Finite Structures with Few Types: Remarks

1 Around 2.4.8

There is a gap in the presentation before Lemma 2.4.8 (which prepares for 2.5.2), which is also rather ill suited to the context in which it finds itself, as the notion of "orthogonality" in force here is not very flexible.

What is missing is Lemma 2 from the article by Hrushovski in the proceedings of the Banff conference: *Finite and Infinite Combinatorics in Sets and Logic*, Nato ASI Series C Volume 411, 1993, pp. 175-187:

Lemma 2 [slightly rephrased:] Let $J_1, \ldots, J_n$ be fully embedded 0-definable normal geometries, orthogonal in pairs. Then $J_1 \cup \ldots \cup J_n$ is stably embedded as a model theoretic disjoint union, in other words without any additional structure.

(NB: This reduces at once to the case $n = 3$.)

The "proof" of 2.4.8 assumes that this result applies in the 0-definable case and deals only with a comparatively minor issue of dependence on parameters. Without something like Lemma 2 (or a notion of orthogonality that incorporates this lemma), it is not comprehensible.

Thanks to Marko Djordjević for pointing out the inaccuracy.

2 Around 2.4.8, again.

Hrushovski sketches a line of argument proving the following in general:

(∗) If P, Q are orthogonal 0-definable sets in a structure M, each stably embedded, then $P \cup Q$ is stably embedded.

"Orthogonal" is taken to mean that over $\mathrm{acl}(\emptyset)$ there is no nontrivial relation between P and Q.

The idea of the proof is to arrive at an ∞-definable group interpreted in both P and Q in two different incarnations; to derive from this a canonical map between their conjugacy classes (really: conjugacy classes of n-tuples); and to derive a purely group-theoretic contradiction from this, which is a strong form of $\aleph_0$-categoricity.

Sketch proof: Suppose on the contrary that relative to some parameter c there is a new relation between P and Q. Passing to imaginary elements, we may suppose it is a bijection f_c between an infinite P-definable set U, and an infinite Q-definable set V.

We can work over a base set $B \subseteq P^{\mathrm{eq}} \cup Q^{\mathrm{eq}}$ such that $\mathrm{dcl}(Bc) \cap [P^{\mathrm{eq}} \cup Q^{\mathrm{eq}}] \subseteq \mathrm{dcl}(B)$; then $\mathrm{tp}(c/B)$ determines $\mathrm{tp}(c/BP)$ and $\mathrm{tp}(c/BQ)$.

Let R be the locus of $\mathrm{tp}(c/B)$. For $c, d \in R$ the map $g = f_d^{-1} f_c : U \to U$ is conded by an element $e = e(c, d) \in P^{\mathrm{eq}}$; write $g_e = g$.

Consider $R_2 = \{e(c_1, c_2) : c_1, c_2 \in R\}$. If $c_1, c_2 \in R$, $e_1 = e(c_1, d_1) \in R_2$, and $c_1' \in R$, then $\mathrm{tp}(c_1 e/B) = \mathrm{tp}(c_1' e/B)$ and hence there is d_1' with $f_{d_1'}^{-1} f_{c_1} = g_{e'}$. Accordingly R_2 is closed under composition: if $g_1 = f_1^{-1} f_2$ and $g_2 = f_3^{-1} f_4$, we may suppose that $f_2 = f_3$ and then cancel terms. So R_2 is a group.

So we have an ∞-definable infinite group R_2 inP^{eq} (isomorphic to $\mathrm{Aut}(R/[B, P, Q])$). Similarly there is an ∞-definable group R_3 in Q^{eq}.

Fix $c \in R$. This provides a definable isomorphism

$$h_c : R_2 \to R_3$$

Any two such isomorphisms differe by conjugation. Thus there is a canonical map h between conjugacy classes in R_2 and R_3; and the same applies to powers R_2^n, R_3^n. By the orthogonality of P and Q, these two sets are finite, for all n.

So we have a group $G = R_2$ such thatn G^n has finitely many conjugacy classes under the action of G, for each n. In particular:

(!) G is an infinite locally finite group with finitely many conjugacy classes

This however is impossible. G has finitely many normal subgroups and so we may suppose first that G has no normal subgroup of finite index, by passing to a subgroup of finite index, and then that G is simple, by factoring out a maximal normal subgroup.

At this point G is a simple locally finite group of bounded exponent. By Kegel and Wehrfritz and the classification of the finite simple groups, G is linear, and by S. Thomas it is at worst twisted Chevalley, a contradiction.

3 Model theoretic terminology

Free usage is made of some model theoretic terms not defined in the text, notably acl, dcl, and M^{eq}. We meant to include some definition or reference for the essential notions taken over from model theory.

For these particular terms, the definitions are:

acl(A): the union of the finite A-definable sets;

dcl(A): the set of A-definable elements;

M^{eq}: an extension of the model M in which n-tuples are treated as elements of another "sort", and more generally the same applies to equivalence classes of n-tuples modulo 0-definable equivalence relations

A supplementary reference for terminology is Marker, *Model Theory: An Introduction*, mainly Chapter 1.