3(1, 1, 2)$: Connectedness.

Lemma 6.1. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Let A be a geodesic of type $(1, 1, 2)$. Then $\Gamma_3(A)$ is connected*

Proof. It suffices to show that $\Gamma_3(A)$ contains geodesics of type $(1, 1, 2)$, $(1, 2, 3)$, and $(2, 2, 4)$, and Lemmas 5.22 and 5.17 cover the first two. So we need to embed the configuration $A \perp^{(3)} B$ isometrically in Γ, where $A = \{a_1, a_2, a_3\}$ is a geodesic of type $(1, 1, 2)$ with midpoint a_2, and $B = \{b_1, b_2, b_3\}$ is a geodesic of type $(2, 2, 4)$ with midpoint b_2.

Adjoin points c_1, c_2 with

$$d(c_1, a_i) = 1, 2, 1 (i = 1, 2, 3) \qquad d(c_2, a_i) = 1, 2, 1 \ (i = 1, 2, 3)$$
$$d(c_1, b_i) = 2 \ (i = 1, 2, 3) \qquad d(c_2, b_i) = 4, 4, 2$$
$$d(c_1, c_2) = 2$$

View the resulting configuration as an amalgamation problem in which the distances between a_1, a_3 and b_1, b_2 are to be determined. The points c_1, c_2 ensure that these distances are equal to 3. So it suffices to show that the factors $Ab_3c_1c_2$ and $a_2Bc_1c_2$ embed isometrically into Γ.

The factor $Ab_3c_1c_2$:

We adjoin a point c_3 with

$$
\begin{aligned}
&d(c_3,b_3) = d(c_3,c_1) = d(c_3,c_2) = 1 \\
&d(c_3,a_1) = d(c_3,a_3) \qquad\qquad\qquad\qquad = 2 \\
&d(c_3,a_2) = 3
\end{aligned}
$$

View the resulting configuration as an amalgamation problem in which the distances between b_3 and c_1,c_2 are to be determined. The points a_1,c_3 ensure that these distances are equal to 2. So it suffices to show that the subfactors Ab_3c_3 and $Ac_1c_2c_3$ embed isometrically into Γ.

Relative to the base point b_3, the configuration Ab_3c_3 consists of a copy of the geodesic A in Γ_3 together with a point of Γ_1 with respective distances $2,3,2$ from the points of A. This may be obtained as follows.

Claim 1. *There is a triple u_2,u_3,u_4 at mutual distance 2 with $u_i \in \Gamma_i$ for $i = 2,3,4$.*

We adjoin a point c in Γ_3 at distance 1 from u_2,u_3,u_4 and view the resulting configuration as a 2-point amalgamation problem with the distance $d(u_2,u_4)$ to be determined. The point c ensures that this distance is 2. The factors of this amalgamation consist of two adjacent points in Γ_3 together with one further point in Γ_2 or Γ_4 making a triangle of type $(1,1,2)$. As both distances 1 and 2 occur between Γ_3 and either Γ_2 or Γ_4, and Γ_3 is connected, these configurations embed isometrically into Γ. This gives the required configuration (u_2,u_3,u_4).

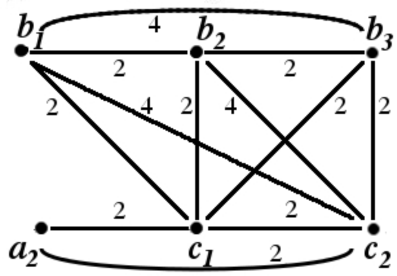
Claim 2. *There is a triple v_1,v_2,v_3, a geodesic of type $(1,2,3)$ with midpoint v_2 and $v_i \in \Gamma_i$ for $i = 1,2,3$.*

Let a be the base point. Relative to the base point v_3, the configuration $(av_1v_2v_3)$ consists of two adjacent points in Γ_3 and a point in Γ_2 making a triangle of type $(1,1,2)$. As both distances 1 and 2 occur between Γ_2 and Γ_3 and Γ_3 is connected, this is easily achieved.

Now fix v_1,v_2,v_3 as in the last claim. Extend v_2,v_3 to a triple v_2,v_3,v_4 at mutual distance 2, with $v_4 \in \Gamma_4$. Now we claim that there are points v,v' adjacent to v_2,v_3,v_4 with $d(v,v') = 2$. To see this, begin with v,v' at distance 2 and apply Fact 1.1.

Now consider the configuration v_1vv_3v'. Here $v_1 \in \Gamma_1$, v,v_3,v' are in Γ_3 forming a geodesic of type $(1,1,2)$ with midpoint v_3, and $d(v_1,v_3) = 3$. As v,v' are adjacent to v_2 and v_3, we find $d(v_1,v) = d(v_1,v') = 2$, as required. Thus we have the configuration Ab_3c_3.

The factor $a_2Bc_1c_2$*:*



Adjoin a point c_3 with

$$d(c_3, b_3) = d(c_3, c_1) = d(c_3, c_2) = 1$$
$$d(c_3, b_1) = d(c_3, b_2) = d(c_3, a_2) = 3$$

View the resulting configuration as the amalgamation of three factors with base $(a_1b_1b_2c_3)$, and with all distances among b_3, c_1, c_2 to be determined. The point c_3 bounds these distances by 2. The points b_1, b_2 then ensure that the distances are equal to 2.

So we must consider separately the factors

$$(1)\ a_2Bc_3;\ (2)\ a_2b_1b_2c_1c_3;\ (3)\ a_2b_1b_2c_2c_3$$

(1) *The subfactor* a_2Bc_3 is $(a_2) \perp^{(3)} Bc_3$, so it suffices to treat Bc_3. We adjoin a point c_4 with

$$d(c_4, b_1) = d(c_4, b_2) = 1$$
$$d(c_4, c_3) = 2$$
$$d(c_4, b_3) = 3$$

We view the resulting configuration as an amalgamation problem in which the distances between b_1 and b_2, c_3 are to be determined. The point c_4, b_3 ensure that these distances are 2 and 3 respectively. So it suffices to show that the configurations $b_1b_3c_4$ and $b_2b_3c_3c_4$ embed isometrically into Γ.

The configuration $b_1b_3c_4$ is a triangle of type $(1, 3, 4)$. For the configuration $b_2b_3c_3c_4$ adjoin a point c_5 adjacent to c_3, c_4 and at distance 2 from b_2, b_3. View $b_2b_3c_3c_4c_5$ as a 2-point amalgamation with the distance $d(c_3, c_4)$ to be determined. The points b_2, c_5 ensure that this distance is 2. The two factors are isomorphic, so we consider only $b_2b_3c_4c_5$: relative to the base point c_4, this represents a pair of vertices at distance 2 in Γ_1, both at distance 2 from a point of Γ_3. This may be obtained by taking a point in Γ_2 and suitable neighbors in Γ_1, Γ_3.

(2) *The subfactor* $(a_2b_1b_2c_1c_3)$*:* adjoin a point c_4 with

$$d(c_4, b_1) = d(c_4, b_2) = d(c_4, c_1) = 1$$
$$d(c_4, c_3) = 2$$
$$d(c_4, a_2) = 3$$

View the resulting configuration as an amalgamation problem with the distances between b_1, b_2 and c_1 to be determined. The points c_3, c_4 ensure that these distances equal 2. So it suffices to show that the configurations

$$(a_2 b_1 b_2 c_3 c_4) \text{ and } (a_2 c_1 c_3 c_4)$$

embed isometrically into Γ.

As $(a_2 b_1 b_2 c_3 c_4) = (a_2) \perp^{(3)} (b_1 b_2 c_3 c_4)$, this configuration reduces to $(b_1, b_2 c_3 c_4)$. Relative to the base point c_3 this represents a point of Γ_2 adjacent to a pair of points in Γ_3 at distance 2, which is known.

And the configuration $(a_2 c_1 c_3 c_4)$ is the same (with base point a_2).

(3) *The subfactor* $(a_2 b_1 b_2 c_2 c_3)$*:* adjoin a point c_4 with

$$\begin{aligned} d(c_4, b_1) &= d(c_4, b_2) = 1 \\ d(c_4, c_3) &= 2 \\ d(c_4, c_2) &= d(c_4, a_2) = 3 \end{aligned}$$

View the resulting configuration as an amalgamation problem with the distances between b_1 and c_2, c_3 to be determined. The points c_2, c_4 ensure that these distances are 2 and 3 respectively. So it suffices to show that the configurations $(a_2 b_1 c_2 c_4)$ and $(a_2 b_2 c_2 c_3 c_4)$ embed isometrically into Γ.

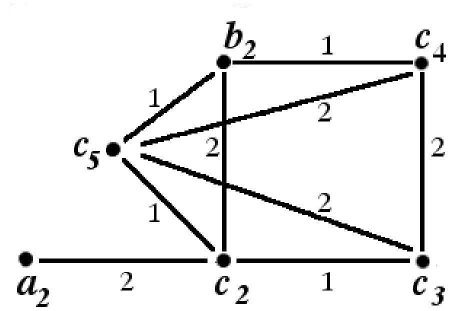
Relative to the base point a_2, the configuration $(a_2 b_1 c_2 c_4)$ consists of a pair of adjacent points in Γ_3, and a point of Γ_2 at distances $3, 4$ from the given points. As the distances $3, 4$ occur between Γ_2 and Γ_3 and Γ_3 is connected, this is easily arranged.

This leaves the configuration

$$(a_2 b_2 c_2 c_3 c_4)$$

for consideration. Adjoin a point c_5 with

$$\begin{aligned} d(c_5, b_2) &= d(c_5, c_2) = 1 \\ d(c_5, c_3) &= d(c_5, c_4) = 2 \\ d(c_5, a_2) &= 3 \end{aligned}$$



View the resulting configuration as a 2-point amalgamation problem with the distance $d(b_2, c_2)$ to be determined. The points c_4, c_5 ensure that this distance is 2. So it suffices to show that the configurations

$$(a_2b_2c_3c_4c_5) \text{ and } (b_2c_2c_3c_4c_5)$$

embed isometrically into Γ.

The configuration $(a_2b + 2c_3c_4c_5)$ is $(a_2) \perp^{(3)} (b_2c_3c_4c_5)$, hence reduces to $(b_2c_3c_4c_5)$. Relative to the base point c_3, the latter consists of a pair of adjacent points in Γ_3 and a point in Γ_2 at distances 1 and 2 from them. Since the distances 1 and 2 occur between Γ_2 and Γ_3, and Γ_3 is connected, this is easily arranged.

Now consider the configuration

$$(b_2c_2c_3c_4c_5)$$

Relative to the base point a_2, this consists of a triple of points in Γ_3 at mutual distance 2 and a point in Γ_2 at distance 2 from two of them at distance 3 from the third.

Take a point $u \in \Gamma_2$, a neighbor v_1 of u in Γ_1, and three neighbors v_2, v_3, v_4 of u in Γ_3 at mutual distance 2. Then the points v_1, v_2, v_3, v_4 are at mutual distance 2.

Claim 3. *Given four points v_1, v_2, v_3, v_4 at mutual distance 2 in Γ there is a point v satisfying*

$$\begin{aligned} &d(v, v_1) = d(v, v_2) = d(v, v_3) = 1 \\ &d(v, v_4) = 3 \end{aligned}$$

Relative to the base point v, we require a point in Γ_3 and three points in Γ_1, so that all four points are at mutual distance 2. We being with the three points in Γ_3, take a common neighbor w in Γ_2, and then a neighbor of w in Γ_1. This proves the claim.

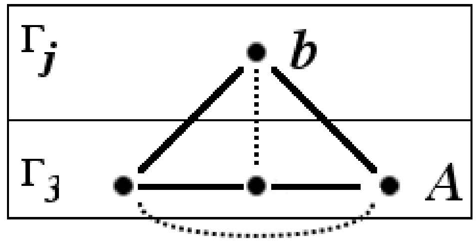
Applying this to our four points v_1, v_2, v_3, v_4 we have v adjacent to v_2, v_3 and at distance 3 from v_4, and $v_2, v_3, v_4 \in \Gamma_3$.

Furthermore as v is adjacent to v_1 and v_2, we have $v \in \Gamma_2$. This is the required configuration. □

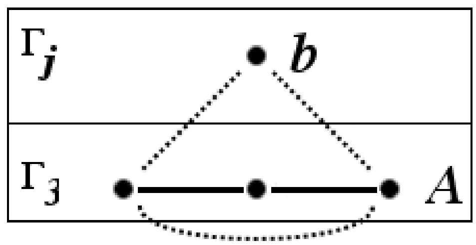
6.2. $\Gamma_3(1,2,3)$**: Connectedness.** Now we turn to $\Gamma_3(1,2,3)$. We first deal with some particular configurations.

Lemma 6.2. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Then the following configurations embed isometrically in Γ.*

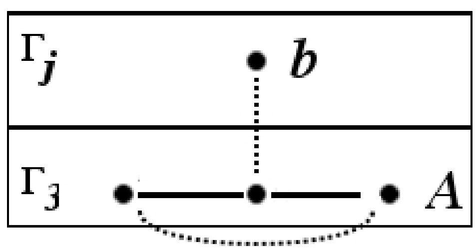
(1) *Ab: A a geodesic of type $(1,1,2)$ in Γ_3, b in Γ_j adjacent to the end-points and at distance 2 from the midpoint; $j = 2$ or 4.*



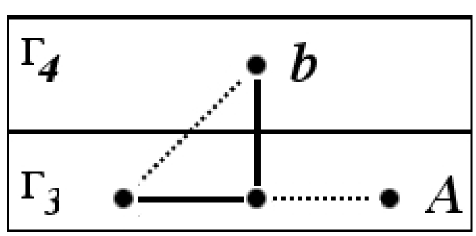
(2) *Ab: A a geodesic of type* $(1,1,2)$ *in* Γ_3*,* $b \in \Gamma_j$ *at distance* 3 *from the midpoint and distance* 2 *from the endpoints;* $j = 1,\ 2$ *or* 4*.*



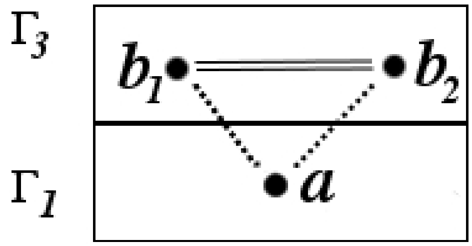
(3) *Ab: A a geodesic of type* $(1,1,2)$ *in* Γ_3*, b a point in* Γ_j *at distance* 2 *from the midpoint and* 3 *from the ends;* $j = 2$ *or* 4*.*



(4) *Ab:* $A = \{a_1, a_2, a_3\}$ *a geodesic of type* $(1,2,3)$ *in* Γ_3*, b a point of* Γ_4 *with distances* $d(b, a_i) = 2, 1, 3$ *respectively.*



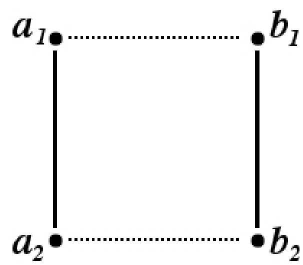
(5) *A point in* Γ_1 *at distance 2 from two points in* Γ_3 *at distance* 4*.*



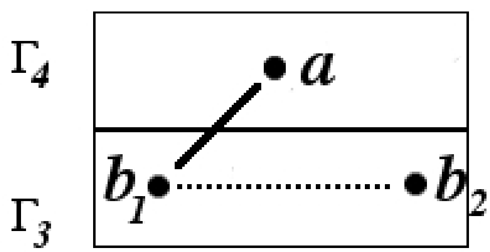
(6) *The configuration* $(a_1 a_2 b_1 b_2)$ *with*

$$d(a_1, a_2) = d(b_1, b_2) = 1$$

$$d(a_i, b_j) = \begin{cases} 2 & (i = j) \\ 3 & (i \neq j) \end{cases}$$



(7) *A point in* Γ_4 *at distance* 1 *and* 3 *from a pair of points of* Γ_3 *at distance* 2.



Proof. We write $A = \{a_1, a_2, a_3\}$ with a_2 the midpoint.
(1):

Adjoin a point c in Γ_3 at distance 2 from the endpoints a_1, a_3 and distance 3 from the midpoint a_2, and adjacent to b. View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, b)$ to be determined. The points a_1, c ensure that this distance is 2. So it suffices to show that the factors Ac and $(a_1 a_3 bc)$ embed isometrically into Γ.

The factor Ac is required in Γ_3, but as Γ_3 satisfies the same hypotheses as Γ it suffices to embed it isometrically into Γ. Relative to the base point a_2, Ac represents a triple of points at mutual distance 2, with two of them in

Γ_1 and one in Γ_3. This configuration may be constructed by taking a point u in Γ_2, and suitable neighbors of u.

The factor a_1a_3bc consists of a point b in Γ_j adjacent to three points of Γ_3 at mutual distance 2. This may be obtained by applying Fact 1.1 to a suitable pair of points b, b' at distance 2.

(2):

Suppose first that

$$j = 2 \text{ or } 4$$

Adjoin a point c in Γ_j adjacent to b, a_1, a_3 and at distance 2 from a_2. View the resulting configuration as an amalgamation problem with the distances between b and a_1, a_3 to be determined. The points a_2, c ensure that these distances are equal to 2. So it suffices to show that the factors Ac and (a_2bc) embed isometrically into Γ.

The factor Ac is covered by (1),

The factor (a_2bc) consists of a point a_2 in Γ_3, an adjacent pair of points b, c in Γ_j, with the distances $2, 3$ between a_2 and b, c. The distances $2, 3$ occur since Γ contains triangles of types $(3,2,2)$, $(3,2,3)$, $(3,4,2)$, $(3,4,3)$. As Γ_j is connected the desired configuration is easily obtained.

Now suppose

$$j = 1$$

Adjoin a point c as above, with c in Γ_2 to reduce to the factors Ac (given by (1)) and a_2bc, where now $b \in \Gamma_1$, $c \in \Gamma_2$, $a_2 \in \Gamma_3$.

Include the base point v_0 to get a configuration of order 4, and view this relative to the base point a_2. We then have a point c in Γ_2 and two adjacent points b, v_0 in Γ_3, with the distances from c to b, v_0 equal to $1, 2$. Since the distances $1, 2$ occur between Γ_2 and Γ_3, and Γ_3 is connected, this configuration may be obtained.

(3): Fix a point u in Γ_j.

The distances $2, 3, 4$ occur between Γ_i and Γ_j as Γ contains triangles of types $(2,3,2)$, $(3,3,2)$, $(4,3,2)$. As Γ_3 is connected we may easily find a triple v_2, v_3, v_4 with $d(u, v_i) = i$ and v_3 adjacent to v_2, v_4. It follows that $d(v_2, v_4) = 2$.

Take v_3' another common neighbor of v_2, v_4 in Γ_3, at distance 2 from v_3. Then we must have $d(u, v_3') = 3$. The configuration $(uv_3v_2v_3')$ is as required.

(4): Adjoin a point c in Γ_3 with

$$d(c, b) = 1$$
$$d(c, a_2) = d(c, a_3) = 2$$
$$d(c, a_1) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, b)$ to be determined. The points a_2 and c ensure that this distance is 2. So it suffices to show that the factors (Ac) and (a_2a_3bc) embed isometrically into Γ.

Now (Ac) is required in Γ_3, but as Γ_3 and Γ satisfy the same conditions it suffices to embed (Ac) in Γ. Relative to the base point a_1, the factor (Ac) consists of a triple of points at mutual distance 2, with two in Γ_3 and one in Γ_1. This may be obtained starting with a point in Γ_2 by taking suitable neighbors.

The factor (a_2a_3bc) is covered by (1).

(5): Let u_1 be the base point, u_2 in Γ_1, and a_1, a_2 the points desired in Γ_3. Adjoin a point c_1 with

$$d(c_1, a_2) = d(c_1, u_2) = 1$$
$$d(c_1, u_1) = 2$$
$$d(c_1, a_1) = 3$$

View the resulting configuration as a 2-point amalgamation problem in which the distance $d(u_2, a_2)$ is to be determined. The points a_1, c_1 ensure that this distance is 2. So it suffices to show that the factors $(a_1a_2u_1c_1)$ and $(a_1u_1u_2c_1)$ embed isometrically into Γ.

The factor $(a_1a_2u_1c_1)$*:*

Adjoin a vertex c_2 with

$$d(c_2, c_1) = 1$$
$$d(c_2, a_1) = d(c_2, a_2) = 2$$
$$d(c_2, u_1) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, c_1)$ to be determined. The points a_2, c_2 ensure that this distance is 3. So it suffices to check that the subfactors $(a_1a_2u_1c_2)$ and $(a_2u_1c_1c_2)$ embed isometrically into Γ.

The subfactor $(a_1a_2u_1c_2) = (a_1a_2c_2) \perp^{(3)} (u_1)$ reduces to the triangle $(a_1a_2c_2)$ of type $(2, 2, 4)$.

Relative to the base point u_1, the subfactor $(a_2u_1c_1c_2)$ consists of a point in Γ_2 adjacent to two points in Γ_3 at distance 2.

The factor $(a_1u_1u_2c_1)$*:*

Relative to the base point a_1, this is a point of Γ_2 adjacent to two points of Γ_3 at distance 2.

This completes the construction of configuration (5).

(6):

Adjoin a point c with

$$d(c, a_2) = d(c, b_2) = 1$$
$$d(c, a_1) = d(c, b_1) = 2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, b_2)$ to be determined. The points a_1 and c ensure that this distance is 2. The two factors $(a_1a_2b_1c)$ and $(a_1b_1b_2c)$ are isomorphic, so it suffices to show that the former emeds isometrically into Γ.

Relative to the base point b_1, the factor $(a_1 a_2 b_1 c)$ consists of a point in Γ_3 adjacent to two points in Γ_2 at distance 2. So this embeds isometrically into Γ.

(7):
Let u be the base point. Adjoin a point c with

$$d(c, b_1) = d(c, b_2) = 1$$
$$d(c, a) = d(c, u) = 2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(u, b_1)$ to be determined. The points a_1 and c ensure that this distance is 3. So it suffices to show that the factors (uab_2c) and (ab_1b_2c) embed isometrically into Γ.

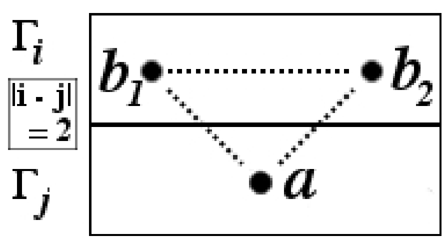
Relative to the base point b_2, the factor (uab_2c) is Configuration (5) above.

The factor (ab_1b_2c) is a geodesic path of length 3.

□

We append something more straightforward which comes up often enough to deserve explicit mention in its own right.

Lemma 6.3. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $C > 10$, and fix $1 \leq i, j \leq 4$ with $j = i \pm 2$. Then there is a triple of points at mutual distance 2 with two in Γ_i and 1 in Γ_j.*



Proof. Take a vertex u in Γ_k where k is between i and j, and suitable neighbors of u in Γ_i and Γ_j.

The main point is that u has two neighbors at distance 2 in Γ_i. This follows from Lemma 1.1 if $i < 4$. If $i = 4$ and $k = 3$ this is given by Lemma 4.18. □

Lemma 6.4. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Let A be a geodesic of type $(1, 2, 3)$. Then $A \perp^{(3)} A$ embeds isometrically in Γ.*

Proof. Label the two copies of A as $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ with midpoints a_2, b_2.

Adjoin two points c_1, c_2 with

$$d(c_1, a_i) = 2, 1, 1 \qquad d(c_2, a_i) = 3, 4, 4$$
$$d(c_1, b_i) = 3, 2, 2 \qquad d(c_2, b_i) = 2, 1, 1$$
$$d(c_1, c_2) = 3$$

View the resulting configuration as an amalgamation problem in which the distances between a_2, a_3 and b_2, b_3 are to be determined. The points c_1, c_2 ensure that all these distances are equal to 3. So it suffices to show that the factors

I. $Ab_1c_1c_2$
II. $a_1Bc_1c_2$

embed isometrically into Γ.

(I): $Ab_1c_1c_2$

View this as a 2-point amalgamation problem with the distance a_2, a_3 to be determined. The points a_1, c_1 ensure that this distance is 2. So it suffices to embed the factors

$$(IA)\text{: } (a_1a_2b_1c_1c_2) \text{ and } (IB)\text{: } (a_1a_3b_1c_1c_2)$$

isometrically into Γ.

(IA): The factor $(a_1a_2b_1c_1c_2)$*:*
Adjoin a point c_3 with

$$d(c_3, b_1) = d(c_3, c_2) = 1$$
$$d(c_3, a_1) = d(c_3, c_1) = 2$$
$$d(c_3, a_2) = 3$$

View the resulting configuration as an amalgamation problem with the distances between c_2 and a_1, c_1 to be determined. The points a_2, c_3 ensure that these distances are equal to 3. So it suffices to show that the subfactors

$$(1)\ (a_1a_2b_1c_1c_3) \text{ and } (2)\ (a_2b_1c_2c_3)$$

embed isometrically into Γ.

(1): $(a_1a_2b_1c_1c_3)$

Relative to the base point b_1, this consists of a point c_3 in Γ_1 and a geodesic $a_1a_2c_1$ of type $(1, 1, 2)$ in Γ_3 as in Lemma 6.2, part (2).

(2): $(a_2b_1c_2c_3)$

Relative to the base point a_2, this consists of a pair of adjacent points b_1, c_3 in Γ_3, and a point c_2 in Γ_4 at distances $1, 2$ from them. As these distances occur and Γ_3 is connected, this is easily obtained.

(IB): The factor $(a_1a_3b_1c_1c_2)$*:*

Adjoin a point c_3 with

$$d(c_3, a_1) = d(c_3, c_1) = 1$$
$$d(c_3, a_3) = d(c_3, b_1) = d(c_3, c_2) = 2$$

View the resulting configuration as an amalgamation problem with the distances between c_1 and a_1, c_2 to be determined. The points a_3, c_3 ensure that these distances will be respectively 2 and 3. So it suffices to show that the subfactors

$$(1)\ (a_1a_3b_1c_2c_3) \text{ and } (2)\ (a_3b_1c_1c_3)$$

embed isometrically into Γ.

(1): $(a_1a_3b_1c_2c_3)$

We adjoin a point c_4 adjacent to b_1, c_2, c_3, a distance 2 from a_1, and at distance 3 from a_3. We view the resulting configuration as a 2-point amalgamation problem with the distance $d(c_2, c_3)$ to be determined. The points a_1, c_4 ensure that this distance is 2. So it suffices to check that the configurations

$$(1a)\ (a_1a_3b_1c_2c_4) \text{ and } (1b)\ (a_1a_3b_1c_3c_4)$$

embed isometrically into Γ.

$(1a)$—$(a_1a_3b_1c_2c_4)$

This is covered by Lemma 6.2, part (4).

$(1b)$—$(a_1a_3b_1c_3c_4)$

This may be viewed as a 2-point amalgamation problem with the distance $d(b_1, c_3)$ to be determined. The points a_1, c_4 ensure that this distance is 2. So it suffices to prove that the configurations $(a_1a_3b_1c_4)$ and $(a_1a_3c_3c_4)$ embed isometrically into Γ.

As $(a_1a_3b_1c_4) = (a_3) \perp^{(3)} (a_1b_1c_4)$, this reduces to $(a_1b_1c_4)$, a geodesic of type $(1, 2, 3)$.

Relative to the base point a_3, the configuration $(a_1a_3b_1c_3)$ consists of a point c_3 in Γ_2 adjacent to two points a_1, c_4 in Γ_3 at distance 2. So this is easily obtained.

(2): $(a_3b_1c_1c_3)$

Relative to the base point b_1, this consists of a point c_3 in Γ_2, and a pair of adjacent points a_3, c_1 in Γ_3, with the distances from c_3 to c_1, a_3 equal to 1 and 2 respectively. As Γ_3 is connected this is easily obtained.

So this concludes the discussion of the factor (I) in our main amalgamation.

(II): $a_1Bc_1c_2$

This may be viewed as a 2-point amalgamation problem with the distance $d(b_2, b_3)$ to be determined. The points b_1, c_2 ensure that this distance is 2.

So it suffices to show that the factors

$$(A)\ (a_1b_1b_2c_1c_2) \text{ and } (B)\ (a_1b_1b_3c_1c_2)$$

embed isometrically into Γ.

(IIA): The factor $(a_1b_1b_2c_1c_2)$:

Relative to the base point a_1, this consists of a point in Γ_2 and a geodesic in Γ_3 of type $(1,1,2)$, with the metric of Lemma 6.2, part (3).

(IIB): The factor $(a_1b_1b_3c_1c_2)$:

Adjoin a point c_3 with

$$\begin{aligned} d(c_3,b_3) &= d(c_3,c_1) = 1\\ d(c_3,b_1) &= d(c_3,c_2) = 2\\ d(c_3,a_1) &= 3 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(b_3,c_1)$ to be determined. The points c_2,c_3 ensure that this distance is 2. So it suffices to show that the subfactors $(a_1b_1b_3c_2c_3)$ and $(a_1b_1c_1c_2c_3)$ embed isometrically into Γ.

The factor $(a_1b_1b_3c_2c_3)$ is $(a_1) \perp^{(3)} (b_1b_3c_2c_3)$ and hence reduces to $(b_1b_3c_2c_3)$. Relative to the base point b_1 this is a point of Γ_3 adjacent toa pair of points in Γ_2 at distance 2, which we have.

For the factor $(a_1b_1c_1c_2c_3)$, adjoin a point c_4 with

$$\begin{aligned} d(c_4,b_1) &= d(c_4,c_2) = d(c_4,c_3) = 1\\ d(c_4,c_1) &= 2\\ d(c_4,a_1) &= 3 \end{aligned}$$

View the resulting configuration as an amalgamation problem with the distances between c_3 and b_1,c_2 to be determined. The points c_1 and c_4 ensure that these distances are equal to 2. So it suffices to show that the configurations

$$(a_1b_1c_1c_2c_4) \text{ and } (a_1c_1c_3c_4)$$

embed isometrically into Γ.

Relative to the base point a_1, the configuration $(a_1b_1c_1c_2c_4)$ consists of a point c_1 in Γ_2 and a geodesic (b_1,c_4,c_2) of type $(1,1,2)$ with the metric of Lemma 6.2, part (3).

Relative to the base point a_1, the configuration $(a_1c_1c_3c_4)$ consists of a pair of adjacent points in Γ_3 and a point in Γ_2 at distances $1,2$ from the given points. As Γ_3 is connected, this is easily obtained.

This completes the construction of the second main factor, and the proof. □

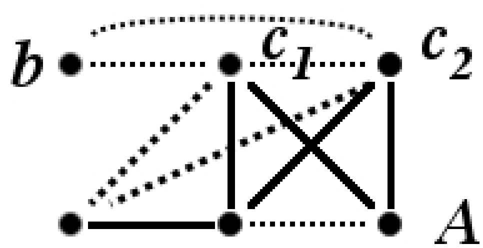
In an amalgamation aimed at constructing $A \perp^{(3)} B$ in which both A and B contain a pair of points at distance 2, one natural way to proceed is by introducing "witnesses" c_1,c_2 to the distances between two such points in A and two such points in B, where c_1 provides paths of type $(1,2,3?)$ from A

to B and c_2 provides paths of type $(1, 4, 3?)$ from A to B; here the question remark refers to the fact that the distances in question are to be forced equal to 3 in the presence of both types of witness.

The next lemma concerns a factor which may occur in such constructions when A is a geodesic of type $(1, 2, 3)$.

Lemma 6.5. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Let $A = \{a_1, a_2, a_3\}$ be a geodesic of type $(1, 2, 3)$ in natural order, and let Abc_1c_2 be an extension with*

$$d(b, a_i) = 3 \;\; (i = 1, 2, 3)$$
$$d(c_1, a_1) = d(c_2, a_1) = 2$$
$$d(c_1, a_i) = d(c_2, a_i) = 1 \;\; (i = 2, 3)$$
$$d(c_i, b) = 2 \;\; (i = 1, 2)$$



Proof. We may view this as a 2-point amalgamation problem with the distance $d(a_2, a_3)$ to be determined. The points a_1, c_1 ensure that this distance is 2. So it suffices to prove that the factors

$$(a_1a_2bc_1c_)\text{ and }(a_1a_3bc_1c_2)$$

embed isometrically into Γ.

The factor $(a_1a_2bc_1c_2)$:
Adjoin a point c_3 with

$$d(c_3, b_1) = d(c_3, c_1) = d(c_3, c_2) = 1$$
$$d(c_3, a_2) = 2$$
$$d(c_3, a_1) = 3$$

View the resulting configuration as an amalgamation problem in which the distances between a_1, b_1 and c_1, c_2 are to be determined. The points a_2, c_3 ensure that all of these distances are equal to 2. So it suffices to show that the subfactors $(a_1a_2b_1c_3)$ and $(a_2c_1c_2c_3)$ embed isometrically into Γ.

Relative to the base point a_1, the subfactor $(a_1a_2b_1c_3)$ consists of a point in Γ_2 at distances $2, 3$ from two adjacent points in Γ_3. This is obtained as usual from the connectedness of Γ_3.

Fact 1.1 affords the configuration $(a_2c_1c_2c_3)$.

The factor $(a_1a_3bc_1c_2)$*:*

Adjoin a point c_3 with

$$d(c_3,a_1)=d(c_3,c_1)=1$$
$$d(c_3,a_3)=2$$
$$d(c_3,b_1)=d(c_3,c_2)=3$$

View the resulting configuration as an amalgamation problem in which the distances between c_1 and a_1,c_2 are to be determined. The points a_3,c_3 ensure that these distances are equal to 2. So it suffices to show that the subfactors

$$(A)\ (a_1a_3b_1c_2c_3) \text{ and } (B)\ (a_3b_1c_1c_3)$$

embed isometrically into Γ.

(A): For the subfactor $(a_1a_3b_1c_2c_3)$, adjoin a point c_4 with

$$d(c_4,a_3)=d(c_4,c_3)=1$$
$$d(c_4,a_1)=d(c_4,b_1)=d(c_4,c_2)=2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_3,c_3)$ to be determined. The points a_1,c_4 ensure that this distance is 2. So it suffices to show that the configurations

$$(A1)\ (a_1a_3b_1c_2c_4) \text{ and } (A2)\ (a_1b_1c_2c_3c_4)$$

embed isometrically into Γ.

$(A1)$: For the configuration $(a_1a_3b_1c_2c_4)$, adjoin a point c_5 with

$$d(c_5,a_1)=d(c_5,c_2)=d(c_5,c_4)=1$$
$$d(c_5,a_3)=2$$
$$d(c_5,b_1)=3$$

View the resulting configuration as an amalgamation problem with the distances between a_1 and c_2,c_4 to be determined. The points a_3,c_5 ensure that these distances are equal to 2. So it suffices to show that the configurations

$$(a_1a_3b_1c_5) \text{ and } (a_3b_1c_2c_4c_5)$$

embed isometrically into Γ.

Now $(a_1a_3b_1c_5)=(b_1)\perp^{(3)}(a_1a_3c_5)$ so this reduces to $(a_1a_3c_5)$, a triangle of type $(1,2,3)$.

Relative to the base point b_1, the configuration $(a_3b_1c_2c_4c_5)$ consists of a pair of points in Γ_2 at distance 2, and a pair of points at distance 2 in Γ_3, the whole forming a complete bipartite graph on four points.

To construct this we first take three points u_1,u_2,u_3 at mutual distance 2 with two in Γ_2 and the third in Γ_4 (taking suitable neighbors of a point in Γ_3), then take a pair of common neighbors v_1,v_2 to u_1,u_2,u_3, at distance 2. Then $(u_1u_2v_1v_2)$ is the required configuration.

To make the extension of u_1, u_2, u_3 by v_1, v_2 it suffices to show that the configuration $(u_1u_2u_3v_1v_2)$ embeds in Γ. This follows by applying Fact 1.1 to v_1, v_2.

So this concludes the discussion of configuration $(A1)$.

$(A2)$: Relative to the base point b_1, the configuration $(a_1b_1c_2c_3c_4)$ consists of a point in Γ_2 and a geodesic of type $(1,1,2)$ in Γ_3, with the metric given in Lemma 6.2, part (2). So this embeds isometrically into Γ.

(B): Relative to the base point b_1, the subfactor

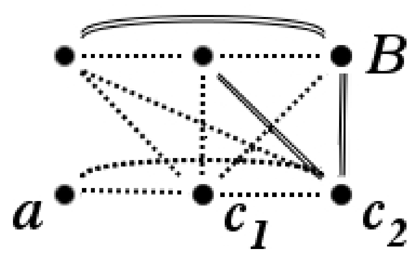
$$(a_3b_1c_1c_3)$$

consists of a point in Γ_2 adjacent to two points of Γ_3 at distance 2, so this embeds isometrically into Γ. □

Now we give the companion factor to the previous one in the case corresponding to the construction of $(1,2,3) \perp^{(3)} (2,2,4)$.

Lemma 6.6. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Let $B = \{b_1, b_2, b_3\}$ be a geodesic of type $(2,2,4)$ in natural order, and let aBc_1c_2 be an extension with*

$$d(a, b_i) = 3 \ (i = 1,2,3)$$
$$d(c_1, b_1) = 2 \qquad d(c_2, a_1) = 2$$
$$d(c_1, a_i) = 2 \ (i = 2,3) \qquad d(c_2, a_i) = 4 \ (i = 2,3)$$
$$d(c_i, a) = 2 \ (i = 1,2)$$



Proof. Adjoin a point c_3 with

$$d(c_3, b_1) = d(c_3, c_1) = d(c_3, c_2) = 1$$
$$d(c_3, a) = d(c_3, b_2) = d(c_3, b_3) = 3$$

View the resulting configuration as an amalgamation of three configurations in which all distances among b_1, c_1, c_2 are to be determined. The points b_2, b_3 ensure that these distances are at least 2, and the point c_3 ensures that these distances are at most 2. So it suffices to show that the three factors

$$(A)\ (aBc_3),\ (B)\ (ab_2b_3c_1c_3),\ \text{and}\ (C)\ (ab_2b_3c_2c_3)$$

all embed isometrically into Γ.

(A) The factor (aBc_3)*:*
This is $(a) \perp^{(3)} (Bc_3)$ and therefore reduces to (Bc_3).
Add a point c_4 to (Bc_3) with

$$d(c_4, b_i) = 1 \ (i = 2, 3)$$
$$d(c_4, c_3) = 2$$
$$d(c_4, b_1) = 3$$

View the resulting configuration as an amalgamation problem in which the distances between b_3 and b_2, c_3 are to be determined. The points b_1, c_4 ensure that these distances are respectively 2 and 3.

So it suffices to show that the subfactors $b_1b_2c_3c_4$ and $(b_1b_3c_4)$ embed isometrically into Γ. The latter is a triangle of type $(1, 3, 4)$, so it suffices to consider

$$(b_1b_2c_3c_4)$$

We add a point c_5 with

$$d(c_5, b_1) = d(c_5, b_2) = 1$$
$$d(c_5, c_3) = d(c_5, c_4) = 2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(b_1, b_2)$ to be determined. The points c_5 and c_3 or c_4 ensure that this distance is 1. So it suffices to show that the two subfactors $(b_1c_3c_4c_5)$ and $(b_2c_3c_4c_5)$ embed isometrically into Γ, and as these are isomorphic it suffices to consider $(b_1c_3c_4c_5)$ alone.

Relative to the base point b_1, the configuration $(b_1c_3c_4c_5)$ consists of three points at mutual distance 2, with two in Γ_1 and one in Γ_3. This may be constructed by taking suitable neighbors of a point in Γ_2.

(B) The factor $(ab_2b_3c_1c_3)$*:*
Adjoin a point c_4 with

$$d(c_4, b_2) = d(c_4, b_3) = d(c_4, c_1) = 1$$
$$d(c_4, c_3) = 2$$
$$d(c_4, a) = 3$$

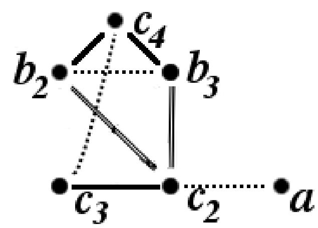
View the resulting configuration as an amalgamation problem in which the distances between c_1 and b_2, b_3 are to be determined. The points c_3, c_4 ensure that these distances are equal to 2. So it suffices to show that the subfactors $(ab_2b_3c_3c_4)$ and $(ac_1c_3c_4)$ embed isometrically into Γ.

The subfactor $(ab_2b_3c_3c_4)$ is $(a) \perp^{(3)} (b_2b_3c_3c_4)$, so reduces to $(b_2b_3c_3c_4)$. Relative to the base point c_3 this is a vertex of Γ_2 adjacent to two vertices of Γ_3 at distance 2.

The subfactor $(ac_1c_3c_4)$ is isomorphic to $(b_2b_3c_3c_4)$ just treated.

(C) The factor $(ab_2b_3c_2c_3)$*:*
Adjoin a point c_4 with

$$d(c_4, b_2) = d(c_4, b_3) = 1$$
$$d(c_4, c_3) = 2$$
$$d(c_4, c_2) = d(c_4, a) = 3$$



View the resulting configuration as an amalgamation problem in which the distances between c_3 and b_2, b_3 are to be determined. The points c_2, c_4 ensure that these distances are equal to 3. So it suffices to show that the factors $(ab_2b_3c_2c_4)$ and $(ac_2c_3c_4)$ embed isometrically into Γ.

The factor $(ab_2b_3c_2c_4)$*:*
Adjoin a point c_5 with

$$d(c_5, c_2) = 1$$
$$d(c_5, c_4) = 2$$
$$d(c_5, b_2) = d(c_5, b_3) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a, c_4)$ to be determined. The points b_2 and c_5 ensure that this distance is 3. So it suffices to check that the subfactors

$$(1)\ (ab_2b_3c_5) \text{ and } (2)\ (ab_2b_3c_4c_5)$$

embed isometrically into Γ.

(1) The subfactor $ab_2b_3c_2c_5$: Adjoin a point c_6 with

$$d(c_6, a) = d(c_6, c_2) = 1$$
$$d(c_6, b_2) = d(c_6, b_3) = d(c_6, c_5) = 3$$

View the resulting configuration as a 2-point amalgamation with the distance $d(a, c_2)$ to be determined. The points c_5, c_6 ensure that this distance is 2. So it suffices to check that the configurations $(ab_2b_3c_5c_6)$ and $(b_2b_3c_2c_5c_6)$ embed isometrically into Γ.

The configuration $(ab_2b_3c_5c_6)$ is $(ac_5c_6) \perp^{(3)} (b_2b_3)$ and is afforded by Lemma 5.17.

Relative to the base point c_2, the configuration $(b_2b_3c_2c_5c_6)$ consists of a pair of points in Γ_1 at distance 2, and another pair in Γ_4 at distance 2, with all distances between them equal to 3. For this, just take adjacent points

u_2, u_3 with $u_2 \in \Gamma_2$, $u_3 \in \Gamma_3$, and suitable neighbors of u_2 in Γ_1, and of u_3 in Γ_4.

This completes the discussion of subfactor (1).

(2) The subfactor $(ab_2b_3c_4c_5)$=$(a) \perp^{(3)} (b_2b_3c_4c_5)$ reduces to $(b_2b_3c_4c_5)$. Relative to the base point c_5, this is a vertex in Γ_2 adjacent to two vertices at distance 2 in Γ_3.

So subfactor (2) occurs.

The factor $(ac_2c_3c_4)$:

Adjoin a point c_5 adjacent to a, c_2 and at distance 2 from c_3, c_4. View the resulting configuration as a 2-point amalgamation problem with the distance $d(a, c_2)$ to be determined. The points c_3 and c_5 ensure that this distance is 2. So it suffices to show that the subfactors $(ac_3c_4c_5)$ and $(c_2c_3c_4c_5)$ embed isometrically into Γ.

Relative to the base point a, the subfactor $(ac_3c_4c_5)$ consists of a point of Γ_1 at distance 2 from two points of Γ_3 at distance 2. This is constructed by taking suitable neighbors of a point in Γ_2.

Relative to the base point c_4, the subfactor $(c_2c_3c_4c_5)$ consists of a point in Γ_3 adjacent to two points of Γ_2 at distance 2. So this also occurs. □

Lemma 6.7. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^*-type with $K_1 \leq 3$ and $C > 10$. Then $\Gamma_3(1,2,3)$ is connected.*

Proof. It suffices to show that $\Gamma_3(1,2,3)$ contains geodesics of types $(1,1,2)$, $(1,2,3)$, and $(2,2,4)$, and the first two are covered by Lemmas 5.17 and 6.4.

So we take $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ geodesics of type $(1,2,3)$ and $(2,2,4)$ respectively, and we must show that $A \perp^{(3)} B$ embeds isometrically into Γ.

We adjoin two points c_1, c_2 with

$$\begin{aligned}
&d(c_1, a_1) = 2 && d(c_2, a_1) = 2\\
&d(c_1, a_i) = 1 \ (i = 2, 3) && d(c_2, a_i) = 1 \ (i = 2, 3)\\
&d(c_1, b_1) = 2 && d(c_2, b_1) = 2\\
&d(c_1, b_i) = 2 \ (i = 2, 3) && d(c_2, b_i) = 4 \ (i = 2, 3)\\
&d(c_1, c_2) = 2 &&
\end{aligned}$$

View the resulting configuration as an amalgamation problem in which the distances between a_2, a_3 and b_2, b_3 are to be determined. The points c_1, c_2 ensure that all these distances are equal to 3.

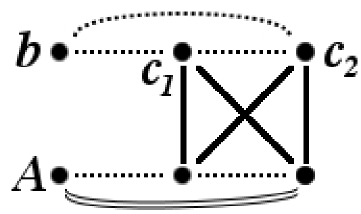
Lemmas 6.5 and 6.6 show that the two factors $(Ab_1c_1c_2)$ and $(a_1Bc_1c_2)$ embed isometrically into Γ. □

6.3. $\Gamma_3(2,2,4)$**: Connectedness.** Now we turn to $\Gamma_3(2,2,4)$, with most of the work already done above.

Lemma 6.8. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and generic type with $K_1 \leq 3$ and $C > 10$. Let $A = (a_1, a_2, a_3)$ be a geodesic* *of type* $(2,2,4)$*, in natural order, and let* Abc_1c_2 *be an extension with*

$$\begin{aligned} d(b,a_i) &= 3 \ (i=1,2,3) \\ d(c_i,a_1) &= 3 \ (i=1,2) \qquad d(c_i,a_j) = 1 \ (i=1,2;\ j=2,3) \\ d(c_i,b) &= 2 \ (i=1,2) \\ d(c_1,c_2) &= 2 \end{aligned}$$

Then Abc_1c_2 *embeds isometrically into* Γ*.*



Proof. View the configuration as a 2-point amalgamation problem with the distance $d(a_2,a_3)$ to be determined. The points a_1,c_1 ensure that this distance is 2. So it suffices to prove that the factors $(a_1a_2bc_1c_2)$ and $(a_1a_3bc_1c_2)$ embed isometrically into Γ.

The factor $(a_1a_2bc_1c_2)$*:*
Adjoin a point c_3 with

$$\begin{aligned} d(c_3,a_1) &= d(c_3,a_2) = 1 \\ d(c_3,b_1) &= d(c_3,c_1) = d(c_3,c_2) = 2 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1,a_2)$ to be determined. The points c_1,c_3 ensure that this distance is 2. So it suffices to prove that the subfactors $(a_1bc_1c_2c_3)$ and $(a_2bc_1c_2c_3)$ embed isometrically into Γ.

Relative to the base point a_1, the subfactor $(a_1bc_1c_2c_3)$ consists of a point in Γ_1 at distance 2 from three points in Γ_3 at mutual distance 2. This is easily obtained by taking suitable neighbors of a point in Γ_2.

Relative to the base point b, the subfactor $(a_2bc_1c_2c_3)$ consists of a point in Γ_3 adjacent to three points in Γ_2 at mutual distance 2, which is easily obtained from Fact 1.1.

The factor $(a_1a_3bc_1c_2)$*:*
Adjoin a point c_3 with

$$\begin{aligned} d(c_3,b) &= d(c_3,c_1) = d(c_3,c_2) = 1 \\ d(c_3,a_1) &= d(c_3,a_3) = 2 \end{aligned}$$

View the resulting configuration as an amalgamation problem with the distances between a_1,b and c_1,c_2 to be determined. The points a_3 and c_3 ensure that the distances $d(a_1,c_i)$ are equal to 3 and the distances $d(b,c_i)$ are equal to 2. So it suffices to show that the subfactors $(a_1a_3bc_3)$ and $(a_3c_1c_2c_3)$ embed isometrically into Γ.

Relative to the base point b, the subfactor $(a_1a_3bc_3)$ is a point in Γ_1 at distance 2 from a pair of points in Γ_3 at distance 4. This is covered by Lemma 6.2, part (5).

The subfactor $(a_3c_1c_2c_3)$ is given by Fact 1.1.

□

Lemma 6.9. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Then $\Gamma_3(2,2,4)$ is connected.*

Proof. It suffices to show that $\Gamma_3(2,2,4)$ contains geodesics of types $(1,1,2)$, $(1,2,3)$, and $(2,2,4)$. The first two follow from Lemmas 6.1 and 6.7.

So it suffices to show that with A, B geodesics of type $(2,2,4)$, the sum $A \perp^{(3)} B$ embeds isometrically into Γ.

We take $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in natural order, and we adjoin vertices c_1, c_2 with

$$
\begin{array}{lll}
d(c_1,a_1) = 3 & d(c_2,a_1) = 3 & \\
d(c_1,a_i) = 1 \ (i=2,3) & d(c_2,a_i) = 1 \ (i=2,3) d(c_1,b_1) & = 3 d(c_2,b_1) \quad = 3 \\
d(c_1,b_i) = 2 \ (i=2,3) & d(c_2,b_i) = 4 \ (i=2,3) d(c_1,c_2) & = 2
\end{array}
$$

We view the resulting configuration as an amalgamation problem in which the distances between a_2, a_3 and b_2, b_3 are to be determined. The points c_1, c_2 ensure that these distances are equal to 3. So it suffices to show that the factors $Ab_1c_1c_2$ and $a_1Bc_1c_2$ embed isometrically into Γ.

The factor $(Ab_1c_1c_2)$ is covered by Lemma 6.8.

The factor $(a_1Bc_1c_2) = (a_1) \perp^{(3)} Bc_1c_2$ reduces to (Bc_1c_2), which is contained in the configuration given by Lemma 6.6. □

7. Γ_2 when $K_1 \leq 2$

For the treatment of the case $K_1 \leq 2$ it will be convenient to prepare some information about the structure of Γ_2.

Lemma 7.1. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Then Γ_2 is a primitive infinite metrically homogeneous graph of diameter* 4.

Proof. Γ_2 is primitive by Lemma 1.5 and the hypothesis $K_1 \leq 2$, that is, Γ_2 contains an edge. The diameter is clearly 4.

As Γ contains an infinite set of points at mutual distance 2, so does Γ_2. In particular Γ_2 is infinite. □

Lemma 7.2. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. If $\tilde{C}$ is the parameter corresponding to C in Γ_2, then*

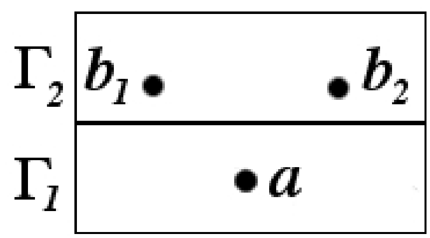
$$\tilde{C} > 10$$

Proof. As Γ_2 is primitive, $C \geq 10$. It suffices to show that a triangle of type $(2,4,4)$ embeds into Γ_2.

Let $B = (b_1b_2b_3)$ be a triangle of type $(2,4,4)$ with $d(b_1,b_3) = 2$. Let aB be the configuration with B in $\Gamma_2(a)$. Relative to the base point b_2, aB consists of a three vertices at mutual distance 2, with two in Γ_4 and one in Γ_2. This is a standard configuration. □

7.1. $\tilde{K}_1 = K_1$.

Lemma 7.3. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let A be a triple of points mutually at distance 3. Then there is an isometric embedding of A into Γ with one point in Γ_1 and two points in Γ_2.*



Proof. Let $A = (a_1a_2a_3)$ where a_1 is to go into Γ_1, let u be the base point, and adjoin a point c with

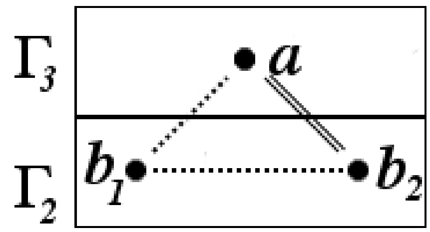
$$d(c,u) = d(c,a_2) = 1$$
$$d(c,a_1) = 2$$
$$d(c,a_3) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(u,a_2)$ to be determined. The points a_3, c ensure that this distance is 2. So it suffices to show that the factors (ua_1a_3c) and Ac embed isometrically into Γ.

Relative to the base point a_3, the factor (ua_1a_3c) consists of a point in Γ_2 adjacent to two points in Γ_3 at distance 2, which is available.

The factor $(Ac) = (a_3) \perp^{(3)} (a_1a_2c)$ with (a_1a_2c) a geodesic. □

Lemma 7.4. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let A be a a geodesic of type $(2,2,4)$. Then there is an isometric embedding of A into Γ with two points at distance 2 in Γ_2 and the third point in Γ_3.*



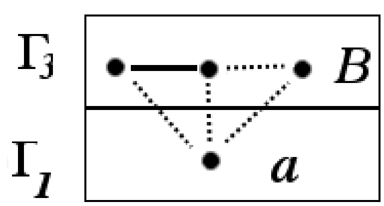
Proof. Let u be the base point, let $A = (a_1, a_2, a_3)$ in natural order (so a_1, a_2 are to go into Γ_2), and adjoin a point c satisfying

$$d(c, u) = d(c, a_1) = d(c, a_2)$$
$$d(c, a_3) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_2)$ to be determined. Then the points a_3 and c ensure that this distance is 2. So it suffices to prove that the factors (ua_1a_3c) and (ua_2a_3c) embed isometrically into Γ.

Relative to the base point a_3, the two factors consist of a pair of adjacent points in Γ_3 with a point either in Γ_2 or in Γ_4 at distances 1 and 2 from them. As Γ_3 is connected and the distance 1 occurs between Γ_3 and either Γ_2 or Γ_4, these configurations are available. □

Lemma 7.5. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Let $B = (a_1a_2a_3)$ be a geodesic of type $(1, 2, 3)$ in natural order and let a satisfy $d(a, b) = 2$ for $b \in B$. Then the configuration aB embeds isometrically in Γ with a in Γ_1 and B in Γ_3.*



Proof. Let u be the base point and adjoin a point c_1 with

$$d(c_1, a) = d(c_1, b_2) = d(c_1, b_3) = 1$$
$$d(c_1, b_1) = d(c_1, u) = 2$$

View the resulting configuration as an amalgamation problem with three factors, with the distances among the points a, b_2, b_3 to be determined. The points u, b_1 ensure that these distances are all at least 2, and then c_1 ensures that these distances are equal to 2. So it suffices to prove that the three factors

$$(uab_1c_1),\ (ub_1b_2c_1),\ (ub_1b_3c_1)$$

embed isometrically into Γ.

Relative to the base point b_1 the factor (uab_1c_1) consists of a pair of adjacent points in Γ_2 with a point in Γ_3 at distances 1 and 2. As Γ_2 is connected and the distances $1, 2$ are represented, this is available.

Relative to the base point u the factor $(ub_1b_2c_1)$ consists of a pair of adjacent points in Γ_3 with a point in Γ_2 at distances 1 and 2, which is similarly available.

The third factor $(ub_1b_3c_1)$ is given by Lemma **??**. □

Lemma 7.6. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 = 1$ and $C > 10$. Let $\tilde{K}_1$ be the corresponding parameter for the graph Γ_2. Then*

$$\tilde{K}_1 = 1$$

Proof. Let $B = (b_1, b_2, b_3)$ be a triangle of type $(1, 1, 1)$. We must embed the configuration aB into Γ, where $d(a, b) = 2$ for $b \in B$.

Adjoin points c_1, c_2 with

$$d(c_1, a) = 1 \qquad d(c_2, a)d = (c_2, b_2) = d(c_2, a_3) = 1$$
$$d(c_2, b_1) = d(c_2, b_2) = 2$$
$$d(c_1, b_i) = 3 \ (i = 1, 2, 3)$$
$$d(c_1, c_2) = 2$$

View the resulting configuration as an amalgamation problem in which the distances between a and b_2, b_3 are to be determined. The points c_1, c_2 ensure that these distances are equal to 2. So it suffices to show that the factors $(ab_1c_1c_2)$ and (Bc_1c_2) embed isometrically into Γ.

The factor $(ab_1c_1c_2)$:

Relative to the base point b_1, this consists of two adjacent points in Γ_2 with a point in Γ_3 at distances 1 and 2 from them. As $K_1 = 1$ there is an edge in Γ_2 and thus Γ_2 is a connected graph of diameter 4. So it suffices to show that the distance 1 occurs between Γ_2 and Γ_3, which simply means that Γ contains a geodesic of type $(1, 2, 3)$.

The factor Bc_1c_2:

Adjoin a point c_3 with

$$d(c_3, b_2) = d(c_3) = 1$$
$$d(c_3, b_1) = d(c_3, c_2) = 2$$
$$d(c_3, c_1) = 4$$

View the resulting configuration as an amalgamation problem with the distances between c_1 and b_2, b_3 to be determined. The points c_2, c_3 ensure that these distances are equal to 3. So it suffices to show that the factors Bc_2c_3 and $(b_1c_1c_2c_3)$ embed isometrically into Γ.

Relative to the base point b_1, the factor Bc_2c_3 consists of a triangle free graph $b_2b_3c_2c_3$ embedded in Γ_1. As $K_1 = 1$ and Γ is not exceptional, this configuration is available.

The factor $(b_1c_1c_2c_3)$ is given by Lemma 7.4. □

Lemma 7.7. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 = 2$ and $C > 10$. Let $\tilde{K}_1$ be the corresponding parameter for the graph Γ_2. Then*

$$\tilde{K}_1 = 2$$

Proof. We consider the configuration $aB = (a) \perp^{(2)} B$ with $B = (b_1, b_2, c_3)$ a triangle of type $(1, 2, 2)$, and $d(b_1, b_3) = 1$. We must embed aB isometrically into Γ.

Adjoin a point c with

$$d(c, a) = d(c, b_1) = d(c, b_2) = 1$$
$$d(c, b_3) = 2$$

View the resulting configuration as an amalgamation problem with the distances between b_1 and a, b_2 to be determined. Since $K_1 = 2$ there are no triangles in Γ, so the point c forces these distances to be equal to 2; note that no identifications are possible.

So it suffices to show that the factors (ab_2b_3c) and (b_1b_3c) embed isometrically into Γ. The first of these is a geodesic of type $(1, 1, 2)$ in $\Gamma_2(b_3)$, and the second is a geodesic of type 2. So both of these are available. □

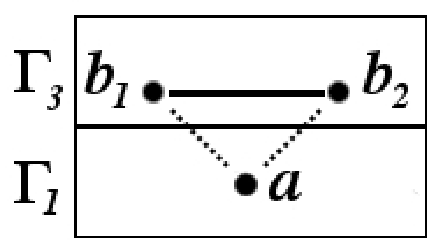
We may sum up as follows.

Lemma 7.8. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Let $\tilde{K}_1$ be the corresponding parameter for the graph Γ_2. Then*

$$\tilde{K}_1 = K_1$$

7.2. $\tilde{K}_2 = K_2$.

Lemma 7.9. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Let $A = (a_1a_2a_3)$ be a triangle of type (122) with $d(a_2, a_3) = 1$. Then A embeds isometrically into Γ with a_1 in Γ_1 and a_2, a_3 in Γ_3.*



Proof. Adjoin a point c in Γ_2 with

$$d(c, a_1) = d(c, a_3) = 1$$
$$d(c, a_2) = 2$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, a_3)$ to be determined. The base point and the point c ensure that this distance is 2. So it suffices to check that the factors (a_1a_2c) and (a_2a_3c) embed isometrically into Γ over the base point.

Taking u as the base point, view the factor (ua_1a_2c) relative to the base point a_3. It then consists of a pair of adjacent points in Γ_2 with a point in Γ_3 at distance 1 and 2. As Γ_2 contains an edge and is connected, and the distance 1 is realized between Γ_2 and Γ_3, this is available.

The factor (a_2a_3c) consists of a pair of adjacent points in Γ_3 with a point in Γ_2 at distance 1 and 2, and is available for similar reasons. □

Lemma 7.10. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \le 2$ and $C > 10$. Let $\tilde{K}_2 = K_2(\Gamma_2)$. Then Then*

$$\tilde{K}_2 = K_2$$

Proof. As Γ_2 is primitive infinite of diameter 4 it satisfies $\tilde{K}_2 \ge 3$. So if $K_2 = 3$ the claim follows. Therefore we suppose $K_2 = 4$, or in other words there is a triangle of type $(1, 4, 4)$ in Γ. Let $B = (b_1, b_2, b_3)$ be such a triangle with $d(b_1, b_3) = 1$ and let a be a point at distance 2 from all $b \in B$. We must embed the configuration aB isometrically into Γ.

Let c_1 be a point with

$$d(c_1, a) = d(c_1, b_2) = 1$$
$$d(c_1, b_i) = 3 \ (i = 1, 3)$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a, b_2)$ to be determined. The points a_1, c_1 ensure that this distance is 2. So it suffices to show that the factors $(ab_1b_3c_1)$ and Bc_1 embed isometrically into Γ.

The factor $(ab_1b_3c_1)$ is afforded by Lemma 7.9.

For the factor Bc_1, adjoin a point c_2 with

$$d(c_2, c_1) = 1$$
$$d(c_2, b_2) = 2$$
$$d(c_2, b_i) = 3 \ (i = 1, 3)$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(c_1, b_3)$ to be determined. The points b_2, c_2 ensure that this distance is 3. So it suffices to show that the subfactors Bc_2 and $(b_1b_2c_1c_2)$ embed isometrically into Γ.

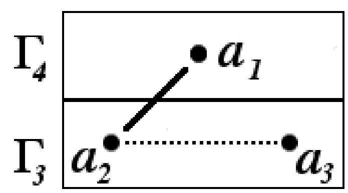
View the subfactor Bc_2 as a 2-point amalgamation problem with the distance $d(c_2, b_1)$ to be determined, with factors two triangles, of types $(1, 4, 4)$ and $(2, 2, 4)$. The points b_2, b_3 ensure that this distance is either 2 or 3. If it is 3 then we have the required subfactor, and it if is 2 then we have an isometric copy of aB.

Relative to the base point b_1, the subfactor $(b_1b_2c_1c_2)$ consists of two adjacent points in Γ_3 and a point in Γ_4 at distance 1 and 2 from them. As Γ_3

is connected and the distance 1 occurs between Γ_3 and Γ_4, this configuration is available. □

7.3. $\tilde{C} = C$ **and** $\tilde{C'} = C'$.

Lemma 7.11. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let $A = (a_1, a_2, a_3)$ be a geodesic of type $(1, 2, 3)$. Then there is an isometric embedding of A into Γ with a_1 in Γ_4 and a_2, a_3 in Γ_3.*



Proof. Let u be the base point and adjoin a point c with

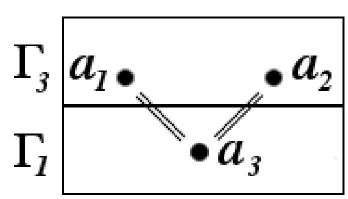
$$d(c, a_2) = d(c, a_3) = 1$$
$$d(c, a_1) = d(c, u) = 2$$

View the resulting configuration as an amalgamation problem with the distances between the points a_2 and u, a_3 to be determined. The points a_1, c_1 ensure that these distances are 3 and 2 respectively. So it suffices to show that the factors $(ua_1a_3c_1)$ and $(a_1a_2c_1)$ embed isometrically into Γ.

Relative to the base point a_3, the factor $(ua_1a_3c_1)$ is the configuration given in Lemma 6.2, part (5).

The factor $(a_1a_2c_1)$ is a geodesic of type $(1, 1, 2)$. □

Lemma 7.12. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 11$. Let $A = (a_1, a_2, a_3)$ be a triangle of type $(3, 4, 4)$ with $d(a_1, a_3) = 3$. Then there is an isometric embedding of A into Γ with a_2 in Γ_1 and a_1, a_3 in Γ_3.*



Proof. Let u be the base point. Adjoin a point c_1 with

$$d(c_1, a_3) = 1$$
$$d(c_1, u) = d(c_1, a_1) = 2$$
$$d(c_1, a_2) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(u, a_3)$ to be determined. The points a_2 and c_1 ensure that this distance is 3. So it suffices to show that the configurations Ac_1 and $(a_1a_2uc_1)$ embed isometrically into Γ.

The factor Ac_1:

Adjoin a point c_2 with

$$d(c_2, a_2) = d(c_2, c_1) = 1$$
$$d(c_2, a_3) = 3$$
$$d(c_2, a_1) = 4$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(c_1, a_2)$ to be determined. The points a_3, c_2 ensure that this distance is 3. So it suffices to show that the subfactors

$$Ac_2 \text{ and } (a_1a_3c_1c_2)$$

embed isometrically into Γ.

We view the subfactor Ac_2 as a 2-point amalgamation problem with the distance $d(a_1, c_2)$ to be determined; here the factors are triangles of types $(1, 3, 4)$ and $(3, 4, 4)$, which we have by hypothesis. The point a_2 ensures that this distance is 3 or 4. If the distance is 4 we have the desired subfactor. If the distance is 3 then relative to the base point c_2, we have the configuration required for the lemma.

Relative to the base point a_1, the subfactor $(a_1a_3c_1c_2)$ is the configuration of Lemma 7.11. □

Lemma 7.13. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 11$. Let $\tilde{C} = C(\Gamma_2)$. Then $\tilde{C} > 11$.*

Proof. We require the configuration $(a) \perp^{(2)} B$ with $B = (b_1, b_2, b_3)$ a triangle of type $(3, 4, 4)$ and $d(b_1, b_3) = 3$.

Adjoin a point c with

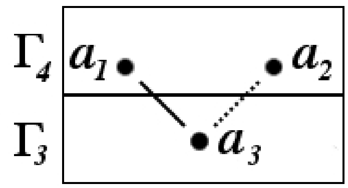
$$d(c, a) = d(c, b_2) = 1$$
$$d(c, b_i) = 3 \ (i = 1, 3)$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a, b_2)$ to be determined. The points b_1, c ensure that this distance is 2. So it suffices to show that the factors $(ab_1b_3c_1)$ and Bc_1 embed isometrically into Γ.

Relative to the base point a, the factor $(ab_1b_3c_1)$ is the configuration of Lemma 7.3.

Relative to the base point c_1, the factor Bc_1 is the configuration of Lemma 7.11. □

Lemma 7.14. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 11$. Let $A = (a_1, a_2, a_3)$ be a geodesic of type $(1, 2, 3)$ in natural order. Then there is an isometric embedding of A into Γ with a_1, a_3 in Γ_4 and a_2 in Γ_3.*



Proof. Let u be the base point. Adjoin a point c with

$$d(c, a_2) = d(c, a_3) = 1$$
$$d(c, a_2) = 2$$
$$d(c, u) = 4$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_2, a_3)$ to be determined. The points a_1, c ensure that this distance is 2. So it suffices to show that the factors (ua_1a_2c) and (ua_1a_3c) embed isometrically into Γ.

Relative to the base point u the factor (ua_1a_2c) consists a point in Γ_3 adjacent to two points of Γ_4 at distance 2, given by Lemma 4.18.

View the factor (ua_1a_3c) as a 2-point amalgamation problem with the distance $d(c, u)$ to be determined; the factors of this are triangles of types $(1, 2, 3)$ and $(3, 4, 4)$, which we have by hypothesis. The point a_3 ensures that this distance is either 3 or 4. If the distance is 4 then we have the required factor, while if it is 3 we have the configuration required for the lemma. □

Lemma 7.15. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Suppose that Γ contains a triangle of type $(4, 4, 4)$. Then Γ_2 contains a triangle of type $(4, 4, 4)$.*

Proof. Let aB consist of a triangle $B = (b_1, b_2, b_3)$ of type $(4, 4, 4)$ and a point a with $d(a, b) = 2$ for $b \in B$. We must embed aB isometrically into Γ.

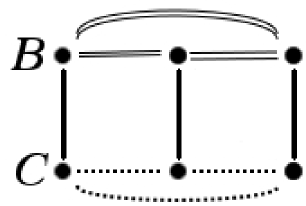
Adjoin points c_1, c_2, c_3 with

$$d(c_i, a) = d(c_i, b_i) = 1 \ (i = 1, 2, 3)$$
$$d(c_i, b_j) = 3 \ (i, j = 1, 2, 3 \text{ distinct})$$
$$d(c_i, c_j) = 2 \ (i, j = 1, 2, 3 \text{ distinct})$$

View the resulting configuration as an amalgamation problem in which the distances between a and B are to be determined. The points c_i ensure that these distances are all bounded by 2 and then the structure of B ensures that these distances are all equal to 2. Writing $C = (c_1c_2c_3)$, it suffices to prove that the factors aC and BC embed isometrically into Γ.

The factor aC embeds into Γ since Γ_1 contains an infinite set of points at mutual distance 2. So we turn to the factor

$$BC$$



View BC as an amalgamation problem with the distances between c_1 and b_2, b_3 to be determined. The points b_1, c_2, c_3 ensure that these distances are both equal to 3. So it suffices to show that the factors

$$Bc_2c_3 \text{ and } b_1C$$

embed isometrically into Γ.

The factor Bc_2c_3:

View this as a 2-point amalgamation problem with the distance $d(c_2, b_3)$ to be determined. The points b_2, c_3 ensure that this distance is 3. So it suffices to show that the subfactors Bc_3 and $(b_1b_2c_2c_3)$ embed isometrically into Γ.

The subfactor Bc_3: first we view this as a 2-point amalgamation problem with the distance $d(b_2, c_3)$ to be determined. The point b_3 ensures that this distance is at least 3. If it is 3, we have the desired configuration. So suppose that it is 4. Then the resulting configuration contains triangles of type $(3, 4, 4)$ and $(1, 4, 4)$.

In this case, adjoin a point c_4 with

$$\begin{aligned} d(c_4, b_2) &= 1 \\ d(c_4, c_3) &= 2 \\ d(c_4, b_3) &= 3 \\ d(c_4, b_1) &= 4 \end{aligned}$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(b_2, c_3)$ to be determined. The points b_3 and c_4 ensure that this distance is 3. So it suffices to check that the configurations Bc_4 and $(b_1b_3c_3c_4)$ embed isometrically into Γ.

The configuration Bc_4 may be viewed as a 2-point amalgamation problem with the distance $d(b_1, c_4)$ to be determined; the factors here are triangles of types $(4, 4, 4)$ and $(1, 4, 4)$, which under our current assumptions are available. The point b_2 ensures that this distance is at least 3. If this distance is 3 then we have a configuration isometric to Bc_3 and we may conclude. If this distance is 4 then we have the required configuration Bc_4.

This disposes of the configuration Bc_3. We have also the configuration $(b_1b_3c_3c_4)$ to deal with. This is covered by Lemma 7.14, bearing in mind that we currently suppose Γ contains a triangle of type $(3, 4, 4)$.

The subfactor $(b_1b_2c_2c_3)$: this is covered by Lemma 7.11.

The factor b_1C:

Relative to the base point b_1, this is the configuration of Lemma 6.3.

□

We may summarize this subsection as follows.

Lemma 7.16. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Let $\tilde{C}_\epsilon = C_\epsilon(\Gamma_2)$ for $\epsilon = 0, 1$. Then*

$$\tilde{C}_\epsilon = C_\epsilon$$

Proof. The value of C_0 or C_1 is determined by the presence or absence of triangles of type $(4, 4, 4)$ or $(3, 4, 4)$ respectively. Thus Lemmas 7.13 and 7.15 settle the issue. □

7.4. **Summary.** The main results of this section that any triangle of a specified type which embeds in Γ also embeds in Γ_2. These are tabulated below.

Type	Lemma	Type	Lemma
(224)	7.2	(144)	7.10
(111)	7.6	(344)	7.13
(122)	7.7	(444)	7.15

All together we have proved the following.

Lemma 7.17. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Then Γ_2 is a primitive metrically homogeneous graph of diameter 4 with the same numerical parameters K_1, K_2, C_0, C_1.*

7.5. $\tilde{\mathcal{S}} = \mathcal{S}$.

Lemma 7.18. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 2$ and $C > 10$. Suppose that Γ contains a clique of order n. Then Γ_2 contains a clique of order n.*

Proof. As $K_1 \leq 2$ this holds if $n \leq 2$, so suppose

$$n \geq 3$$

By Lemma 1.1 Γ_2 contains a copy of Γ_1.

If Γ contains a clique of order $n+1$ then Γ_1 contains a clique of order n and hence Γ_2 contains a clique of order n. So we may suppose

$$\Gamma \text{ contains no clique of order } n+1$$

Let A be a clique of order n and Ab an extension with $d(a, b) = 2$ for $a \in A$. Fix $a_1 \in A$. Let C be a clique of order $n-2$ with

$$d(c, a_1) = d(c, b) = 1 \ (c \in C)$$
$$d(c, a) = 2 \ (a \in A,\ a \neq a_1)$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a_1, b)$ to be determined. The clique C forces this distance to be at most 2 and as there is no clique of order $n+1$, C also forces the distance to be greater than 1. Thus this distance must be 2. So it suffices to show that the factors

$$AC \text{ and } A'bC \text{ with } A' = A \setminus \{a_1\}$$

embed isometrically into Γ.

The factor AC:

Relative to the base point a_1, the factor AC is the graph $A'C$ embedded in Γ_1. As $A'C$ contains no clique of order n, it embeds in Γ_1.

The factor $A'bC$:

This is a graph containing no clique of order n. So it embeds in Γ_1 and a fortiori into Γ. □

⋮

8. Structure of $\Gamma_3(A)$

For A a geodesic of type $(1,1,2)$, $(1,2,3)$, or $(2,2,4)$, we know that $\Gamma_3(A)$ is a connected metrically homogeneous graph. (The same then follows for A of type $(1,3,4)$ but it will not be necessary to consider this case.)

We need to show that this graph has the same parameters $K_1, K_2, C_0, C_1, \mathcal{S}$ as Γ. Until that is proved, we will write $\tilde{K}_1, \tilde{K}_2$ and so on for the parameters associated to whichever graph $\tilde{\Gamma}$, of the form $\Gamma_3(A)$, is under consideration.

We assume Γ to be of K^* type throughout, so that if $\Gamma_3(A)$ does not have the same parameters as Γ, then it is of known type.

Lemma 8.1. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let A be a geodesic triangle of type $(1,1,2)$, $(1,2,3)$, or $(2,2,4)$, and $\tilde{\Gamma} = \Gamma_3(A)$. Then $\tilde{\Gamma}$ is of generic type and the associated parameters $\tilde{K}_1, \tilde{C}$ satisfy*

$$\tilde{K}_1 \leq 3,\ \tilde{C} > 10$$

In particular, $\tilde{\Gamma}$ is primitive.

Proof. Let B be an set of n points at distance 3, with n arbitrary. Then by repeated use of Lemma 4.19, $A \perp^{(3)} B$ embeds isometrically into Γ, so B embeds into $\Gamma_3(A)$. Thus $\Gamma_3(A)$ is infinite and satisfies $\tilde{C} > 9$, and is not bipartite.

As $\Gamma_3(A)$ is infinite and has finite diameter greater than 3, the classification of exceptional metrically homogeneous graphs says that $\Gamma_3(A)$ is of generic type. As $\tilde{C} > 9$, $\Gamma_3(A)$ is not of antipodal type. Thus $\Gamma_3(A)$ is primitive.

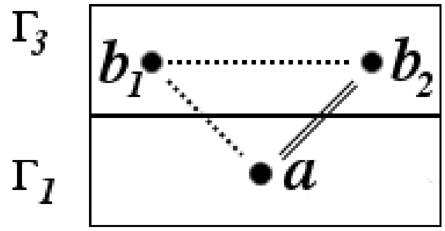
Therefore it will suffice to prove that $\Gamma_3(A)$ contains triangles of types $(3,3,1)$ and $(3,3,4)$.

Let B be a triangle of type $(3,3,i)$ with $i \leq 4$. We will show that $A \perp^{(3)} B$ embeds isometrically into Γ. Write $B = (b) \perp^{(3)} E$ with E a pair of points at distance i. Then by Lemma 4.19 it suffices to show that $A \perp^{(3)} E$ embeds isometrically into Γ. But we know already that $A \perp^{(3)} B'$ embeds isometrically into Γ with B' a geodesic of type $(1,2,3)$ or $(2,2,4)$, and these geodesics contain all possible types of pairs E. So the result follows. ☐

8.1. **Preparation.**

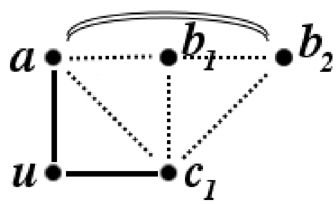
This section is devoted to specific configurations needed in the sequel, or thought to be needed.

Lemma 8.2. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Then a geodesic of type $(2,2,4)$ may be embedded isometrically into Γ with one endpoint in Γ_1 and the other two points in Γ_3.*



Proof. Let u be the base point. Adjoin a point c_1 with

$$d(c_1, u) = 1$$
$$d(c_1, a) = d(c_1, b_i) = 2 \ (i = 1, 2)$$



View the resulting configuration as a 2-point amalgamation problem with the distance $d(u, b_2)$ to be determined. The points a, c_1 ensure that this distance is 3. So it suffices to show that the factors uab_1c_1 and $(a_1b_1b_2c_1)$ embed isometrically into Γ.

The factor uab_1c_1*:*

Relative to the base point a_2, this is a pair of vertices at distance 2 in Γ_2 with a common neighbor in Γ_3, which we have.

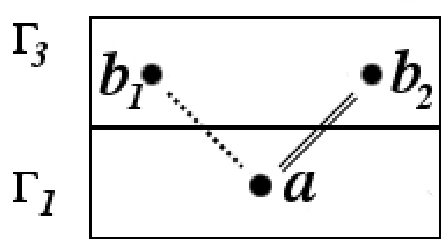
The factor $(ab_1b_2c_1)$*:*

This is contained in the configuration treated in Lemma 6.5. □

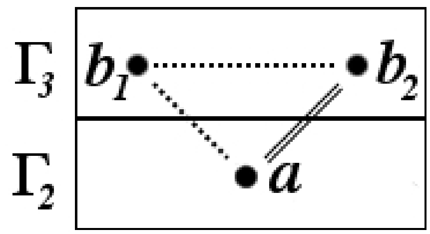
In the next lemma we continue the numbering from Lemma 6.2.

Lemma 8.3. *Let* Γ *be a primitive metrically homogeneous graph of diameter* 4 *and* K^* *type with* $K_1 \leq 3$ *and* $C > 10$*. Then the following configurations embed isometrically into* Γ*.*

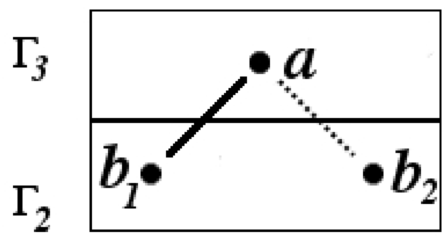
(8) *A triangle of type* $(2, 3, 4)$*, with the edge of length* 3 *in* Γ_3 *and the third point in* Γ_1*.*



(9) *geodesic of type* $(2, 2, 4)$*, with one endpoint in* Γ_2 *and the other two points in* Γ_3*.*



(10) *A geodesic of type* $(1, 2, 3)$*, with the edge of length* 3 *in* Γ_2 *and the third point in* Γ_3*.*



Proof.

(8): Let u be the base point and adjoin a point c with

$$d(c, a) = d(c, b_1) = 1$$
$$d(c, u) = 2$$
$$d(c, b_2) = 3$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(a, b_1)$ to be determined. The points u, c ensure that this distance is 2. So it suffices to show that the factors (uab_2c) and $(ub_1b - 2c)$ embed isometrically into Γ.

Relative to the base point b_2, the factor (uab_2c) consists of a vertex of Γ_4 adjacent to two vertices of Γ_3 at distance 2. This is available.

The factor $ub_1b_2c = (b_2) \perp^{(3)} ub_1c$ reduces to the geodesic ub_1c.

(9):

Let u be the base point and adjoin a point c with

$$d(c, a) = d(c, b_1) = 1$$
$$d(c, b_2) = d(c, u) = 3$$

View the resulting configuration as a 2-point amalgamation problem with $d(a, b_1)$ to be determined. The points b_2, c ensure that this distance is 2. So it suffices to show that the factors (uab_2c) and (ub_1b_2c) embed isometrically into Γ.

Relative to the base point c, the factor (uab_2c) is the configuration (8).

The factor $(ub_1b_2c) = u \perp^{(3)} b_1b_2c$ reduces to the geodesic (b_1b_2c).

(10):
Adjoin a point c in Γ_3 with

$$d(c,b_1)=1$$
$$d(c,a)=2$$
$$d(c,b_2)=4$$

View the resulting configuration as a 2-point amalgamation problem with the distance $d(b_1,b_2)$ to be determined. The points a,c ensure that this distance is 3. So it suffices to show that the factors (ab_1c) and (ab_2c) embed isometrically into Γ (over the base point).

The factor (ab_1c) consists of a point in Γ_2 adjacent to two points of Γ_3 at distance 2. This is available.

The factor (ab_2c) is (9). □

8.2. $\tilde{K}_1 = K_1$, $\tilde{K}_2 = K_2$. We take A to be a geodesic of type $(1,1,2)$, $(1,2,3)$, or $(2,2,4)$, and we aim to show that in $\Gamma_3(A)$ we have $\tilde{K}_1 = K_1$ and $\tilde{K}_2 = K_2$.

Lemma 8.4. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 = 1$ and $C > 10$. Let A be a geodesic triangle of type $(1,1,2)$, and $\tilde{\Gamma} = \Gamma(A)$. Then*

$$\tilde{K}_1 = 1$$

Proof. We let $A = (a_1,a_2,a_3)$ in natural order, B a geodesic of type $(1,1,1)$. Our goal is to embed $A \perp^{(2)} B$ isometrically into Γ.

Let Ab_1b_2 be the extension of A by points b_1,b_2 satisfying

$$d(b_1,a_2)=d(b_1,b_2)=2$$
$$d(b_2,a_2)=4$$
$$d(b_i,a_j)=3\ (i=1,2;\ j=1,3)$$

Suppose that we can embed this configuration isometrically into Γ. Let $\Gamma' = \Gamma_3(a_1,a_3)$. Then we have vertices $b_1,b_2 \in \Gamma'$ at distance 2 and 4 from a_2, respectively.

By Lemma 5.21, Γ' is a connected metrically homogeneous graph of diameter 4, and by Lemma 4.19 Γ' is infinite. It follows that Γ' is of generic type. By Lemma 1.1 the points b_1,b_2 have a geodesic C of type $(1,1,2)$ among their common neighbors in Γ'. Then C must lie in $\Gamma_3(A)$ and our claim will follow.

So we now turn to the construction of Ab_1b_2. We adjoin a point c_1 with

$$d(c_1,a_1)=d(c_1,a_3)=1$$
$$d(c_1,a_2)=d(c_1,b_1)=d(c_1,b_2)=2$$

We view the resulting configuration as an amalgamation problem with the distances between b_2 and a_1,a_3 to be determined. The points a_2,c_1 ensure that these distances are equal to 3. So it suffices to show that the factors (Ab_1c_1) and $(a_2b_1b_2c_1)$ embed isometrically into Γ.

The factor (Ab_1c_1)*:*

Adjoin a point c_2 with

$$d(c_2,a_1)=d(c_2,a_3)=1$$
$$d(c_2,a_2)=d(c_2,c_1)=2$$
$$d(c_2,b_1)=4$$

View the resulting configuration as an amalgamation problem with the distances between b_1 and a_1,a_3 to be determined. The points a_2 and c_2 ensure that these distances are equal to 3. So it suffices to show that the factors (Ac_1c_2) and $(a_2b_1c_1c_2)$ embed isometrically into Γ.

Relative to the base point a_1, the factor Ac_1c_2 is a graph without triangles embedded into Γ_1. So this is available.

The factor $(a_2b_1c_1c_2)$ is contained in the configuration of Lemma 6.6.

The factor (Ab_1c_1)*:*

Again, this is contained in the configuration of Lemma 6.6. □

⋮

9. Direct Sums

$\vdots$

9.1. **The $(2,3)$-Embedding Principle.**

Lemma 9.1. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and generic type with $K_1 \leq 3$ and $C > 10$. Then any finite $(2,3)$-space A embeds isometrically into Γ.*

Proof. For each pair u, v in A with $d(u,v) = 2$ attach witnesses $b_1 = b_1(u,v)$ and $b_2 = b_2(u,v)$ with the following metric.

$$
\begin{aligned}
d(b_1, u) &= 1 & & & d(b_2, u) &= 1 \\
d(b_1, v) &= 1 & & & d(b_2, v) &= 3 \\
d(b_1(u_1,v_1), b_1(u_2,v_2)) &= 2 & &\text{if } u_1, v_1 \text{ meets } \{u_2, v_2\} \\
d(b_1(u_1,v_1), b_2(u_2,v_2)) &= 2 & &\text{if } u_2 \in \{u_1, v_1\} \\
d(b_2(u_1,v_1), b_2(u_2,v_2)) &= 2 & &\text{if } u_1 = u_2 d(b_i, x) & &= 3 \text{ otherwise}
\end{aligned}
$$

Note that this gives a Γ-constrained configuration as triangles of type $(3,3,3)$ are allowed.

We may view our configuration as a 2-point amalgamation problem in which one of pairs at distance 2 in A has its distance determined by the corresponding witnesses, and pass to the corresponding factors. Passing to another such pair in such a factor and continuing, we arrive eventually at factors of the form $A_0 \cup B$ where A_0 is a set of points in A which are pairwise at distance 3, and B is the full set of witnesses adjoined at the outset.

At this stage, each $b \in B$ is at distance 1 from at most one of the vertices of A_0. So set $B_a = \{b \in B \mid d(b,a) = 1\}$ for $a \in A_0$.

Now $A_0 \cup B$ is the $\perp^3$-sum of all the sets $\{a\} \cup B_a$, together with the residue $B \setminus \bigcup_a B_a$. By Lemma 4.12 it suffices to show that the factors B_a and the residue $B \setminus \bigcup_a B_a$ embed isometrically into Γ.

Here B_a may be thought of as a base point together with a set of points at mutual distance 2 in Γ_1. So this embeds isometrically in Γ. The residue breaks up into even simpler components: it is the 3-direct sum of sets of points at mutual distance 2. □

10. Temporary Documentation

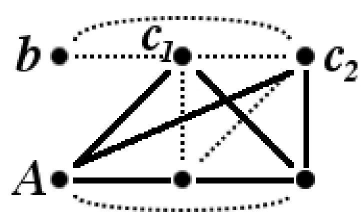
In this section we tabulate some of the useful configurations that have been dealt with. First we give a reserve of constructions that may not be needed.

10.1. **Workspace ...** Configurations that may not be needed, and for which the proofs may not have been worked out either. (I.e., issues that seemed to be on the main path but have not yet materialized.)

Lemma 10.1. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let $A = (a_1, a_2, a_3)$ be a geodesic of type $(1,1,2)$ in natural order and bc_1c_2 a triangle of type $(2,2,2)$ with*

$$d(c_i, a_j) = 1 \ (i = 1,2;\ j = 1,3)$$
$$d(c_i, a_2) = 2 \ (i = 1,2)$$
$$d(b, a_i) = 3 \ (i = 1,2,3)$$

Then Abc_1c_2 embeds isometrically into Γ.



Proof. Adjoin a point c_3 with

$$d(c_3, b) = d(c_3, c_1) = d(c_3, c_2) = 1$$
$$d(c_3, a_1) = d(c_3, a_2) = 2$$
$$d(c_3, a_2) = 3$$

View the resulting configuration as an amalgamation problem with the distances between a_2, b and c_1, c_2 to be determined. The points a_1, c_3 ensure that these distances are equal to 2. So it suffices to show that the subfactors

$$(Abc_3) \text{ and } (a_1a_2c_1c_2c_3)$$

embed isometrically into Γ.

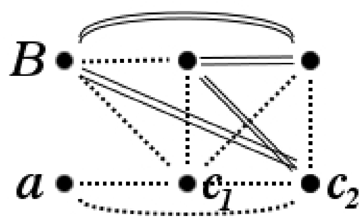
Relative to the base point b the subfactor (Abc_3) consists of the configuration of Lemma 6.2 part (2).

The configuration $(a_1a_3c_1c_2c_3)$ is obtained by applying Lemma 1.1 to the points c_1, c_2. □

Lemma 10.2. *Let Γ be a primitive metrically homogeneous graph of diameter 4 and K^* type with $K_1 \leq 3$ and $C > 10$. Let $B = (b_1, b_2, b_3)$ be a triangle* *of type* $(2, 4, 4)$ *with* $d(b_1, b_2) = 2$ *and let* ac_1c_2 *be a triangle of type* $(2, 2, 2)$ *with*

$$d(c_1, b_i) = 2 \ (i = 1, 2, 3) \qquad d(c_2, b_i) = 4 \ (i = 1, 2)$$
$$d(c_2, b_3) = 2$$
$$d(a, b_i) = 3 \ (i = 1, 2, 3)$$

Then ac_1c_2B *embeds isometrically into* Γ.



Proof. Adjoin a point c_3 with

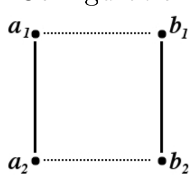
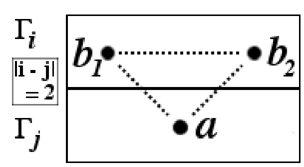
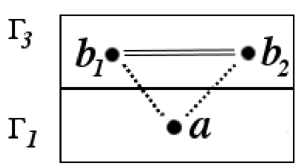
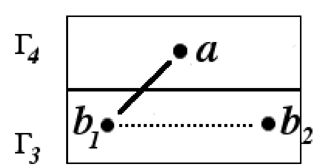
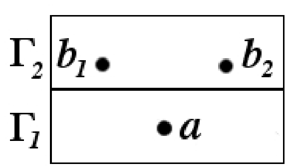
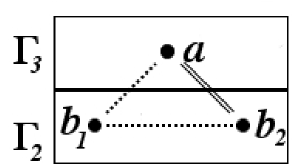
$$d(c_3, b_1) = d(c_3, b_2) = d(c_3, c_1) = 1$$
$$d(c_3, a) = d(c_3, c_2) = d(c_3, b_3) = 3$$

View the resulting configuration as an amalgamation problem in which the distances between c_1 and b_1, b_2 are to be determined.

Rest omitted as may not be needed and seems to involve some auxiliary configurations we have not yet documented.

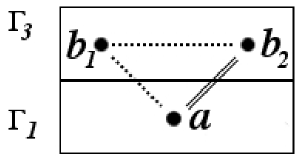
□

10.2. **Table of General Configurations.** Here the assumptions are $C > 10$ and $K_1 \leq 3$.

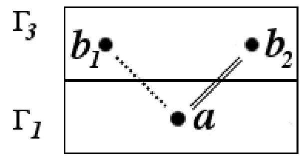
Lemma	Configuration	Lemma	Configuration
6.2 (6)		6.3	
6.2 (5)		6.2 (7)	
7.3		7.4	

Lemma — Configuration

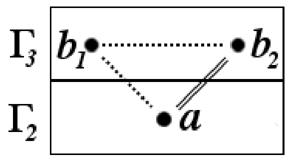
8.2



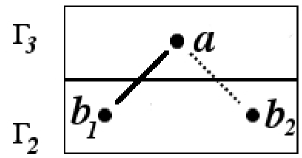
8.3 (8)



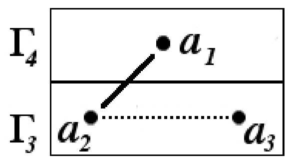
8.3 (9)



8.3 (10)

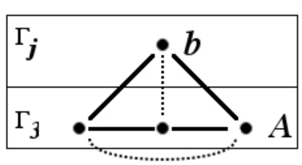


7.11

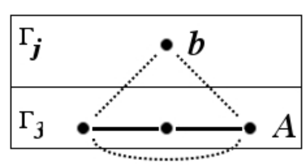


Lemma — Configuration

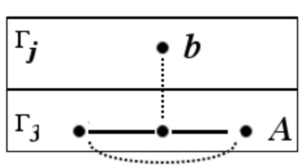
6.2 (1)



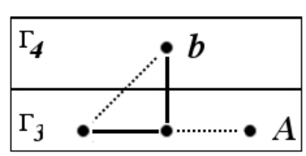
6.2 (2)

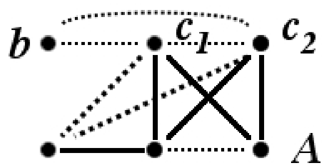
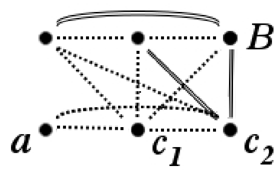
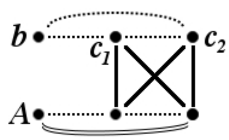
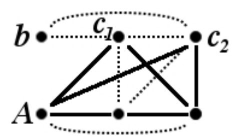
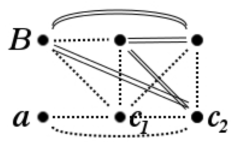


6.2 (3)



6.2 (4)



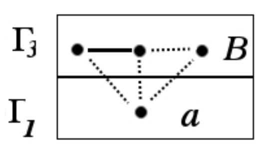
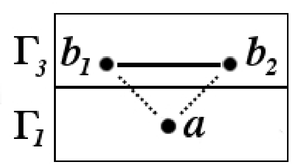
Lemma	Configuration	Lemma	Configuration
6.5		6.6	
6.8		10.1	
10.2			

10.3. **Table of Configurations, Special Cases.** Here we make additional assumptions.

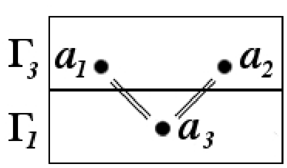
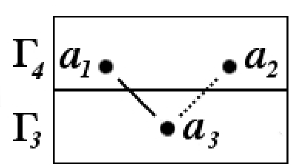
$K_1 = 1$

Lemma	Configuration	Lemma	Configuration

$K_1 \leq 2$

Lemma	Configuration	Lemma	Configuration
7.5		7.9	

$C > 11$

Lemma	Configuration	Lemma	Configuration
7.12		??	

References

[ACM13] D. Amato, G. Cherlin, and H. D. Macpherson, "Metrically homogeneous graphs of diameter 3," Preprint, September 2013.

[Che98] G. Cherlin, "The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments," *Mem. Amer. Math. Soc.* **131** (1998), no. 621,
xiv+161 pp.

[Che11] G. Cherlin, "Two problems on homogeneous structures, revisited," *in* **Model Theoretic Methods in Finite Combinatorics,** M. Grohe and J. A. Makowsky eds., Contemporary Mathematics **558**, American Mathematical Society, 2011

[Che13] G. Cherlin "Metrically Homogeneous Graphs: A Catalog." Preprint, 2013.

[Fra54] R. Fraïssé, "Sur l'extension aux relations de quelques propriétés des ordres," *Ann. Ecole Normale Sup.* **7** (1954), 361–388.

[GGK96] M. Goldstern, R. Grossberg, M. Kojman, "Infinite homogeneous bipartite graphs with unequal sides" *Discrete Math.* **149** (1996), 69–82.

[Hen71] C. W. Henson, "A family of countable homogeneous graphs," *Pacific J. Math.* **38** (1971), 69–83.

[LW80] A. Lachlan and R. Woodrow, "Countable ultrahomogeneous undirected graphs," *Trans. Amer. Math. Soc.* **262** (1980), 51-94.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854 USA

E-mail address: `cherlin@math.rutgers.edu`