3

Smooth Approximability

3.1 ENVELOPES

We defined standard systems of geometries at the end of the last section. These provide a framework for the construction of Zilber/Lachlan envelopes.

Definition 3.1.1. *Let $\mathcal{M}$ be Lie coordinatized.*

1. *A* regular expansion *of $\mathcal{M}$ is the structure obtained by adjoining to $\mathcal{M}$ finitely many sorts of $\mathcal{M}^{\mathrm{eq}}$ with the induced structure.*
   Note that a regular expansion of $\mathcal{M}$ is Lie coordinatizable but not literally Lie coordinatized, since the additional sorts are disjoint from the tree structure.
   A regular expansion of $\mathcal{M}$ is adequate *if it contains a copy of each canonical projective which is nonorthogonal to a coordinatizing geometry of $\mathcal{M}$.*
   The remainder of this definition should be applied only to adequate regular expansions of Lie coordinatized structures (as will be seen on inspection of the definition of envelopes, below).
2. *An* approximation *to a geometry of a given type is a finite or countable dimensional geometry of the same type.*
   This includes, of course, a nondegeneracy condition on the bilinear or quadratic forms involved, and in the case of a quadratic geometry, the quadratic part must be present (a symplectic space with Q empty is not an approximate quadratic space), and ω in the finite dimensional case must actually be the Witt defect.
3. *A* dimension function *is a function μ defined on equivalence classes of standard systems of geometries, with values isomorphism types of approximations to canonical projective geometries of the given type. (This is actually determined by a dimension, and the type.)*
   By the usual abuse of notation, we construe these functions as functions whose domain consists of all standard systems.
4. *If μ is a dimension function, then a* μ-envelope *is a subset E satisfying the following three conditions:*

(i) *E is algebraically closed in $\mathcal{M}$ (not $\mathcal{M}^{\mathrm{eq}}$);*

(ii) *For $c \in M - E$, there is a standard system of geometries J with domain A and an element $b \in A \cap E$ for which $\operatorname{acl}(E, c) \cap J_b$ properly contains $\operatorname{acl}(E) \cap J_b$;*

(iii) *For J a standard system of geometries defined on A and $b \in A \cap E$, $J_b \cap E$ has the isomorphism type given by $\mu(J)$.*

5. *If μ is a dimension function and E is a μ-envelope we write $\dim_J(E)$ for $\mu(J)$ when E meets the domain of J, and otherwise we write $\dim_J(E) = -1$; in the latter case the value $\mu(J)$ is irrelevant to the structure of E.*

Our goals are existence, finiteness, and homogeneity of envelopes.

Lemma 3.1.2. *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatized structure. Suppose that E is algebraically closed, and satisfies (iii) with respect to the standard system of geometries J. Suppose that J' is an equivalent standard system of geometries and that J, J' are in $\mathcal{M}$ (not just $\mathcal{M}^{\mathrm{eq}}$). Then E satisfies (iii) with respect to J'.*

Proof. We note that as $E \subseteq \mathcal{M}$ it would not make a great deal of sense to attempt to say something substantial about its intersection with a geometry lying partly outside $\mathcal{M}$.

Condition (iii) for J' means that if $b' \in E \cap A'$, where A' is the domain of J', then $E \cap J'_{b'}$ has the structure specified by $\mu(J') = \mu(J)$. The element b' corresponds to an element b of $E \cap A$, with A the domain of J, and there is a 0-definable bijection between $E \cap J_b$ and $E \cap J'_{b'}$ which is an isomorphism of weak unoriented structures. This may involve twisting by a field automorphism or switching the sides of a polar geometry, but does not affect the isomorphism type. If we use canonical orientations, it will preserve them.

Lemma 3.1.3 (Existence). *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatized structure.*

1. Let $E_0 \subseteq \mathcal{M}$ be algebraically closed in $\mathcal{M}$ and suppose that for each standard system of geometries J with domain A and each $b \in E_0 \cap A$, $J_b \cap E_0$ embeds into a structure of the isomorphism type $\mu(J)$. Then E_0 is contained in a μ-envelope.

2. In particular, for any μ, μ-envelopes exist.

Proof. Let $\mathcal{J}$ be a representative set of standard systems of geometries. By the previous lemma it suffices to work with $\mathcal{J}$. We may take E containing E_0 maximal algebraically closed such that

(∗) For $J \in \mathcal{J}$ with domain A, and $b \in E \cap A$,
$J_b \cap E$ embeds into a structure of the type specified by $\mu(J)$.

We need to check both (ii) and (iii) for E.

We begin with (ii). Suppose $c \in M - E$. Let $E' = \operatorname{acl}(E \cup \{c\})$. Then we have some $J \in \mathcal{J}$ with domain A, and some $b \in E' \cap A$, for which $J_b \cap E'$ does not embed into a structure of the type specified by $\mu(J)$. If $b \in A \cap E$ then $J_b \cap E$ does embed in such a structure, and (ii) follows. Now suppose that $b \notin A \cap E$. In this case we show that $J_b \cap E = \emptyset \neq J_b \cap E'$, so that (ii) holds also in this case. As E is definably closed it is a subtree of $\mathcal{M}$ with respect to the coordinatizing tree. As b is not definable over E, J_b is orthogonal to the geometries associated with this tree. Thus by induction over this tree, $\operatorname{acl}(E) \cap J_b = \emptyset$.

We turn to (iii), and we need only concern ourselves here with $J \in \mathcal{J}$. Suppose that J has domain A, and $b \in E \cap A$, and let B be an extension of $J_b \cap E$ inside J_b of the desired isomorphism type $\mu(J)$. Our claim is that $B \subseteq E$. Let $E' = \operatorname{acl}(E \cup B)$. We will argue that E' also has the property $(*)$, so $E' = E$.

If $J' \in \mathcal{J}$ has domain A', and $b' \in A'$, then unless $J' = J$ and $b' = b$, the geometries $J_b, J_{b'}$ are orthogonal and $J_{b'} \cap E' = J_{b'} \cap E$. On the other hand, by Lemma 2.3.3 any element of J_b^{eq} algebraic over E is algebraic over $J_b \cap E$. This applies in particular to any E-definable formula $\varphi(x, y)$ such that for some elements $\mathbf{b} \in B$, $\varphi(x, \mathbf{b})$ isolates an algebraic type over $E \cup B$ in J_b. Thus $J_b \cap E' = J_b \cap \operatorname{acl}((E \cap J_b) \cup B) = B$.

Lemma 3.1.4 (Finiteness). *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatized structure. Suppose that for each standard system of geometries J the dimension function μ is finite. Then every μ-envelope E is finite.*

Proof. E is algebraically closed in $\mathcal{M}$ and hence inherits a coordinatizing tree from $\mathcal{M}$. It suffices, therefore, to check that for any $a \in E$, its successors in the tree form a finite set. We may suppose the successors are of the form $E \cap P_a$ with P_a an a-definable geometry in $\mathcal{M}$, nonorthogonal to some canonical projective geometry J_b with $b \leq a$ in the tree. The size of $J_b \cap E$ is controlled by μ and there is an a-definable bijection between the localization of J_b at $\operatorname{acl}(a) \cap J_b$ and the projective version of P_a, so this goes over to E as well. Thus $E \cap P_a$ is finite.

3.2 HOMOGENEITY

Definition 3.2.1

1. *Let (V, A) be an affine space (a linear space with a regular action) defined over the set B. A is* free *over B if there is no projective geometry J defined over B for which $A \subseteq \operatorname{acl}(B, J)$. An element a, or its type over B, is said to be* affinely isolated *over B if a belongs to the affine component A of an affine space (V, A) defined and free over B.*

   Note: As a copy of V is definable over A in A^{eq}, it can and will be suppressed in this context.

2. *Let A and A be two affine spaces free over B. They are* almost orthogonal *if there is no pair $a \in A, a' \in A'$ with $\operatorname{acl}(a, B) = \operatorname{acl}(a', B)$.*

Lemma 3.2.2 (Uniqueness of Parallel Lines) *Let (V, A), (V', A') be almost orthogonal affine spaces defined and free over the algebraically closed set B, with PV and PV' complete 1-types over B. Let J be a projective geometry defined over B, not of quadratic type, and stably embedded in $\mathcal{M}$. For $a \in A$, $a' \in A'$, and $c \in J - B$, the triple (a, a', c) is algebraically independent over B.*

Proof. We have (V, A), (V', A'), and J all defined over B. Our definitions amount to the hypothesis that the elements (a, a', c) are pairwise independent over B, so if two of these geometries are orthogonal there is nothing to prove. We suppose therefore that they are all nonorthogonal. In particular, the projectivization PV of V can be identified with part of J.

We consider the structure $J \cup A$. For $a \in A$, A is definable over $J \cup \{a\}$ and hence $J \cup A$ is stably embedded in $\mathcal{M}$. As PV can be identified with part of J, $J \cup A$ carries a modular geometry over B.

Now suppose toward a contradiction that $\operatorname{rk}(aa'c/B) = 2$. Take independent conjugates a_1, c_1 of a, c over a'. Then $\operatorname{rk}(aca_1c_1/B) = 3$. This takes place in $J \cup A$, so there is $d \in (J \cup A) - B$ algebraic over acB and a_1c_1B, hence in $\operatorname{acl}(a', B)$. Thus $\operatorname{acl}(dB) = \operatorname{acl}(a'B)$ and either $d \in A$, and A, A' are not almost orthogonal, or $d \in J$, and A' is not free over B.

Lemma 3.2.3. *Let $\mathcal{M}$ be Lie coordinatized. Let A be an affine space defined and free over the algebraically closed set B. Let $B \subseteq B' = \operatorname{acl}(B')$ with B' finite, and let J be a canonical projective geometry associated with A. Assume*

1. *$J \cap B' \subseteq B$;*

2. *$J \cap B$ is nondegenerate (if there is some form or polarity present);*
3. *If J is a quadratic geometry, then its quadratic part Q meets B.*

Then A either meets B', or is free over it.

Proof. We remark that if A does not meet B', A need not remain a geometry over B', but will split into a finite number of affine pregeometries over B'. We will call A free over B' if this applies to each of the associated geometries over B'.

The claim will be proved by induction with respect to the coordinatization of the algebraically closed set B' relative to B, inherited from $\mathcal{M}$. Accordingly we may take $B' = \operatorname{acl}(B, a')$, where a' comes from an affine, quadratic, or projective geometry A' defined over B.

Assume that $A \cap B' = \emptyset$ and some affine part A_0 of A relative to B' is contained in $\operatorname{acl}(B, a', J')$ with $J' = J_{b'}$ projective and defined over B'. As $J' \subseteq \operatorname{acl}(J, b')$ the same applies with J' replaced by J, that is: $A_0 \subseteq \operatorname{acl}(B, a', J)$, while $A \cap \operatorname{acl}(B, J) = \emptyset$. It follows that A' and J are nonorthogonal, and that $A' \cap \operatorname{acl}(B, J) = \emptyset$. In view of (iii) we have A' affine, and easily free over B.

If A and A' are not almost orthogonal over B, then B' meets A. Suppose therefore that A and A' are almost orthogonal over B. Then we will apply the previous lemma. Choose $a \in A$. As $a \in \operatorname{acl}(B, a', J)$, and the geometry of (A, J) is modular, there is $c \in J \cap \operatorname{acl}(Baa')$ with $a \in \operatorname{acl}(Ba'c)$. Then $c \notin B$, and in view of (iii) we may suppose c is not in the quadratic part of J, if there is a quadratic part.

Let J_B be the localization of J over B. By hypothesis (iii) this is not a quadratic geometry. By hypothesis (ii) J is in the algebraic closure of $B \cup J_B$; normally over B, J would break up into a number of pregeometries, at least one ($(J \cap B)^{\perp}$) sitting over the localization, while some of the cosets would be affine pregeometries. However, since $J \cap B$ is nondegenerate, all elements of J lie in translations by elements of B of $(J \cap B)^{\perp}$. Of course, when forms are absent, the situation is trivial.

Replacing c momentarily by an element of J_B having the same algebraic closure over B, we may apply the previous lemma to a, a', c, reaching a contradiction.

Lemma 3.2.4. *Let $\mathcal{M}$ be an adequate regular explansion of a Lie coordinatized structure, let μ be a dimension function, and let E and E' be μ-envelopes. If $A \subseteq E$, $A' \subseteq E'$ are finite, and $f : A \longrightarrow A'$ is elementary in $\mathcal{M}$, then f extends to an elementary map from E to E'. In particular, μ-envelopes are unique, and (taking $E = E'$) homogeneous.*

Proof. It suffices to treat the case in which E and E' are finite, as the existence and finiteness properties then suffice for a back-and-forth argument using finite envelopes. What we must show is that if $A \neq E$ then there is an extension of f to $\operatorname{acl}(A \cup \{b\})$ for some $b \in E - A$. There are essentially two cases, depending on whether we are trying to add a point to the domain coming from a canonical projective geometry, or we are extending to the other points of the envelope. We may suppose A and A' are algebraically closed.

Case 1. There is a standard system of geometries J and an $a \in A$ for which $J_a \cap E$ is not contained in A.

Expand J_a to a basic projective geometry $J^\circ_{a^*}$ defined over $a^* = \operatorname{acl}(a)$. Let L and L' be finite dimensional linear geometries covering $J^\circ_{a^*} \cap E$ and $J^\circ_{fa^*} \cap E'$, respectively. Then L and L' are isomorphic, and their isomorphism type is characterized by its type, dimension, and Witt defect (if applicable).

As f is elementary, it gives a partial isomorphism between some $J_a \cap E$ and $J_{fa} \cap E'$, which lifts to an elementary map between the corresponding parts of L and L'. Let $\hat{f}$ be an extension of f by an isomorphism of L with L'. The existence of such a compatible extension is trivial in the absence of forms and given by Witt's theorem [**Wi**] otherwise, with the exception of the polar and quadratic cases. The polar case is quite straightforward. In the quadratic case one first extends f so that its domain meets Q, and then the problem reduces to the orthogonal case, in other words to Witt's theorem.

By weak elimination of imaginaries and stable embedding, since $A = \operatorname{acl} A$, we find that $\operatorname{tp}(A/L \cap A)$ determines $\operatorname{tp}(A/L)$; similarly, $\operatorname{tp}(A'/L' \cap A')$ determines $\operatorname{tp}(A'/L')$. Implicit in this determination is knowledge of the type of L or L' over $\emptyset$. Since $\hat{f}$ preserves the two relevant types, it preserves $\operatorname{tp}(A/L)$ and is thus elementary.

Case 2. For any standard system of geometries J, and any $a \in A$, $J_a \cap E \subseteq A$.

It follows that the same applies to A'. We extend f to a minimal element a in the coordinatization tree for E, not already in the domain. So the tree predecessor b of a is in A, and a is not algebraic over b. Accordingly a belongs to a geometry J_b which is nonorthogonal to a canonical projective geometry. As we are not in Case 1, J_b is affine, and free over A. If f is extended to $\operatorname{acl}(A) \cap \mathcal{M}^{\mathrm{eq}}$ we may take J_b basic.

In E' we have, correspondingly, J_{fb} affine and free over A'. However, as E' is an envelope, the maximality condition (clause (ii)) implies that J_{fb} cannot be free over E'. Lemma 3.2.3 applies in this situation, to the affine space J_{fb} and the algebraically closed sets A' and E', in view of the hypothesis for Case 2. Thus the conclusion is that J_{fb} meets E'.

We will next find an element a' of $J_{fb} \cap E'$ satisfying the condition

$$(a, \lambda) = (a', f\lambda) \text{ for all } \lambda \in J^*_b \cap A \text{ (the affine dual).}$$

Here one should, strictly speaking, again extend f to the algebraic closure of A in $\mathcal{M}^{\mathrm{eq}}$. We consider a stably embedded canonical projective geometry P associated with J_b. Then P is b-definable and the projectivization of the linear space V_b which acts regularly on J_b is definably isomorphic to one of the sorts of the localization P/b of P at b. By our case assumption $P \cap E$ is as specified by μ and is, in particular, nondegenerate. The same applies to $P' \cap E'$. Thus the action of the definable linear dual of $V_{b'}$ is represented, in its action on $V_{b'} \cap E'$, by elements of A' (or $\mathrm{acl}(A') \cap \mathcal{M}^{\mathrm{eq}}$, more precisely). As E' meets the affine space $J_{b'}$, the same applies to the affine dual. Again by the linear nondegeneracy and the fact that E' meets $J_{b'}$, the specified values for $(a', f\lambda)$ can be realized in $E' \cap J_{b'}$. We extend f by $f(a) = a'$.

Now the type of A over (PV_b, J_b, J_b^*) is determined by its type over its algebraic closure in that geometry, and this applies in particular to the type of A over a. So in order to see that f remains elementary, it suffices to check that a and a' have corresponding types over $A^{\mathrm{eq}} \cap (PV_b, J_b, J_b^*)$ and its f-image; and this is what we have done.

Corollary 3.2.5. *Let $\mathcal{M}$ be an adequate regular expansion of a Lie coordinatized structure. Then a subset E of $\mathcal{M}$ is an envelope if and only if the following conditions are satisfied:*

(i) *E is algebraically closed;*
(ii) *For any $c \in M - E$, there is a projective geometry J defined over E, not quadratic, and an element $c' \in (J \cap \mathrm{acl}(Ec)) - E$;*
(iii) *If $c_1, c_2 \in E$ are conjugate in $\mathcal{M}$ and $D(c_1)$, $D(c_2)$ are corresponding conjugate definable geometries, then $D(c_1) \cap E$ and $D(c_2) \cap E$ are isomorphic.*

This does not depend on a particular coordinatization of $\mathcal{M}$.

3.3 FINITE STRUCTURES

In this part we summarize some useful facts applying to finite geometries and their automorphism groups, notably the result of [**KLM**].

Definition 3.3.1. *A* simple *Lie geometry L is either a weak linear geometry of any type other than polar or quadratic, the projectivization of such a geometry, or the affine or quadratic part of a geometry; in the latter case the "missing," linear part is to be considered as encoded into L^{eq}.*

These do not have the best properties model theoretically, and a polar geometry cannot be recovered at all from a single simple Lie geometry, but apart from this, at the level of $\mathcal{C}^{\mathrm{eq}}$ there is little difference between simple Lie geometries and the geometries considered previously.

Definition 3.3.2

1. *A* coordinatizing structure *of type (e, K) and dimension d is a structure $\mathcal{C}$ with a transitive automorphism group, carrying an equivalence relation E with $e < \infty$ classes, such that each class carries the structure of a simple Lie geometry over the finite field K, of dimension d. (One could include the type of the geometry as well in the type of $\mathcal{C}$.)*
2. *Let $\mathcal{C}$ be a coordinatizing structure of type (e, K) and dimension d, and let τ be the type over the empty set of a finite algebraically closed subset (not sequence) t of $\mathcal{C}$. The* Grassmannian $\Gamma(\mathcal{C}, \tau)$ *is the set of realizations of the type τ in $\mathcal{C}$, with the structure induced by $\mathcal{C}$. It is said to have* type (e, K, τ) *and* dimension d.
3. *Let $\mathcal{C}$ be a coordinatizing structure. $\mathcal{C}$ is* proper *if each equivalence class of $\mathcal{C}$ as a geometry is canonically embedded in $\mathcal{C}$, or equivalently if the automorphism group induced on each class is dense in its automorphism group as a geometry (in the finite dimensional case, dense means equal). If $\mathcal{C}$ is finite dimensional, it is* semi-proper *if the automorphism group of $\mathcal{C}$ induces a subgroup of the automorphism group G of the geometry which contains $G^{(\infty)}$.*
4. *A structure is* primitive *if it has no nontrivial 0-definable equivalence relation.*

Fact 3.3.3 [**KLM**]. *For each k there is n_k such that for any finite primitive structure $\mathcal{M}$ of order at least n_k, if $\mathcal{M}$ has at most k 5-types then $\mathcal{M}$ is isomorphic to a semiproper Grassmannian of type (e, K, τ) with $e, |K|, |\tau| \le k$, where $|\tau|$ has the obvious meaning.*

As noted in the introduction, D. Macpherson found [**Mp2**] that the method of proof of [**KLM**] suffices to prove the same fact with 5 reduced to 4. The statement is quite false for 3.

The next set of facts is standard in content, though not normally phrased precisely as follows.

Fact 3.3.4 [**CaL**].

1. *Let k be an integer. There is a $d = d_k$ such that for any finite basic simple projective Lie geometry L of dimension at least d we have*

   (i) *The socle G of* $\mathrm{Aut}\,(L)$*, is simple and nonabelian, and* $\mathrm{Aut}\,(L)/G$ *is solvable of class at most 2;*

   (ii) *G and* $\mathrm{Aut}\,L$ *have the same orbits on L^k;*

   (iii) *The automorphism group of L* as a weak geometry *coincides with* $\mathrm{Aut}\,G$*. with one exception: if L is a pure vector space then the automorphism group of L is a subgroup of index 2 in* $\mathrm{Aut}\,G$*, and the full group* $\mathrm{Aut}\,G$ *is realized geometrically as the automorphism group of the weak polar geometry (L, L^*).*

2. *If J_1, J_2 are nondegenerate basic projective geometries, not quadratic, of large enough dimension, and their automorphism groups have isomorphic socles, then they are isomorphic as weak geometries.*

Here our policy of leaving the degenerate case to fend for itself may be too lax; but the statement certainly applies also in the context of $\mathrm{Sym}(n)$ and $\mathrm{Aut}\,(n)$.

Remarks 3.3.5

Note that the automorphism groups of the basic geometries are classical groups with no Galois action. In the first statement we ignore 4-dimensional symplectic groups over fields of characteristic 2 and 8-dimensional orthogonal groups of positive Witt defect by taking $d > 8$. The polar geometry implements a "graph automorphism," of the general linear group in any dimension. The graph automorphism of order 2 for Chevalley groups of type D_n is part of the geometric automorphism group. G is usually equal to the commutator subgroup of $\mathrm{Aut}\,L$, with exceptions in the orthogonal case (and a few small exceptions that can be ignored here).

Fact 3.3.6. *For any finite basic simple linear geometry V of dimension at least 5, if $G = (\mathrm{Aut}\,V)^{(\infty)}$ acts on an affine space A over V so as to induce its standard action on V, then either G fixes a point of A or the characteristic is 2, G is the symplectic group operating on its natural module V, and the action of G on A is definably equivalent to its action on Q, the space of quadratic maps on V associated to the given form.*

Proof. Taking any point $a \in A$ as a base point, the function $f(g) = a^g - a$ can be construed as a function from G to V, and is a 1-cocycle. Change of base point gives a cohomologous cocycle. If this cocycle is trivial, it means we may choose the base point so that this cocycle vanishes, and a is a fixed point for the action of G.

Typically the first cohomology group for a (possibly twisted) Chevalley group on its natural module vanishes; see the tables in [**JP**], for example. Rather large counterexamples are associated with exceptional Chevalley groups, but for the classical types ($A - D$, possibly twisted) restricted to dimension greater than 4, the only counterexamples involving natural modules are 1-dimensional cohomology groups for symplectic groups in characteristic 2 (listed twice in [**JP**], once as C_n and once as B_n, since the natural module for the odd dimensional orthogonal groups in even characteristic corresponds to a representation of this group as the symplectic group in one lower dimension). This is the case in which we have Q, or more exactly αQ for $\alpha \in K^\times$. The latter can be thought of most naturally as the space of quadratic forms inducing $\alpha\beta$, where β is the given symplectic form on V, but can also be viewed as the space Q with the action $q \mapsto q + \lambda_v^2$ replaced by $q \mapsto q + \lambda^2_{\alpha^{1/2}v}$.

Thus we can either consider A as isomorphic to Q, by an isomorphism which is not the identity on V, or as definably equivalent to Q over V, holding V fixed and rescaling the regular action on A; our formulation of the result reflects the second alternative.

Remark 3.3.7.

It seems advisable to remember that the "Q," alternative in the preceding statement is in fact αQ for some unique $\alpha \in K$.

Fact 3.3.8 [**CaK**]. *Let G be a subgroup of a classical group acting naturally on a finite basic simple classical projective geometry P, and suppose that G has the same action on P^3 as* $\operatorname{Aut} P$. *Then G contains* $(\operatorname{Aut} P)^{(\infty)}$ *(the iterated derived group).*

This iterated derived group is at worst $(\operatorname{Aut} P)^{(2)}$ and is a simple normal subgroup with solvable quotient.

Remark 3.3.9.

In this connection our general policy of leaving the degenerate case to fend for itself is definitely too lax. A similar statement does apply also in the context of $\operatorname{Sym}(n)$ and $\operatorname{Aut}(n)$, with 6-tuples in place of 3-tuples, but one needs the classification of the finite simple groups to see this.

Fact 3.3.8 is phrased rather differently in [**CaK**], as the result is considerably sharper in more than one respect. Here we ignore low dimensional examples and also invoke a significantly stronger transitivity hypothesis. A somewhat more complete statement of the result of [**CaK**] goes as follows.

Fact 3.3.10 [[]**cf. Theorem IV**]**CaK.** *Let $G \le \Gamma L(n,q)$, $n \ge 3$, and suppose G is 2-transitive on the corresponding projective space. Then either $G \ge \mathrm{SL}(n,q)$ or $G \le \mathrm{SL}(4,2)$.*

Fact 3.3.11 [[]**cf. Theorem IV**]**CaK.** *Let $G \le H = \Gamma Sp(n,q)$, $\Gamma O^\epsilon(n,q)$, or $\Gamma U(n,q)$ with $n > 13$ and suppose that G has the same orbits on 2-dimensional spaces as H. Then $G \ge H^{(\infty)}$.*

Theorem IV of [**CaK**] varies from Fact 3.3.10 in the following respects. The transitivity hypothesis is weaker, amounting to transitivity on pairs consisting of two isotropic nonorthogonal lines. This allows three low dimensional exceptions and two families defined over the field F_2, where G normalizes a classical subgroup with coefficients in F_4, so that G has more than one orbit on totally isotropic planes.

Lemma 3.3.12. *Let H be a normal subgroup of a product*

$$G = G_1 \times \cdots \times G_n$$

such that H projects surjectively onto each product of the form $G_i \times G_j$. Then G/H is nilpotent of class at most $n-2$. In particular, if G is perfect then $G = H$.

Proof. Let σ_i for $1 \le i \le n-1$ be a sequence of elements of G_n and for each i let $\sigma_i^* \in G$ be an element of H which projects onto σ_i in the nth coordinate, and 1 in the i-th coordinate. Then any iterated commutator $\gamma(\sigma_i^*)$ in the elements σ_i^* will project onto $\gamma(\sigma_i)$ in G_n, and 1 in the other coordinates. It follows easily that any iterated commutator of length $n-1$ belongs to H, and our claim follows.

Remark 3.3.13. The proof of Fact 3.3.6 actually involves a great deal of calculation, somewhat disguised by the fact that the reference [**JP**] presents the final outcome in tabular form. A qualitative version of this, sufficient for our purposes, can be obtained by postponing the issue somewhat and making use of our later results. We will indicate this approach.

View (A,V) as a structure by endowing it with all invariant relations. Replacing the bound "5" by "sufficiently large," we may take V to have a nonstandard dimension. If we show that A has either a 0-definable point, or quadratic structure, then the same follows for sufficiently large finite dimensions.

The induced structure on V is that of a standard linear geometry. Let V' be the structure induced on V by (V,A,a) with a a point of A. Note that V' interprets the triple (V,A,a). One cannot expect V' to be stably embedded, in view of the characteristic 2 case, but we still expect

$$(*) \qquad V' \text{ is Lie coordinatized.}$$

Given $(*)$, one deduces Fact 3.3.6 from the theorem on reducts and the recognition lemmas: by Proposition 7.4.4 (V, A) is weakly Lie coordinatized. By Lemma 6.2.11 V is part of a basic linear geometry in this structure, and Proposition 7.1.7 recognizes A.

The theorem on reducts can also be used in the proof of $(*)$ itself. Note that any two unstable linear geometries interpret each other, provided only that the characteristics of the base fields are equal. Once reducts are under control, one can expand the geometry to a polar geometry over a field of size greater than 2. This has the effect of reducing all cases of $(*)$ to the simplest case of Fact 3.3.6, namely the general linear group, which can be handled by a direct argument.

3.4 ORTHOGONALITY REVISITED

For simplicity we will work for some time in a nonstandard extension of the set theoretic universe in which we have infinite integers. This gives a rigorous basis for the treatment of sequences of finite structures of increasing size in terms of one infinitely large structure of integral cardinality. In this context it will be important to distinguish internal and external objects, notably in connection with the languages used, and the supply of automorphisms available.

Definition 3.4.1. *Let $\mathcal{M}$ be an internally finite structure with internal language L_0 in a nonstandard extension of the universe of set theory. Then $\mathcal{M}^*$ is the structure with the same universe, in a language whose atomic relation symbols consist of names for all the relations in finitely many variables defined on $\mathcal{M}$ by L_0-formulas.*

Observe that $\mathcal{M}^*$ is not an element of the nonstandard universe. If $\mathcal{M}$ is a nonstandard finite model of a standard theory T in the language L, then the corresponding language L_0 (normally called L^* in this case) is the language corresponding to L in the nonstandard universe; this has more variables than L (x_n for all integers n, standard or nonstandard), and more importantly, consists of arbitrary internally finite well-formed formulas in its language. This includes formulas with infinitely many (but internally finitely many) free variables; these are discarded in forming the language for $\mathcal{M}^*$, so $\mathcal{M}^*$ is a reduct of $\mathcal{M}$ from the nonstandard language L^*, one which is in general richer than the reduct of $\mathcal{M}$ to the standard language L. For a concrete example, consider a discrete linear order of nonstandard finite length: among the predicates of $\mathcal{M}^*$ one has, for example, the distance predicates $D_n(x, y)$ in two variables, for every n up to the (nonstandard) size of the order. Of course, in this case there are no nontrivial internal automorphisms of $\mathcal{M}$; in fact, there are no nontrivial automorphisms of $\mathcal{M}^*$.

Lemma 3.4.2. *Let $\mathcal{M}$ be an internally finite structure, and J a finite disjoint union of basic 0-definable projective simple Lie geometries with no additional structure. Let G be* $\mathrm{Aut}\, J$ *and let G_1 be* $(\mathrm{Aut}\, J)^{(\infty)}$ *(the iterated derived group), where both* $\mathrm{Aut}\, J$ *and* $\mathrm{Aut}\, J^{(\infty)}$ *are understood internally (the latter coinciding with the internal socle here), and automorphisms are taken with respect to the geometric structure. Let H be the group of automorphisms of J which are induced by internal automorphisms of $\mathcal{M}$. Then J is canonically embedded in $\mathcal{M}^*$ if and only if H contains G_1.*

Proof. Suppose first that J consists of a single projective geometry. J is canonically embedded in $\mathcal{M}^*$ if and only if for each finite n, G and

H have the same orbits on n-tuples in J; applying Fact 3.3.4, part 1(ii), this means that G_1 and H have the same orbits on n-tuples in J. This certainly holds if $H \supseteq G_1$. Conversely if H has the same orbits on J as G, it contains G_1 by Fact 3.3.8.

The argument is similar in the general case, but we must justify further the claim that if H acts on n-tuples of J as does G, then it contains G_1. Arguing inductively, it suffices to show that the pointwise stabilizer of J_1 in H acts on m-tuples from $J_2 \times \cdots \times J_n$ as G_1 does. Let $B \subseteq J_2 \times \cdots \times J_n$ have cardinality m, and let $g \in G_1$. By the argument of the first part, the action of the pointwise stabilizer H_{B^g} on J_1 induces the action of g on J_1. Hence in its action on $J_2 \times J_n$, H_{J_1} has the same orbits on m-tuples as G; by induction then H_{J_1} induces the action of G_1 on $J_2 \times J_n$. It follows that H induces G_1.

Lemma 3.4.3. *Let $\mathcal{M}$ be an internally finite infinite structure. Let J_1, J_2 be a pair of basic pure projective geometries (with no forms) defined and orthogonal over the algebraically closed set A in the sense that $(J_1, J_2; J_1 \cap A, J_2 \cap A)$ is canonically embedded in $\mathcal{M}$. Let $J = J_1 \cup J_2$, $A_J = A \cap J$. Then the permutation group G induced on J by the internal automorphism group of $\mathcal{M}$ contains* $\mathrm{Aut}\,(J; A_J)^{(\infty)}$ *(which in this case is just the commutator subgroup of* $\mathrm{Aut}\,(J; A_J)$*). All group theoretic notions are to be understood internally here.*

Proof. For notational definiteness let us assume that $A \cap J_i$ is nonempty for each i. In the linear model we have vector spaces V_i with $PV_i = J_i$ and we will take $U_i = \mathrm{acl}\,(A) \cap V_i$, and decompose $V_i = U_i \oplus W_i$. Then we may check

$$\mathrm{Aut}\,(J_i; A \cap J_i) \simeq \mathrm{Hom}(W_i, U_i) \rtimes \mathrm{GL}(W_i).$$

Our claim is that the group G contains the product of the two groups $\mathrm{Hom}(W_i, U_i) \rtimes \mathrm{SL}(W_i)$ acting on J. We know that on the localizations $\mathrm{Aut}\,\mathcal{M}$ induces $\mathrm{PSL}(W_1) \times \mathrm{PSL}(W_2)$ as these geometries are orthogonal. Let H_1 be the kernel of the natural map from G to $\mathrm{Aut}\,J_2/(A \cap J_2)$. Then H_1 covers at least $\mathrm{PSL}(W_1)$ and is normal in G. It follows that the same applies to the perfect subgroup $H_1^{(\infty)}$. Now $H_1^{(\infty)}$ projects trivially into the second factor and may therefore be thought of as a normal subgroup of $\mathrm{Aut}\,(J_1; A \cap J_1)$ covering $\mathrm{PSL}(W_1)$; any such subgroup contains $\mathrm{Hom}(W_1, A \cap J_1) \rtimes \mathrm{SL}(W_1)$, by inspection.

Remark 3.4.4. *We are working here with automorphisms of* pointed *projective geometries, in which constants have been added. It is not always possible to reduce their analysis to a localization. In a similar vein, Lemma 3.4.2 may be proved for pointed pure projective geometries as well, or for that matter for any pointed projective geometries,* *if we are willing to write out the stabilizers of various sets.*

Definition 3.4.5. *A collection of A-definable sets S_i is said to be* jointly orthogonal *over A in $\mathcal{M}$ if the disjoint union of the structures $(S_i,\ \mathrm{acl}(A) \cap S_i)$ is canonically embedded in $\mathcal{M}$.*

Lemma 3.4.6. *Let J_i be defined over A in $\mathcal{M}$, with weak elimination of imaginaries, and let $B \subseteq J = \bigcup_i J_i$. Then the J_i are jointly orthogonal in $\mathcal{M}$ over A if and only if they are jointly orthogonal in $\mathcal{M}$ over $A \cup B$.*

Proof. If they are jointly orthogonal over A and R is a relation on J definable from $A \cup \bigcup_i \mathrm{acl}(AB) \cap J_i$, then R is the specialization of a 0-definable relation S over J to parameters from $\bigcup_i \mathrm{acl}(AB) \cap J_i$. Accordingly S is a boolean combination of products of $(\mathrm{acl}(A) \cap J_i)$-definable relations on J_i, and after specialization the same applies to R over AB.

Conversely, assuming orthogonality over $A \cup B$, let R be A-definable on J. This is definable by hypothesis in J, with respect to parameters from $\bigcup_i \mathrm{acl}(AB) \cap J_i$. Viewing R as an element of J^{eq}, let $e = \mathrm{acl}(R) \cap J$. By weak elimination of imaginaries, R is e-definable and $e \subseteq \mathrm{acl}(A) \cap J$.

Lemma 3.4.7. *Let $\mathcal{M}$ be an internally finite structure. Let J_i $(i \in I)$ be canonically embedded projective Lie geometries in $\mathcal{M}^*$, defined over, and orthogonal in pairs over, the set A in $\mathcal{M}^*$. Then they are jointly orthogonal over A in $\mathcal{M}^*$.*

Proof. Let $A_i = \mathrm{acl}(A) \cap J_i$. The assumption is that $(J_i \cup J_j; A_i \cup A_j)$ is canonically embedded in $\mathcal{M}^*$. Extend A by finite subsets B_i of J_i containing A_i so that B_i is a nondegenerate subspace containing a quadratic point, if possible. In the pure projective case $B_i = A_i$. Replace A by $B = A \cup \bigcup_i B_i$. Then A_i is replaced by B_i, the geometries continue to be pairwise orthogonal, and it suffices to prove joint orthogonality over AB. For this, by the choice of B_i, except in the pure projective case it suffices to go to the (nondegenerate) localizations, which are definably equivalent over B_i to the previous structures. Now we consider the group H of permutations induced by $\mathrm{Aut}\,\mathcal{M}$ on $\bigcup_i (J_i; B_i)$. Write G_i for $\mathrm{Aut}\,(J_i; B_i)^{(\infty)}$. Applying Lemma 3.4.2 of §3.3 to $H^{(\infty)}$, using the pairwise orthogonality, we find $H \supseteq \prod_i G_i$. By Lemma 3.4.2 and the remark following Lemma 3.4.3 (used in the more straightforward of the two directions) our claim follows.

Lemma 3.4.8. *Let $\mathcal{M}$ be an internally finite structure. Let J_1, J_2 be 0-definable basic simple projective Lie geometries canonically embedded in $\mathcal{M}^*$. Then in $\mathcal{M}^*$ we have one of the following:*

1. *J_1 and J_2 are orthogonal.*
2. *There is a 0-definable bijection between J_1 and J_2.*
3. *J_1 and J_2 are of pure projective type—that is, with no forms—and there is a 0-definable duality between them making the pair (J_1, J_2) a polar space.*

Proof. Let S be the internal permutation group induced on $J = J_1 \cup J_2$ by internal automorphisms of $\mathcal{M}$ and let G_i be the internal automorphism group of the geometry J_i. Set $S_1 = S \cap (G_1 \times G_2)^{(\infty)}$, again working internally (as we will throughout). As S projects onto G_i, $S^{(\infty)} \subseteq S_1$ projects onto $G_i^{(\infty)}$ for $i = 1, 2$. As $G_i^{(\infty)}$ is simple, $S^{(\infty)}$ is either the full product or the graph of an isomorphism between $G_1^{(\infty)}$ and $G_2^{(\infty)}$.

In the first case J_1 and J_2 are orthogonal by Lemma 3.4.2. In the second case, by Fact 3.3.11, the geometries J_1 and J_2 are isomorphic as weak geometries, and if we identify them by an isomorphism, thereby identifying their automorphism groups, S_1 is then the graph of an automorphism. With the exception of the pure projective case, this automorphism is an inner automorphism with respect to the full automorphism group of the geometry, by Fact 3.3.4, 1(iii); in the exceptional case it may be the composition of an inner automorphism and a graph automorphism. If S_1 is the graph of an inner automorphism corresponding to an isomorphism $h : J_1 \simeq J_2$, then as S_1 is normal in S, this isomorphism is S-invariant, hence 0-definable. In the exceptional case S_1 can be viewed as an isomorphism of J_1^* and J_2; in particular, J_1^* is interpretable in $\mathcal{M}$, and is 0-definably isomorphic with J_2.

Lemma 3.4.9. *Let $\mathcal{M}$ be an internally finite structure. Let A be a 0-definable basic affine space, with corresponding linear and projective geometries V and J. Suppose that J is canonically embedded in $\mathcal{M}^*$. Then one of the following holds in $\mathcal{M}^*$:*

1. *A is canonically embedded in $\mathcal{M}^*$.*
2. *There is a 0-definable point of A in $\mathcal{M}^*$.*
3. *J is of quadratic type and there is a 0-definable bijection of A with αQ for some unique α.*

Proof. As usual all computations with automorphisms will be taken relative to the internal automorphism groups.

We argue first that V is canonically embedded in $\mathcal{M}^*$. Let V_1 be V with all 0-definable relations from $\mathcal{M}$. Then J is canonically embedded in (J, V_1), and stably embedded since $V_1 \subseteq \operatorname{acl}(J)$. For $a \in V$, $V_1 \subseteq \operatorname{dcl}(Ja)$, and hence $(V_1, a) = (V, a)$ as structures. By weak elimination of imaginaries for V, it follows that $V_1 = V$ as structures.

Now consider

$$U = \{v \in V : \text{ Translation by } v \text{ is an automorphism of } A \text{ over } V\}.$$

For v in U let τ_v be the corresponding translation map on A. Then for $\alpha \in \operatorname{Aut} \mathcal{M}^*$ we have $\tau_v^\alpha = \tau_{\alpha^{-1}v}$. Thus U is $(\operatorname{Aut} \mathcal{M}^*)$-invariant, and hence also 0-definable in $\mathcal{M}^*$, since $\mathcal{M}$ is internally finite. But V is canonically embedded in $\mathcal{M}^*$, so $U = V$ or $U = (0)$.

If $U = V$ then A is canonically embedded in $\mathcal{M}^*$, since V is. Suppose that $U = (0)$. Every automorphism of V extends to $\mathcal{M}^*$ and hence to A; as $U = (0)$, this extension is unique, and $\operatorname{Aut} V$ acts on A. By Fact 3.3.6, we have either a fixed point or a bijection with αQ, as in possibilities (2,3) above, fixed by $(\operatorname{Aut} V)^{(\infty)}$. Furthermore, the fixed point or bijection, as the case may be, is unique, as otherwise this $(\operatorname{Aut} V)^{(\infty)}$ would fix correspondingly either a point of V, or a nonidentity bijection of αQ with itself. The first alternative is obviously impossible. In the second case, if $q \in \alpha Q$ is moved by the bijection, say $q \mapsto q + \alpha\lambda_v^2$, then v is fixed by the corresponding orthogonal group, which is again a contradiction. Thus the unique fixed point, or the unique bijection with αQ, is fixed by $\operatorname{Aut} \mathcal{M}^*$.

3.5 LIE COORDINATIZATION

In this section we introduce the notion of a locally Lie coordinatized structure, which is approximately a structure coordinatized in the manner of [**KLM**] (in other words, without concern for stable embedding), and we check that the internally finite structures associated with 4-quasifinite structures are bi-interpretable with locally Lie coordinatized structures, which is another way of phrasing the results of [**KLM**] (with 5 reduced to 4). Then to complete the proof of the equivalence of the first five conditions given in Theorem 2, we show that 4-quasifinite locally Lie coordinatized structures are Lie coordinatizable. See the discussion at the end of the present section for a review of the situation up to this point.

Definition 3.5.1. *A structure $\mathcal{M}$ in some nonstandard set theoretic universe is* locally Lie coordinatized *if it has nonstandard finite order, has finitely many 1-types, carries a tree structure of finite height whose unique root is 0-definable, and has a collection $\mathcal{J}$ of pairs (b, J) with $b \in \mathcal{M}$, J a b-definable component of a b-definable basic semiprojective, linear, or affine geometry, $J \subseteq \mathcal{M}$, satisfying the following conditions:*

1. *If a is not the root, then there is $b < a$ such that either $a \in \operatorname{acl}(b)$ or there is a pair $(b, J_b) \in \mathcal{J}$ with $a \in J_b$.*
2. *If $(b, J) \in \mathcal{J}$ with J semiprojective or linear then J is canonically embedded in $\mathcal{M}$.*
3. *Affine spaces are preceded in the tree by their linear versions.*

Lemma 3.5.2. *Let Γ be an infinite dimensional proper Grassmannian of type (e, K, τ), and $a \in \Gamma$. Then there are elements $a_0, \ldots, a_n \in \Gamma^{\mathrm{eq}} \cap \operatorname{acl}(a)$ and Lie geometries J_i, possibly affine, with J_i 0-definable and canonically embedded relative to the structure $(\Gamma; a_0, \ldots, a_i)$, such that $a_0 \in \operatorname{acl}(\emptyset)$, $a_{i+1} \in J_i$, and $a \in \operatorname{acl}(a_0, \ldots, a_n)$.*

Proof. The components J of the underlying coordinatizing structure $\mathcal{C}$ can be recovered from equivalence relations on pairs from Γ. Let a_0 consist of these components as elements of Γ^{eq}, together with enough elements of $\operatorname{acl}(\emptyset)$ in $\mathcal{C}^{\mathrm{eq}}$ to make them all basic. We define a_i inductively, stopping when $a \in \operatorname{acl}(a_0, \ldots, a_i)$. Given $(a_0, \ldots, a_i)$, with a not algebraic over them, pick a component J meeting $\operatorname{acl}(a) - \operatorname{acl}(a_0, \ldots, a_i)$ and let a' be a point of the intersection. Consider the localization $\bar{J} = J/(a_0, \ldots, a_i)$. This is not in general the full quotient of J modulo algebraic closure relative to $(a_0, \ldots, a_i)$, but just a part of that. The remainder consists of various geometries which are either 0-definably equivalent to the localization, or affine over it. In particular, we may take a' to represent either an element of this localization or an element of an affine geometry over the localization. More precisely, there is an element a'' lying either in the localization $\bar{J}$, or in an affine geometry over it, for which $\operatorname{acl}(a_0, \ldots, a_i, a') = \operatorname{acl}(a_0, \ldots, a_i, a'')$. We set $a_{i+1} = a''$ and correspondingly $J_i = \bar{J}$ or an affine geometry over $\bar{J}$.

The localizations are canonically embedded in $(\Gamma; a_0, \ldots, a_i)$. In the affine case Lemma 3.4.9 applies. If the affine space is actually a copy of Q, then a'' is taken in Q (which is part of the semiprojective model).

Lemma 3.5.3. *Let $\mathcal{M}$ be a structure, k an integer, and let Ψ be a finite set of first order formulas in four free variables. Suppose that for every first order sentence φ true in $\mathcal{M}$ there is a finite model $\mathcal{M}'$ satisfying*

1. *$\mathcal{M}' \models \varphi$.*
2. *$\mathcal{M}'$ has at most k 4-types.*
3. *Every 0-definable 4-ary relation on $\mathcal{M}'$ is defined by one of the formulas in Ψ.*

Then $\mathcal{M}$ is bi-interpretable with a locally Lie coordinatized structure $\mathcal{M}'$ which forms a finite cover of $\mathcal{M}$: $\mathcal{M}'$ has $\mathcal{M}$ as a 0-definable quotient with finite fibers (see §4.5 for a formal discussion of covers).

Proof. These conditions imply that $\mathcal{M}$ itself has at most k 4-types, and that every 4-ary relation on $\mathcal{M}$ is defined by one of the formulas in Ψ. In particular, one can select a maximal chain $E_0 < \ldots < E_d$ of 0-definable equivalence relations on $\mathcal{M}$ and we may suppose that in all the models $\mathcal{M}'$ this chain remains a maximal chain of 0-definable equivalence relations (making use, among other things, of condition (1)). We take $E_i < E_{i+1}$ to mean that E_{i+1} is coarser than E_i.

For i fixed, and $a \in \mathcal{M}$, we consider the E_{i+1}-class C of a, and its quotient C/E_i. It will suffice to prove that C/E_i is either finite or a proper Grassmannian, as we can then coordinatize $\mathcal{M}$ by coordinatizing each infinite section C/E_i, starting from the coarsest, using Lemma 3.5.2; of course, if C/E_i is finite, then its elements are algebraic over C. When projective geometries occur they can be replaced by semiprojective ones in $\mathcal{M}^{\mathrm{eq}}$.

If C/E_i is infinite, then by [**KLM**], specifically by Fact 3.3.3, above, we may suppose that in the finite structures $\mathcal{M}'$ approximating $\mathcal{M}$ in the sense of clauses (1–3) above, the corresponding set C'/E'_i carries the structure of a semiproper Grassmannian of fixed type. There are 4-place relations R_i which encode the components of the coordinatizing structure underlying the Grassmannian, as well as the geometric structure on this coordinatizing structure. Primarily, the R_i should be equivalence relations on pairs, so as to encode the elements of the coordinatizing structure; one can also encode, e.g., a ternary addition relation, with some care, by using four variables in the Grassmannian.

There is also a statement $\gamma(R_1, \ldots, R_n)$ expressing the fact that C'/E_i is a Grassmannian of the given type for this coordinatizing structure. Accordingly in view of our hypotheses, a formula of the same type will apply to C/E_i, for some choice of the R_i, and C/E_i is the Grassmannian of a coordinatizing structure.

To conclude we must check properness: that is, in C/E_i, we claim that each 0-definable relation S is geometrically definable (i.e., definable from the structure with which the Grassmannian inherits from the coordinatizing structure) over $\operatorname{acl}(\emptyset)$. For fixed S this will hold in sufficiently large finite approximations $\mathcal{M}'$ and by (1) this property passes to $\mathcal{M}$.

Corollary 3.5.4. *If $\mathcal{M}$ is strongly 4-quasifinite, then $\mathcal{M}$ is bi-interpretable with a locally Lie coordinatized structure which forms a finite cover of $\mathcal{M}$.*

Lemma 3.5.5. *Let $\mathcal{M}$ be an internally finite structure and suppose that $\mathcal{M}^*$ has a finite number k of 4-types. Then $\mathcal{M}^*$ is bi-interpretable with a locally Lie coordinatized structure which forms a finite cover of $\mathcal{M}^*$.*

Proof. We apply the previous lemma. Let Ψ be a set of representatives for the internally 0-definable formulas in 4 variables in $\mathcal{M}^*$. Let φ be a first order statement true in $\mathcal{M}^*$. Let $L \supseteq \Psi$ be a finite language contained in the language of $\mathcal{M}^*$ such that φ is a formula of L. We seek a finite structure $\mathcal{M}'$ for the language L such that

1. $\mathcal{M}' \models \varphi$.
2. $\mathcal{M}'$ has at most k 4-types.
3. Every $(\operatorname{Aut} \mathcal{M}')$-invariant 4-ary relation on $\mathcal{M}'$ is defined by one of the symbols in L.

Note that properties (1–3) taken jointly constitute a standard property of a finite language, and are satisfied (in the internal sense) in a nonstandardly finite structure, hence also in some finite structure.

Lemma 3.5.6. *Let J be a semiprojective or basic linear Lie geometry, $C \subseteq J$ finite, and suppose that $(J; C)$ (C treated as a set of constants) is canonically embedded in the structure $(\mathcal{M}; A)$. Let $C' = \operatorname{acl}_{\mathcal{M}}(A) \cap J$. Then C' is finite and $(J; C')$ is canonically embedded in $(\mathcal{M}; A)$.*

Proof. $C' \subseteq \operatorname{acl}(C)$ in the sense of J, so C' is finite.

Let R be an A-definable relation on J. Then R is C-definable and thus $R \in J^{\mathrm{eq}}$. It follows from weak elimination of imaginaries that R is C'-definable.

Lemma 3.5.7. *Let $\mathcal{M}$ be internally finite, J a semiprojective or linear geometry, B-definable, and $C \subseteq J$ finite with (J/C) canonically embedded in $(\mathcal{M}^*; B)$. Assume that C is nondegenerate if J involves a form, and otherwise, if J is pure projective, then assume that in $\mathcal{M}^*$ the definable dual of the linear model V is trivial. Then the group G induced on J by the internal automorphism group of $\mathcal{M}$ over B contains $(\mathrm{Aut}\,(J;C))^{(\infty)}$.*

Proof. In the nondegenerate case, dealing with J over C is equivalent to dealing with J/C and Lemma 3.5.2 of §3.4 applies. In the pure projective case $(\mathrm{Aut}\,(J;C))^{(\infty)}$ has the form $\mathrm{Hom}(W,U) \rtimes \mathrm{SL}(W)$ relative to a decomposition of the linear model $V = W \oplus U$ with U covering C, and all we learn from looking at the localization is that G induces at least $\mathrm{SL}(W)$ on the localization; thus the subgroup of $\mathrm{Hom}(W,U) \rtimes \mathrm{SL}(W)$ induced by G is $H \rtimes \mathrm{SL}(W)$ with H an $\mathrm{SL}(W)$-invariant subgroup of $\mathrm{Hom}(W,U)$. Then H will be $\mathrm{Hom}(W,U_0)$ for some subspace U_0 of U and $P(W \oplus U_0)$ is the unique minimal $G^{(\infty)}$-invariant subspace of J. Thus this space is G-invariant. But as we are in the pure projective case there can be no definable subspace of finite codimension, so $U_0 = U$ and $H = \mathrm{Hom}(W,U) \rtimes \mathrm{SL}(W)$.

Lemma 3.5.8. *Let $\mathcal{M}$ be an internally finite locally Lie coordinatized structure with respect to the coordinate systems in $\mathcal{J}$ and suppose that*

1. *Whenever $J_b \in \mathcal{J}$ is pure projective, with linear model V, the definable dual V^* is (0).*
2. *Whenever $J_b \in \mathcal{J}$ is symplectic of characteristic 2, there are no definable quadratic forms on J_b compatible with the given symplectic form.*

Then for any finite subset A of $\mathcal{M}$ closed downwards with respect to the coordinatizing tree, we have

3. *For $b \in A$, if J_b is nonaffine, then for some finite subset $C \subseteq J_b$, the structure $(J;C)$ is canonically embedded in $\mathcal{M}^*$ over A.*
4. *For $J_1, J_2 \in \mathcal{J}$ nonaffine, with defining parameters in A, if $C_i = \mathrm{acl}_{\mathcal{M}^*}(A) \cap J_i$, then either $(J_1;C_1)$ and $(J_2;C_2)$ are orthogonal over A, or else there is an A-definable bijection of J_1/C_1 with J_2/C_2.*

Proof. We prove (3, 4) simultaneously by induction on the size of A.

Let A, b be given. We prove (3). If A is the branch below b then (3) holds by definition of local lie coordinatization. So we may suppose that A contains elements not on the branch below b; let $c \in A$ be maximal among such elements, and $B = A - \{c\}$. Induction applies to B. In particular $(J_b;C_0)$ is canonically embedded in $\mathcal{M}^*$ over B, for some finite $C_0 \subseteq J_b$. We may take C_0 nondegenerate when a form is present. Then the internal automorphism group of $\mathcal{M}^*$ over B induces at least $(\mathrm{Aut}\,(J_b; C_0))^{(\infty)}$ on J_b.

If c is algebraic over B, then its stabilizer in the internal automorphism group of $(\mathcal{M}^*; B)$ has finite index, hence also covers $(\mathrm{Aut}\,(J_b; C_0))^{(\infty)}$. Thus in this case $(J_b; C_0)$ is canonically embedded in $\mathcal{M}^*$ over A.

Suppose therefore that c is not algebraic over B. Thus there is a geometry J_2 associated to a parameter d of B, with $c \in J_2$. We will write J_1 for J_b. Let $C_i = \mathrm{acl}_{\mathcal{M}^*}(B) \cap J_i$. Then $(J_i; C_i)$ is canonically embedded in $\mathcal{M}^*$ over B by Lemma 3.5.6, and (4) applies to this pair if J_2 is also nonaffine.

Case 1. J_2 is nonaffine, and $(J_2; C_2)$ is orthogonal to $(J_1; C_1)$.

Then $(J_1, J_2; C_1C_2)$ is canonically embedded in $\mathcal{M}^*$ over B and hence $(J_1; C_1)$ is canonically embedded in $\mathcal{M}^*$ over A.

Case 2. J_2 is affine, with corresponding linear geometry V_2, and the projectivization $P_2 = P(V_2/B)$ is orthogonal to J_1/B over B.

As the orthogonality statement is preserved by adding parameters from J_1, and this does not affect the desired conclusion (3), we may take C_1 to be nondegenerate, or J_1 to be pure projective. We now work with the internal automorphism groups.

Let G be the automorphism group of $(J_1; B)$, H the automorphism group of J_2, and $G(X)$ and $H(X)$ the pointwise stabilizers. Then $G(P_2) = G$ since the geometries are orthogonal and basic. Thus $G/G(J_2) \simeq H(P_2, B)/H(P_2, J_1, B)$. On the right hand side we have a solvable group and hence $G(J_2)$ contains $G^{(\infty)}$. Thus $(J_1; B)$ is canonically embedded in $(J_1; BJ_2)$ and in particular is canonically embedded over $B \cup \{c\} = A$.

Case 3. J_2 is nonaffine and is nonorthogonal to J_1 over B.

Find $J' = J_{b'}$ with $b' \leq b$ minimal such that J' and J_1 are nonorthogonal. By the induction hypothesis (4) applied to the branch below b, there s a b-definable bijection between J'/b and J_1, which must be an isomorphism of weak geometries. Accordingly, we may replace J_1 by J', and if $b' < b$ conclude by induction. Thus we now assume J_1 is orthogonal to every earlier geometry. In much the same way we may assume that J_2 is orthogonal to every earlier geometry.

As these geometries are nonorthogonal, they are now assumed orthogonal to every geometry associated with a parameter below b or d. It follows that $\mathrm{acl}\,(bd) \cap J_i = \emptyset$ for $i = 1, 2$. The induction hypothesis (4) applies to the union of the branches up to b and d, and gives a bd-definable bijection between J_1 and J_2. Thus $c \in \mathrm{dcl}\,(Bc')$ for some $c' \in J_1$, and (3) follows.

Case 4. J_2 is affine, with corresponding linear geometry V_2; and the projectivization $P(V_2/B)$ is nonorthogonal to J_1 over B.

We minimize parameters as in the previous case, taking J_1 orthogonal to its predecessors, and taking P_2 to be a (nonaffine) geometry nonorthogonal to $P(V_2/B)$ and minimal below d. Then P_2 and J_1 can be identified, as in the previous case, and we apply Lemma 3.5.8 to J_2 and P_2/B. There are then three possibilities.

If J_2 has a 0-definable point in $\mathcal{M}^*$, then $\operatorname{dcl}(A) = \operatorname{dcl}(B, c')$ for some $c' \in V_2$ and we may replace c by c' and return to the previous case.

If in $\mathcal{M}^*$ we have a B-definable bijection of J_2 with Q, then by hypothesis (2) Q is also part of V_2, and again we reduce to the previous case.

Suppose finally that J_2 is canonically embedded in $\mathcal{M}^*$. Now P_2/B is geometrically definable over J_2, so $(P_2/B, J_2)$ is canonically embedded in $\mathcal{M}^*$. Furthermore, P_2/B is canonically embedded in $(P_2/B, J_2; c)$ (one affine parameter). Thus P_2/B is embedded in $(\mathcal{M}^*; c)$. As P_2 and J_1 are B-definably identified, we wish to show that P_2 is itself canonically embedded in $(\mathcal{M}^*; c)$. When P_2 carries a form then P_2 is geometrically definable from P_2/B and additional parameters from P_2. When P_2 is pure projective it follows from Lemma 3.5.7 that it is canonically embedded in $\mathcal{M}^*$.

This exhausts the cases and proves (3). We now consider (4): so we have J_1, J_2 nonaffine, with defining parameters in A, and $C_i = \operatorname{acl}_{\mathcal{M}^*}(A) \cap J_i$.

We apply Lemma 3.5.8 of §3.4. By hypothesis (1) if the geometries involved are pure projective, the polar case cannot arise between them. So either we have an A-definable bijection of J_1/C_1 with J_2/C_2, or these localizations are orthogonal over A.

Suppose therefore that J_1/C_1 and J_2/C_2 are orthogonal over A. Our claim is that then $(J_1; C_1)$ and $(J_2; C_2)$ are orthogonal over A. If J_1 is pure projective then Lemma 3.5.7 applies to give the orthogonality of $(J_1; C_1)$ and J_2/C_2. If J_1 involves a form then consider $G = \operatorname{Aut}(J_1; C_1)$ and the pointwise stabilizer $G(J_1/C_1)$. The quotient $G/G(J_1/C_1)$ is solvable and as in Case 2 above it follows that $(J_1; C_1)$ and J_2/C_2 are orthogonal over A. In this case they remain orthogonal over a nondegenerate extension C_1' of C_1 and $(J_1; C_1')$ is definably equivalent to J_1/C_1'.

If J_2 is pure projective the same argument gives us that $(J_1; C_1)$ or $(J_1; C_1')$, as the case may be, is orthogonal to $(J_2; C_2)$. Otherwise, we may suppose that both J_1 and J_2 involve forms, and that $(J_1; C_1')$ is definably equivalent to J_1/C_1, so that repetition of the first argument gives the orthogonality of $(J_1; C_1')$ and $(J_2; C_2)$, using the solvability of the relative automorphism group for $(J_2; C_2)$ over J_2/C_2. By Lemma 3.5.6 the orthogonality holds over A.

Lemma 3.5.9. *Let $\mathcal{M}$ be an internally finite locally Lie coordinatized structure. Then $\mathcal{M}^*$ is Lie coordinatizable. If in addition $\mathcal{M}$ is strongly 4-quasifinite then $\mathcal{M}$ is Lie coordinatizable.*

Proof. We will apply the previous lemma. The first point is that without loss of generality we may suppose that the coordinatizing family $\mathcal{J}$ satisfies the following:

(i) whenever $J_b \in \mathcal{J}$ is pure projective, with linear model V, the definable dual J^* is (0);
(ii) whenever $J_b \in \mathcal{J}$ is symplectic of characteristic 2, there are no definable quadratic forms on J_b compatible with the given symplectic form.

In other words, if the definable dual J^* is nontrivial, then J is part of a polar geometry encoded in $\mathcal{M}$ which may be used in place of J, and if a symplectic space carries a nontrivial form (and is acted on by the full symplectic group) then it may be replaced by the corresponding quadratic geometry, interpreted in $\mathcal{M}$.

So we have, in particular, the following conclusion from Lemma 3.5.8 for any finite subset A of $\mathcal{M}$:

For $b \in A$, if J_b is nonaffine then for some finite subset $C \subseteq J_b$, the structure $(J; C)$ is canonically embedded in $\mathcal{M}^*$ over A]

Varying A, this implies that the nonaffine geometries are stably embedded in $\mathcal{M}^*$. By Lemma 3.5.8 of §3.4 the same is true for the affine geometries. Thus after replacing the semiprojective geometries with projective ones, $\mathcal{M}^*$ is Lie coordinatized.

If in addition $\mathcal{M}$ is strongly 4-quasifinite, then the Lie coordinatization can be defined using formulas in the language of $\mathcal{M}$.

There has been a certain amount of vacillation between projective and semiprojective geometries visible. The orthogonality theory is simpler for projectives, and elimination of imaginaries holds for the semiprojectives. Furthermore, they are bi-interpretable, so in a sense both theories are available for either version.

We recall the statements of Theorems 2 and 2′ of §1.2.

Theorem 8 (1.2.2: Characterizations)
The following conditions on a model $\mathcal{M}$ are equivalent:

1. *$\mathcal{M}$ is smoothly approximable.*
2. *$\mathcal{M}$ is weakly approximable.*
3. *$\mathcal{M}$ is strongly quasifinite.*
4. *$\mathcal{M}$ is strongly 4-quasifinite.*

5. *$\mathcal{M}$ is Lie coordinatizable.*
6. *The theory of $\mathcal{M}$ has a model $\mathcal{M}^*$ in a nonstandard universe whose size is an infinite nonstandard integer, and for which the number of internal n-types $s^*_n(\mathcal{M}^*)$ satisfies:*

$$s^*_n(\mathcal{M}^*) \le c^{n^2}$$

for some finite c, and in which internal n-types and n-types coincide. (Here n varies over standard natural numbers.)

Theorem 9 (1.2.2′: Reducts). *The following conditions on a model $\mathcal{M}$ are equivalent:*

1. *$\mathcal{M}$ has a smoothly approximable expansion.*
2. *$\mathcal{M}$ has a weakly approximable expansion.*
3. *$\mathcal{M}$ is quasifinite.*
4. *$\mathcal{M}$ is 4-quasifinite.*
5. *$\mathcal{M}$ is weakly Lie coordinatizable.*
6. *The theory of $\mathcal{M}$ has a model $\mathcal{M}^*$ in a nonstandard universe whose size is an infinite nonstandard integer, and for which the number of internal n-types $s^*_n(\mathcal{M}^*)$ satisfies*

$$s^*_n(\mathcal{M}^*) \le c^{n^2}$$

for some finite c. (Here n varies over standard natural numbers.)

We remarked in §2.1 that weak approximability implies strong quasifiniteness; thus the implications $1 \implies 2 \implies 3 \implies 4$ in Theorem 2 all hold. Furthermore, by existence, finiteness, and homogeneity of envelopes, Lie coordinatizability gives smooth approximation. In the present section we showed that 4-quasifinite structures are Lie coordinatizable. Thus the equivalence of the first five conditions in Theorem 2 has been verified; the estimate needed for the sixth clause will be found in §5.2. One can also verify the equivalence of the first five conditions in Theorem 3 if one replaces "*weakly Lie coordinatizable*" by "*reduct of a Lie coordinatizable structure.*" However, the proof that these two conditions are equivalent is subtle and is the subject of Chapter 7.