4

Finiteness Theorems

4.1 GEOMETRICAL FINITENESS

As Ahlbrandt and Ziegler showed, the key combinatorial property of coordinatizing geometries depends on Higman's lemma, itself a special case of the Kruskal tree lemma. This was given an additional degree of flexibility in [HrTC], adequate to our present purposes, once we verify that the geometries we are using possess the following property. The proof is very much the same as in the pure linear case.

Definition 4.1.1. *A countable structure $\mathcal{M}$ is* geometrically finite *with respect to an ordering $<$ of type ω, if for each n the following holds:*

> *For any sequence of n-tuples $\mathbf{a}_i$ $(i = 1, 2, \ldots)$ in $\mathcal{M}$ there is an order-preserving elementary embedding $\alpha : \mathcal{M} \to \mathcal{M}$ taking $\mathbf{a}_i$ to $\mathbf{a}_j$ for some $i < j$.*

Lemma 4.1.2. *Suppose that $\mathcal{M}$ is $\aleph_0$-categorical and geometrically finite with respect to the ordering $<$. Let $\mathbf{a}$ be a finite sequence of elements of $\mathcal{M}$, and suppose that for each $i = 1, 2, \ldots$ there are given k finite initial segments $S_{i1}, \ldots, S_{ik}$ of $(\mathcal{M}; <)$. Then there is an automorphism α of $\mathcal{M}$, a finite initial segment S of $\mathcal{M}$, and a pair $i < i'$ such that*

1. $\mathbf{a} \subseteq S$*;* $S_{ij} \subseteq S$ *for* $j = 1, \ldots, k$.
2. $\alpha \restriction S$ *is order preserving.*
3. α *fixes* $\mathbf{a}$.
4. $\alpha[S_{ij}] \subseteq S_{i'j}$ *for* $j = 1, \ldots, k$.
5. $\alpha(\max S_{ij}) = \max S_{i'j}$ *for* $j = 1, \ldots, k$.

Proof. Set $b_{ij} = \max S_{ij}$ for all i, j and apply geometrical finiteness to the sequences $(\mathbf{a}, \mathbf{b}_i)$ with $\mathbf{b}_i = (b_{i1}, \ldots, b_{ik})$. The result is an order-preserving elementary map $\beta : \mathcal{M} \to \mathcal{M}$ fixing $\mathbf{a}$ and carrying some $\mathbf{b}_i$ to $\mathbf{b}_{i'}$ with $i < i'$. Restrict β to a large initial segment S of $\mathcal{M}$, and then extend the restriction to an automorphism of $\mathcal{M}$.

In proving the geometrical finiteness of a geometry we first deal with linear models. We work with the following orderings.

Definition 4.1.3. *The* standard *orderings of basic linear (or degenerate) geometries are defined as follows.*

1. *Any ordering of a pure set in order type ω is standard.*
2. *If X is an ordered basis for a vector space V and $<_K$ is an ordering on the base field, with 0 as the first element, then the* induced ordering *on V is derived from the reverse lexicographic ordering on words in the alphabet K as follows. To any vector v we assign the word consisting of the sequence of its coordinates, truncated after the last nonzero coordinate. A standard ordering of the pure vector space V is any ordering induced by such a pair $(X, <_K)$, where the order type of X is ω.*
3. *If V is a vector space carrying a nondegenerate symplectic or hermitian form, or a nondegenerate quadratic form Q with an associated symmetric form, then an ordered basis X for V will be considered standard if it has the form $(e_1, f_1, e_2, f_2, \ldots)$ where in all cases $(e_i, e_i) = (f_i, f_i) = 0$, $(e_i, f_i) = 1$, the subspaces $H_i = (e_i, f_i)$ are pairwise orthogonal, and in the presence of a quadratic form Q we require furthermore that $Q(e_i) = Q(f_i) = 0$.*
   In this case an ordering on V is considered standard if it is induced by a pair $(X, <_K)$ where X is a standard ordered basis.
4. *A standard ordering of the linear polar geometry (V, W) is defined as in the previous case, using the appropriate version of a standard basis for $V \cup W$; here the e_i form a basis for V, and the f_i form a basis for W.*

We remark that given any standard ordering on a vector space derived from an ordered basis X, the subspaces generated by initial segments of X will be initial segments of V with respect to the induced ordering. We note also that we include the polar case here because it does not reduce to the pure projective case, but we exclude the quadratic case for notational convenience since its underlying set is not a vector space; however, this is a triviality, since after fixing a point of the quadratic geometry it can be treated as an orthogonal geometry.

We review the combinatorial lemma on which geometrical finiteness depends.

Definition 4.1.4. *Let Σ be a finite set.*

1. *A* word *in the alphabet Σ is a finite sequence of elements of Σ. $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$ is the set of all words in the alphabet Σ.*
2. *The* embeddability ordering *on Σ^* is the partial ordering defined as follows: $w \leq w'$ if there is an order-preserving embedding of w into w'.*

3. *A partially ordered set* $(X, <)$ *is* well quasi-ordered *if it has no decreasing sequences and no infinite antichains; by Ramsey's theorem, an equivalent condition is that any infinite sequence of distinct elements of* X *contains an infinite strictly increasing subsequence.*

Fact 4.1.5 (Higman's Lemma [Hi]). *If* Σ *is a finite alphabet, then the partially ordered set* $(\Sigma, <)$*, under the embeddability ordering, is well quasi-ordered. Thus for any infinite sequence of words* $w^{(i)} \in \Sigma^*$*, there is a pair* i, j *with* $i < j$ *such that* $w^{(i)}$ *embeds in* $w^{(j)}$*.*

We note that this fact is proved more generally in a relative form, for words in any alphabet which is well quasi-ordered, with an appropriately modified embeddability relation. Only the finite case is used here.

Lemma 4.1.6. *The countably infinite versions of the linear and degenerate geometries—a pure set, a pure vector space, a symplectic, hermitian, or orthogonal space, or a polar pair—are geometrically finite with respect to their standard orderings.*

Proof. It will suffice to treat the cases of nondegenerate symplectic, hermitian, or orthogonal spaces, where the notation is uniform. The other nondegenerate cases are simple variations.

We fix a standard ordering $<$ on V with respect to a standard basis $X = (e_1, f_1, \ldots)$ for V and an ordering of K with 0 as initial element. Let $H_i = \langle e_i, f_i \rangle$; this is a nondegenerate plane of the same type as V.

With n fixed we consider n-tuples $\mathbf{a}^{(i)} = (v_{i1}, \ldots, v_{in})$ from V. For each i, expanding relative to the basis X, think of $\mathbf{a}^{(i)}$ as a matrix with n semi-infinite rows, and entries in K. Let $\mathbf{b}^{(i)} = (w_{i1}, \ldots, w_{im_i})$ be the corresponding matrix in reduced row echelon form, and let M_i be the $n \times m_i$ matrix over K connecting the two forms by: $\mathbf{a}^{(i)} = M_i \mathbf{b}^{(i)}$. Without loss of generality, the numbers $m_i = m$ and the matrices $M_i = M$ are independent of i, and we may also suppose that the maps $\mathbf{b}^{(i)} \to \mathbf{b}^{(i')}$ defined by $w_{ij} \mapsto w_{i'j}$ are all isometries with respect to whatever forms are present.

Now we will make the reduction to Higman's lemma, encoding the sequences $\mathbf{b}^{(i)}$ by a word in an appropriate alphabet. We expand each vector w_{ij} as $\sum_r h_{ijr}$ where $h_{ijr} \in H_r$. As the H_r are all isometric we will identify them all with a single plane $H = \langle e, f \rangle$ and consider h_{ijr} to be an element of H. We say that r is the *leading index* for w_{ij} if r is maximal such that $h_{ijr} \neq 0$; we say that the leading index r for w_{ij} is of *type* e if $h_{ijr} \in \langle e \rangle$, and of *type* f otherwise. We associate to $\mathbf{b}^{(i)}$ a sequence $w^{(i)} = (h_{i1}, h_{i2}, \ldots, h_{ir})$ with r the maximal leading index of the w_{ij} in such a way that h_{is} encodes the following sequence of data for $1 \le j \le m$:

The value of $h_{ijs} \in H$; Whether s is the leading index of w_{ij} (yes/no).

Clearly this information can be expressed by a finite alphabet.

By Higman's lemma we have a pair $i < i'$ such that $w^{(i)}$ embeds in $w^{(i')}$. We will now write out exactly what this means. Let l, l' be the lengths of $w^{(i)}$ and $w^{(i')}$ respectively. There is an increasing function $\iota : \{1, \ldots, l\} \to \{1, \ldots, l'\}$ such that

$$h_{i'\iota(s)} = h_{is} \text{ for } s \le l. \tag{1}$$

or more explicitly, in terms of the data encoded, for $s \le l$ we have:

$$h_{i'j\iota(s)} = h_{ijs} \text{ for } j \le m \tag{1.1}$$

(1.2) If s is the leading index for w_{ij},
then $\iota(s)$ is the leading index for $w_{i'j}$.

Set $y_j = \sum\{h_{i'js} : s \notin im\,\iota\}$. The leading index of y_j is less than the leading index of $w_{i'j}$, by (1.2).

We now associate with ι a linear map β, which is defined on the span of $e_1, f_1, \ldots, e_l, f_l$, as follows:

(2.1) $\beta(e_s) = e_{\iota(s)}$ unless
s is the leading index of some w_{ij} and is of type e for it.
(2.2) $\beta(f_s) = f_{\iota(s)}$ unless
s is the leading index of some w_{ij} and is of type f for it.
(2.3) $\beta(h_{ijs}) = h_{i'j\iota(s)} + y_j$ if
s is the leading index of w_{ij}.

By the initial reduction to row echelon form, a given index s can occur at most once as the leading index of a given type (e or f) for one of the w_{ij}. If s is the leading index for w_{ij} and is of type e for it, then (2.3) and linearity determine $\beta(e_s)$, while if, on the other hand, s has type f for w_{ij}, then (2.3), linearity, and the value of $\beta(e_s)$ determine $\beta(f_s)$. So (2.1–2.3) determine some linear function β. For any r let $H'_r = \bigoplus\{H_s : s < \iota(r), s \notin im\,\iota\}$. Then β has the following properties:

$$\beta(h_{ijr}) \in h_{i'j\iota(r)} + H'_r \tag{3.1}$$

$$\beta(w_{ij}) = w_{i'j} \tag{3.2}$$

From (3.1) it follows that β is order preserving: if u_1, u_2 have their last difference in the rth component, then $\beta(u_1)$ and $\beta(u_2)$ will differ last in their $\iota(r)$th component, and in the *same manner*. By (3.2) and the relations $\mathbf{a}^{(i)} = M_i \mathbf{b}^{(i)}$, we find $\beta(v_{ij}) = v_{i'j}$.

It remains to check that β is an isometry. We make use of a basis $X_1 \cup X_2$ for $\langle e_1, f_1, \ldots, e_l, f_l \rangle$ of the following form: X_1 consists of all w_{ij} for $j \leq m$; X_2 consists of the e_r and the f_r for which r is not a leading index of corresponding type for any of the w_{ij}. Then by (3.2) β is an isometry on $\langle X_1 \rangle$, and β is also an isometry on $\langle X_2 \rangle$. So we need only check that β preserves inner products between X_1 and X_2 (even in the orthogonal case, this now suffices). In view of the orthogonality of the spaces H_s, the relation (3.1), and the definition of β, this follows.

Corollary 4.1.7. *The basic geometries are geometrically finite.*

Proof. Let J be the geometry, and V the corresponding linear model, equipped with a standard ordering.

If J is projective order it as follows: $a < b$ if the first representative u for a in V precedes the first representative v of b in V.

If J is affine, then call one element 0, place it first, and order the remainder of J as in V. Similarly, if J is of quadratic type pick one element q of the space Q of quadratic forms on V compatible with the symplectic structure, place it first, and then identify (V, Q) with the orthogonal space $(V; q)$; order it as two copies of a standard orthogonal space.

4.2 SECTIONS

We will establish the notation used in proving that Lie coordinatized structures have finite languages and quasifinite axiomatizations. A particular coordinatization is fixed throughout. The coordinatizing tree, together with some relevant data, will be called the skeleton of the model.

It will be convenient to coordinatize using semiprojectives in place of projectives from this point on.

Definition 4.2.1

1. *A* skeletal type *consists of the following data:*

   a parameter h (the height of a tree);
   an assignment τ associating to each i with $1 \le i \le h$ the type of a basic semiprojective or affine-with-dual Lie geometry, or a finite structure;
   a partial function σ from $\{1, \ldots, h\}$ to $\{1, \ldots, h\}$. Here σ satisfies the following conditions:

   (i) *the domain and range of σ are disjoint and their union is contained in the set of indices i for which $\tau(i)$ is not a finite structure;*
   (ii) *$\sigma(i) < i$;*
   (iii) *the domain of σ contains the set of indices i for which $\tau(i)$ is a basic affine-with-dual Lie geometry.*

   A level i for which $\tau(i)$ is a semiprojective type geometry and i is not in the domain of σ is said to be a level of new type.

2. *The* skeletal language L_{sk} *and* skeletal theory T_{sk} *associated with a given skeletal type (not shown in the notation) are defined as follows.*

   L_{sk} contains symbols $\le$ and P_i $(0 \le i \le h)$ which are asserted by T_{sk} to constitute a tree ordering of height h with levels given by the unary predicates $P_0, \ldots, P_h$; P_0 consists of the root alone. There should also be predecessor functions for the tree order, so that a substructure will be a subtree.

   L_{sk} contains several additional symbols. In the first place, it contains languages suitable to the description of structures of the types specified by the τ-component of the skeletal type. T_{sk} asserts, using these symbols, that the tree successors of a given point at level $i-1$ form a structure of the type specified by $\tau(i)$, that is, either a specific finite structure or an infinite dimensional basic geometry of specified type. It will be convenient to write $P_i(a)$ for the successors of a point a at level $i-1$; so T_{sk} controls the type of each $P_i(a)$.

   Finally, and crucially, the σ-component of the skeletal type furnishes nonorthogonality information. L_{sk} contains function symbols *in several variables f_{ij} whenever $j = \sigma(i)$ representing a parametrized family of functions f_{ija}, where a varies over the points at level $i - 1$, providing a bijection between the projectivization of $P_i(a)$ and a localization of the projectivization of $P_j(a')$ relative to some finite subset, where a' is the element lying below a at height $j - 1$.*

It is not quite necessary to fix the skeletal data, as long as the various variables involved, such as the sizes of the finite structures, are kept bounded. However, we can analyze more general situations of this type by dealing with each possible refinement to full skeletal data.

Definition 4.2.2. *Let the skeletal data (h, τ, σ) be fixed, hence also the skeletal language L_{sk} and the skeletal theory T_{sk}. Let L be an expansion of L_{sk}.*

1. *A* skeleton *with given skeletal data is a model for T_{sk}.*
2. *A* skeletal expansion *is a structure for the language L whose reduct to L_{sk} is a model of T_{sk}. It has* true dimensions *if not only the type of the geometry, but its isomorphism type, is determined by the atomic type of its controlling parameter.*
3. *A fully* proper *model for the language L is a skeletal expansion which satisfies*
   - (i) *The L_{sk}-reduct of each layer $P_i(a)$ with i in the range of σ (that is, the pure geometry) is fully embedded in $\mathcal{M}$.*
   - (ii) *If $a' \le a$ in the tree lie at level $i-1$ and $j-1$ respectively, with i, j in the range of σ, then $P_i(a')$ and $P_j(a)$ are orthogonal*
   - (iii) *The dual affine part of an affine-with-dual geometry is the full definable affine dual.*

Lemma 4.2.3. *The class of fully proper L-structures relative to a given skeletal theory is an elementary class.*

Proof. The point that requires care is the axiomatization of stable embeddedness of a given geometry J in $\mathcal{M}$, since in order to state in first-order terms the definability of the relativization of a formula φ to J using parameters of J, it is necessary to give an a priori bound on the number of parameters needed in J.

So let $D_b = \{x \in J : \varphi(x, b)\}$ be an $\mathcal{M}$-definable subset of J with parameters b (containing defining parameters for J). If this is J-definable, it is definable using parameters in $J \cap acl(b)$, by weak elimination of imaginaries. This is a finite dimensional subspace of J of dimension at most $rk(b)$, and $rk(b)$ is at most the height h times the number of entries in the sequence b.

We now deal at length with skeletons and expansions of skeletons.

View $L_{\rm sk}$ and $T_{\rm sk}$ as fixed for the present.

Definition 4.2.4. *Let $\mathcal{M}$ be a countable skeletal expansion.*

An Ahlbrandt–Ziegler enumeration *(more specifically, a breadth-first Ahlbrandt–Ziegler enumeration) is an enumeration of $\mathcal{M}$ derived from some data of the following type, according to the recipe following. The data will be*

1. *A standard enumeration of the projectivization of each one of the semiprojective layers at level i where i is a level of new type;*
2. *An enumeration of each of the finite structures found in the coordinate tree;*
3. *A set $C_i(a)$ of at most $|K|$ elements (K is the base field) in each of the components $P_i(a)$ of the ith layer, whenever $P_i(a)$ is not finite, chosen so that*

   if $P_i(a)$ is semiprojective, then $C_i(a)$ is the set of semiprojective points above some point of the projectivization of $P_i(a)$ (in the sense explained below); if $P_i(a)$ is affine then $C_i(a)$ enumerates an affine line in $P_i(a)$.

Relative to these data, we order $\mathcal{M}$ as follows. Enumerate successive layers of the tree; the order in which the ith layer is enumerated is determined first by the enumeration of the previous layer, and for a fixed element a of layer $i-1$, either

- *the enumeration of $P_i(a)$ is given as part of the data, using one of the clauses (1, 2), or*
- *in the event that $j = \sigma(i)$ is defined, the enumeration of $P_i(a)$ is determined by the enumeration of $P_j(a')$ where a' lies below a at level $j-1$, as follows. We have by hypothesis a specific identification of the projectivization P_a of $P_i(a)$ with a localization $P_{a'}$ of $P_j(a')$. If $P_i(a)$ is semiprojective then enumerate the points of $C_i(a)$ first; then over these points there is a definable function from the projectivization onto $P_i(a)$, so an ordering of the rest of $P_i(a)$ is determined by an ordering on the corresponding localization of $P_j(a')$ where $j = \sigma(i)$ and a' lies below a at level $j-1$. Such an ordering on the localization of $P_j(a')$ can be induced from the ordering of $P_j(a')$ using first representatives, as in the original discussion of geometrical finiteness. If $P_i(a)$ is affine-with-dual then the dual part is enumerated first, following the enumeration of the projective dual (which is part of the corresponding projective geometry), and then the affine part is enumerated by taking the affine line $C_i(a)$ of (3) first, after which one follows the enumeration of its projectivization as in the semiprojective case.*

Definition 4.2.5. *Let $\mathcal{M}$ be a countable skeletal expansion.*

A section *of $\mathcal{M}$ is an initial segment of $\mathcal{M}$ with respect to an Ahlbrandt–Ziegler enumeration. The* height *h of a section is the least level not completely contained in the section. According to this definition the height of $\mathcal{M}$ itself should be considered to be undefined.*

Definition 4.2.6. *Let $\mathcal{M}$ be a countable skeletal expansion and U a section of $\mathcal{M}$ of height h.*

A support *for U consists of the following data (B, a, C):*

1. *The sequence $B = (B_1, \ldots, B_h)$, with B_i consisting of all points a at level i for which a lies below some point of U at level h, and the tree predecessor of a lies below some point at level h not in U;*
2. *If $i \leq h$ is maximal such that B_i is nonempty: let $a = (a_0, a_1, \ldots, a_{i-1})$ be the (unique) branch leading to B_i;*
3. *If $P_i(a_{i-1})$ is finite let $C_i(a)$ be the complete enumeration of $P_i(a_{i-1})$; if $P_i(a)$ is semiprojective or if B_i meets the affine part, let $C_i(a)$ be the finite subset chosen originally in the construction of the order from which U was derived; if $P_i(a)$ is an affine-with-dual pair and B_i is contained in the affine dual, let $C_i(a)$ be an enumeration of the points of B_i which lie over the last point of the projectivization (the point being that the ordering of the projectivization does not define a unique ordering of the affine dual, but knowing $C_i(a)$ and the projective ordering, the initial segment of the affine dual is determined).*

Note here that a section does not quite determine its support, since the same section may be derivable from different orderings; this is just an abuse of language, and in any case in practice supports are used to determine sections, rather than the reverse.

Lemma 4.2.7. *Let (B, a, C) be given with $B = (B_1, \ldots, B_h)$ a sequence of subsets of the first $h+1$ layers of a countable skeletal expansion $\mathcal{M}$, $a = (a_0, a_1, \ldots, a_{h'-1})$ the branch leading to $B_{h'}$, where h' is maximal such that this is nonempty, and $C = (C_1, \ldots, C_{h'})$ a sequence of finite enumerated subsets C_i of $P_i(a_{i-1})$. Then whether (B, a, C) is a section support or not is determined by its type in L_{sk}, and if this is so, then the section U supported by it consists of everything of level less than h together with everything of level h that lies above an element of one of the sets B_i.*

Furthermore, a and C are of bounded size, and allow B to be recovered from data of the form $(B'_i; B_{ij})_{i \text{ newtype}}$, where B'_i is a finite subset of $P_i(a)$ for i of new type and B_{ij} is a sequence of subsets of B'_i.

Proof. The last paragraph is really the key. In the case in which we are in fact dealing with a section support, the B_{ij} should be the initial segments at level i gotten by projecting the B_j when $h(j) = i$ (but in the affine-with-dual case B_j is either a finite subset of the dual part, or the whole dual component plus a finite subset of the affine part, and in the present context one should throw away the affine dual part if it is completely contained in B_j), and B'_i should be their union (i.e., the longest one).

To determine whether we actually have a section support, what we must determine is whether a candidate sequence B_{ij} of finite subsets of a geometry does, in fact, constitute a sequence of initial segments of that geometry with respect to some standard ordering.

An initial segment of a standard ordering on one of the projective geometries contains an initial segment of the standard basis from which the ordering was defined; conversely, if such a finite basis is found in the set B'_i, isomorphic to an initial segment of a standard basis, and making all B_{ij} initial segments in the induced ordering (relative to some ordering of the base field), then it can be completed to a standard basis for the whole space, for which the given sets constitute initial segments.

Definition 4.2.8. *A* reduced section support *is a sequence B of sequences $B_i = (B_{ij})$ for i of new type and $j = i$ or $\sigma(j) = i$, together with auxiliary data (of bounded size) $\mathbf{a}$, $C_i(a)$ ($a \in \mathbf{a}$) as in the previous lemma, and the maximal elements a_{ij} of the B_{ij} in a standard ordering of B_i. The $C_i(a)$, $\mathbf{a}$, and a_{ij} will be called the* bounded part *of the section support.*

Remarks 4.2.9

When the standard ordering on the projectivizations of the $P_i(a)$ is fixed, the B_{ij} are determined by B_i and the bounded part, specifically the a_{ij}.

Sections are atomically L_{sk}-definable from their reduced section supports. We may speak also of sections and section supports in envelopes of Lie coordinatized structures, as they can be described in terms of their atomic L_{sk} types.

4.3 FINITE LANGUAGE

Definition 4.3.1. *Let $\mathcal{M}$ be a fully proper countable skeletal expansion.*

Triples (E, X, a) with E an envelope for $\mathcal{M}$, $X \subseteq E$, and a a finite sequence of elements of E, will be partially ordered by the following relation:

$(E, X, a) \leq (E', X', a')$ *if and only if*
there is an elementary map $f : E \to E'$
for which $f[X] \subseteq X'$ *and* $f(a) = a'$.

The partial orderings of interest to us here will be restrictions of this ordering to the sets $\mathcal{U}_n$ and $\mathcal{S}_n$ of triples in which a has length n and X is, respectively, a section U of E or a reduced section support S for E.

Lemma 4.3.2. *Let $\mathcal{M}$ be a proper countable skeletal expansion. Let $(a_0, a_1, \ldots, a_h)$ be a branch of the tree, and let α_i be an automorphism of the $P_i(a_{i-1})$ for i of new type. Then the union of the α_i is an elementary map in $\mathcal{M}$.*

Proof. Full embedding and orthogonality. The orthogonality theory applies directly to the projectivizations, but the semiprojective geometries are definable over them and have the same automorphism group.

Lemma 4.3.3. *Let $\mathcal{M}$ be a proper countable skeletal expansion. The partial orderings defined above on $\mathcal{U}_n$ and $\mathcal{S}_n$ are well quasi-orderings.*

Proof. The result for reduced section supports implies the result for sections, so we focus on $\mathcal{S}_n$. We can drop the envelope E from the triple, since given (E, B, a) and (E', B', a') with E a μ-envelope, E' a μ'-envelope, and $\mu(J)$ embedding in $\mu'(J)$ everywhere, and an elementary map f with $f[B] \subseteq B'$ and $f(a) = a'$, there is an elementary map $E \to E'$ extending it, by (essentially) Lemma 3.2.4. We may thin the original sequence so that the condition on comparability of μ and μ' holds everywhere.

We treat the case of reduced section supports. This is done as in [HrTC, Lemma 2.10], which, however, makes use of rather abstract notation for part of the situation.

Increasing n slightly, we may suppose that the bounded part of the reduced section support is encoded in a. Now take a sequence $S_k = (B^{(k)}, a^{(k)})$ of reduced section supports with auxiliary data. Adjusting by automorphisms of the geometries, using the previous lemma, we may suppose that the orderings used on the projective geometries of new type are fixed standard orderings, so that the terms B_i (which initially are sequences (B_{ij})) can be thought of as initial segments of these geometries. Moving up through the levels i which are of new type, and thinning the sequence S_k at each stage, we will construct the desired elementary maps in stages. What we require at stage i is that the maps be defined through the ith level, be order-preserving on each projective geometry associated with a level of new type, and fix the data in $a^{(k)}$ occurring up to the ith level. We require of the sequence S_k that the type of $a^{(k)}$ over $\bigcup_j P_j(b^{(k)}_{j-1})$ (with $b^{(k)}$ the branch being followed by the $B^{(k)}_i$) be fixed. If this is the case at a given stage, it can be preserved without difficulty up to the next new level i. At such a new level i, the elementary maps will have to be chosen carefully to preserve the types of $a^{(k)}$ over the union including the ith level.

Let $A_k = \bigcup_{j<i} P_j(b^{(k)}_{j-1})$. The type of $a^{(k)}$ over $A_k \cup P_i(b^{(k)}_{i-1})$ is determined by its (known) type over A_k and its type over $c_k = acl(a^{(k)}) \cap P_i(b^{(k)}_{i-1})$. So we impose on our elementary maps the additional constraint that they preserve the c_k. Exactly this condition is allowed by geometrical finiteness, after thinning the sequence S_k (and applying Ramsey's theorem): for $k < l$ we may carry $B^{(k)}_i$ into $B^{(l)}_i$ by an order-preserving elementary map which carries c_k to c_l. Thinning down so that the types of the $a^{(k)}$ over the c_k correspond, this completes the ith stage.

Lemma 4.3.4. *Let E be an envelope, U a section of E, and E' an envelope contained in E, with the support S of U contained in E'. Then $E' \cap U$ is the section of E' supported by S.*

Proof. The statement is a bit misleading; the issue is not so much whether S supports $E' \cap U$, but rather whether S fulfills the definition of section support relative to E' in the first place. This is essentially one of the points made in Lemma 4.2.7. In the present version, the statement is that if B is an increasing sequence of initial segments of a projective Lie geometry J, with respect to some standard ordering, and lies in a subgeometry J' of J, then B is also a sequence of initial segments of J' with respect to a standard order, the point being that an initial segment of an appropriate standard basis can be extracted from B and completed in J or J'.

Lemma 4.3.5. *Let $\mathcal{M}$ be a Lie coordinatized structure. Then there is an integer k with the following properties:*

1. *For any envelope E, any section U of E, and any $a \in E$, if $a \in acl(U)$ then for some subset C of U of size at most k, a is algebraic over C and its multiplicity over U and over C coincide.*
2. *For any envelope E, any section support S in E, and any $a \in E$, if $a \in acl(S)$ then for some subset C of S of size at most k, a is algebraic over C and its multiplicity over S and over C coincide.*

Proof. The contrary to (1) would yield as a counterexample a sequence (E_k, U_k, a_k) refuting the claim for each k. After passing to a subsequence and applying Lemma 4.3.3 we get a single element a algebraic over an increasing chain of sets U_{k_i} but whose type over U_{k_i} cannot be fixed by k_i elements. The multiplicity m of a over $\bigcup_i U_{k_i}$ is of course the same as its multiplicity over some finite set C contained in all U_{k_i} from some point on, and once $k_i > |C|$ we reach a contradiction.

The failure of (2) is refuted similarly.

Definition 4.3.6. *The* standard language *for a Lie coordinatized structure will be the language L containing all 0-definable $(k+1)$-ary predicates with k (minimal) furnished by the preceding lemma. Note that $k \geq 2$.*

Proposition 4.3.7. *Every Lie coordinatized structure $\mathcal{M}$ admits a finite language L. The standard language will do. The standard language also satisfies the following homogeneity conditions:*

1. *Every section of any envelope of $\mathcal{M}$ is L-homogeneous: if E is an envelope of $\mathcal{M}$, U a section of E, and $f : U \to \mathcal{M}$ an L-map, then f is elementary.*
2. *Every section support of any envelope of $\mathcal{M}$ is L-homogeneous in the same sense.*

Proof. Let L be the standard language for $\mathcal{M}$. Part (1) includes the statement that the language L is adequate for $\mathcal{M}$. We use semiprojectives rather than projectives in the coordinatization.

Both (1) and (2) reduce to finite envelopes, using Lemma 4.3.4. We can enumerate the envelope E so that any initial segment of E is a section. Here we are viewing the envelope as a subset of a coordinatized structure (in the construction of envelopes, we added some sorts of $\mathcal{M}^{\text{eq}}$). Whenever we encounter an affine point the whole dual-affine part is already in the part enumerated. For (1) it suffices to show

(1′) For any section U of an envelope E of $\mathcal{M}$, and a the next element of E, the L-type of a over U determines its type over U.

In the algebraic case this holds by the choice of k. In the nonalgebraic case the L-type of a over U ensures that a is nonalgebraic, again by the choice of k. Let P be the component of the coordinatizing tree in which a lies. We claim that

$$(*) \qquad acl(U) \cap P \subseteq U.$$

As a is not algebraic over U, P is neither finite nor a semiprojective geometry "repeating" an earlier one. Thus it is either a semiprojective geometry of new type or an affine-with-dual pair. Consider the affine case. Again by the nonalgebraicity assumption, U will contain no affine point of P, while a is affine; as a is the next point of the enumeration, U contains the full dual-affine part of P in E, and as E is itself algebraically closed in $\mathcal{M}$, the claim $(*)$ holds in this case. Suppose now that P is semiprojective of new type, so orthogonal to all projective geometries J' at lower levels. Then $acl(U) \cap P = acl(U \cap P) \cap P$. This reduces our claim to the corresponding claim $(*)$ in a single geometry, where it is a property of standard enumerations.

This gives $(*)$. Now in $\mathcal{M}$ as P is fully embedded, the type of a over $acl(U) \cap P$ implies its type over U, and by $(*)$ $acl(U) \cap P$ is $U \cap P$. To conclude, then, it suffices to observe that $tp_k(a/U \cap P)$ proves $tp(a/U \cap P)$, which holds since $k \geq 3$ and P is a-definable (directly from the tree language, in fact).

For (2) we may proceed similarly, extending f over an enumeration of E.

Lemma 4.3.8. *Let $\mathcal{M}$ be Lie coordinatized, L the standard language for $\mathcal{M}$. Then for any section U of any envelope E, the theory of U is model complete.*

Proof. We must show that any type in U is equivalent to an existential type. We show by induction on the section U:

$(*)$ For any finite sequence c in U there is a finite sequence c' in U such that $tp_L(cc')$ implies $tp_{\mathcal{M}}(c)$.

Granted this, if c is expanded first to contain a support for U, then the type of c in $\mathcal{M}$ will determine its type in U, and our claim follows.

This statement passes through at limit stages, so we deal with the case $U = U_1 \cup \{a\}$. We may suppose $c = c_1 a$ with c_1 from U_1. We need first a finite set C such that $tp_L(a/C)$ determines $tp(a/c_1)$. This is a consequence of $(1')$ from the previous proof. (C will grow with c_1 in general, when a is the first affine point.) We may suppose $c_1 \subseteq C$.

It is useful at this stage to make the statement "$tp_L(a/C)$ determines $tp(a/c_1)$" more explicit. This is a statement belonging to the type of C; another way of putting it is that the type of C and the L-type of a over C determine the type of $c_1 a$.

We let C' be chosen by applying $(*)$ inductively to C and U'. We claim that $tp_L(CaC')$ determines $tp_{\mathcal{M}}(c_1 a)$. Given $tp_L(CaC')$, we first recover $tp_{\mathcal{M}}(C)$. Then we know that $tp_L(aC)$ determines $tp_{\mathcal{M}}(c_1 a)$.

4.4 QUASIFINITE AXIOMATIZABILITY

In this section we provide reasonably explicit axiomatizations of theories of Lie coordinatized structures, modulo certain information which is determined only qualitatively by the geometrical finiteness of the coordinatizing geometries.

Definition 4.4.1. *Let $\mathcal{M}$ be Lie coordinatized and L a specified language for $\mathcal{M}$. A* characteristic sentence *for $\mathcal{M}$ is an L-sentence whose countable models which are skeletal expansions with true dimensions are exactly the envelopes of $\mathcal{M}$ and their isomorphic images.*

Lemma 4.4.2. *Let a skeletal type and corresponding skeletal language L_{sk} be fixed. For any k there is a (uniformly computable) integer k^* such that any $2k$ elements of a section U of a skeleton $\mathcal{M}$ for L_{sk}, with support S, are contained in a subsection U' whose support S' has the same bounded part and satisfies $|S'| \le k^*$.*

Proof. Note that the subsection will be taken with respect to a different ordering.

This statement reduces to the same statement in a single projective geometry. The existence of k^* follows from the geometric finiteness. Its computability follows from the decidability of the theory of the geometry.

Proposition 4.4.3. *Let a skeletal type and corresponding skeletal language L_{sk} be fixed, and let L be a finite language containing L_{sk}. Then there is a recursive class Ξ of (potential) characteristic sentences, which can be found uniformly in the data L_{sk}, L, with the following properties:*

1. *If $\mathcal{M}$ is a skeletal expansion with true dimensions relative to L_{sk}, and $\mathcal{M} \models \xi$ (some $\xi \in \Xi$), then every countable model of ξ with true dimensions is isomorphic with an envelope of $\mathcal{M}$.*
2. *Any Lie coordinatized structure with coordinatizing skeleton M_{sk} satisfies one of the sentences in Ξ.*

In particular, every Lie coordinatized structure has a characteristic sentence.

Proof. We form the set Ξ^* of sextuples $(\xi, k, k^*, k^{**}, L', \Sigma)$ satisfying the following six conditions, and then take Ξ to consist of the sentences ξ for which some suitable k,k^*,k^{**},L',and Σ can be found; this will make Ξ recursively enumerable but by a standard device any r.e. set of sentences is equivalent to a recursive set: it suffices to replace each sentence ξ by a logically equivalent one whose length is at least the time taken to enumerate ξ.

The conditions on $(\xi, k, k^*, k^{**}, L', \Sigma)$ are as follows:

(i) L' is a list of formulas of L, each with at most $k+1$ free variables. L' is to be thought of as a new language, and the given formulas will be called L'-atomic formulas. These formulas will include the atomic formulas of L. Σ is a finite set of existential L'-formulas.

(ii) ξ implies the skeletal theory $T_{\rm sk}$, apart from the clause asserting infinite dimensionality of certain geometries.

(iii) ξ asserts that certain quantifier free L'-formulas in $k+1$ free variables are algebraic in the last k variables, that is for each choice of these k variables, the formula has only finitely many solutions (with a specified bound). These formulas will be called *explicitly algebraic*.

(iv) For any $\forall\exists$ L'-sentence with k^* universal quantifiers and $k+1$ existential ones, ξ specifies the truth or falsity of the statement.

(v) For any section support S of size $l \leq k^*$ whose atomic L'-type is r (in l variables), and for any L'-formula φ in these l variables with at most $k+1$ quantifiers, ξ implies that either all realizations of r satisfy φ, or all realizations of r satisfy $\neg\varphi$.

(vi) For any section U of a model $\mathcal{M}$ of ξ with support S of size at most k^*, and any $a \in \mathcal{M}$, ξ asserts that one of the following occurs (to be elucidated more fully below):

(vi.a) There is a set $B \subseteq U$ of order at most k for which the quantifier-free L'-type of a over B is explicitly algebraic and "implies its L'-type over U";

(vi.b) a lies in an affine-with-dual geometry J whose dual affine part D (if present) lies in U, and the geometric type of a over D "implies its L'-type over U."

(vi.c) a lies in a semiprojective geometry of new type J and the geometric type of a over $J \cap U$ "implies its L'-type over U."

It remains to formalize condition (*vi*) more completely, and in so doing to explain the role of the formulas in Σ. We are dealing with expressions of the form "ξ states that $tp^*(a/X)$ determines $tp(a/U)$" where the second type is an atomic L'-type and the first type is some part of an atomic L'-type.

To formalize (vi.a) we consider a formula $\alpha(x; y)$ expressing the atomic L'-type of a over B, $|B| \leq k$, with x standing for a and y for B, and we consider any other formula $\beta(x; y')$ in $l \leq k$ variables. We are trying to formalize (and to put into ξ) the statement $(\alpha \Longrightarrow \beta)$, whenever this is true. This is done as follows, elaborating on the model completeness:

(vi.a') For any $B' \subseteq U$ with $|B'| = l$ (enumerated as a sequence of length l), and any section support $S' \subseteq S$ with $|S'| \leq k^*$

such that the section U' supported by S' contains $B \cup B'$: if $\beta(a, B')$ holds then there is an existential formula $\sigma(z, y, y')$ in Σ where z corresponds to an enumeration of S', true in U', such that ξ implies that $[\sigma(z, y, y') \& \alpha(x, y)] \Longrightarrow \beta(x, y')$.

The existential quantifiers in σ will refer to the section supported by z. We treat (vi.b) and (vi.c) similarly, e.g.:

(vi.b') For any $B' \subseteq U$ with $|B'| = l$ (enumerated as a sequence of length l), and any section support $S' \subseteq S$ with $|S'| \le k^{**}$ such that the section U' supported by S' contains the affine dual of the component of a and B': if $\beta(a, B')$ holds then there is an existential formula $\sigma(z, y, y')$ in Σ where z corresponds to an enumeration of S' and y enumerates some elements of the affine dual part, that σ holds in U' and ξ implies that $[\sigma(z, y, y') \& \alpha(x, y)] \Longrightarrow \beta(x, y')$.

We require of course that for every β involving k variables there should be a suitable α for which the corresponding version of (vi) holds. This can be viewed as a condition on k^* and k^{**}, particularly when we wish to verify point (2).

We claim that with this choice of ξ, (1,2) hold. We begin by commenting on (2), which amounts to an elaboration of the proof of the existence of a finite language. The parameter k is the one used to define a standard language, and L' is the standard language, given in terms of 0-definable relations in the specified language L. Clause (iii) is natural in view of the definition of k; given $\mathcal{M}$, all the formulas of the given type which are algebraic in $\mathcal{M}$ will be made explicitly algebraic. Point (v) reflects the homogeneity of section supports. Finally, point (vi) reflects the control of types over envelopes, and the model completeness of the theory of the envelopes. Part (vi.a) is an accurate reflection of the role of k as a bound for the base of algebraicity over an envelope. Point (vi.b) requires further elucidation. We will have in general $tp_G(a/D) \vdash tp_{L'}(a/U)$ ("G" for "geometric"). Now $tp_{L'}(a/U)$ consists of formulas β of the appropriate form for (vi.b'). The formulas $\alpha(x, y)$ coming from $tp_G(a/D)$ may require more than k variables. However, given $\mathcal{M}$, there will be a bound k_1 for the number of variables needed, and a corresponding bound k^{**} for the size of a section support needed to capture $k_1 + k$ variables. Then (vi.b') expresses (vi.b).

We turn to (1): $\mathcal{M}$ is a proper L-structure relative to $L_{\rm sk}$ and $\mathcal{M} \models \xi$ (some $\xi \in \Xi$). We claim that every countable model $\mathcal{M}'$ of ξ is isomorphic with an envelope of $\mathcal{M}$ (or with the restriction of an envelope in an adequate expansion of $\mathcal{M}$, to the sorts of $\mathcal{M}$).

If $\mathcal{M}^*$ is an $\aleph_1$-saturated elementary extension of $\mathcal{M}$ then $\mathcal{M}$ is the countable envelope for $\mathcal{M}^*$ with all μ-invariants infinite dimensional. It suffices to show that $\mathcal{M}'$ is isomorphic with an envelope in $\mathcal{M}^*$.

We enumerate $\mathcal{M}'$ so that each initial segment is a section of the skeleton, and we define a map $F : \mathcal{M}' \to \mathcal{M}^*$ by induction. An *approximation* to F will be a pair (f, U) satisfying the following three conditions:

(a) U is a section of $\mathcal{M}'$ with support S;
(b) f is an L'-embedding of U into $\mathcal{M}^*$;
(c) If J_b is a semiprojective component of $\mathcal{M}'$ of new type, with $b \in U$, $J_b \subseteq U$, then $acl(f[U]) \cap J_{f(b)}$ is $f[J_b]$.

Condition (c) essentially rules out "accidents" in which as f is extended, some new value generates a coordinate in a geometry which has already been dealt with. Since we have been rather more careful in the axiomatization to specify what is algebraic than we have been to avoid algebraicity, there is something to be concerned with.

If we are able to carry out the inductive step in which a single element is added to U, then the construction passes smoothly through limit stages and produces a total $(F, \mathcal{M}')$ satisfying the conditions (b, c) with $U = \mathcal{M}'$. By (c) the image of F will be algebraically closed in each semiprojective component of new type coded by an element of the image. It follows easily that $F[\mathcal{M}']$ is algebraically closed in $\mathcal{M}^*$. Also if $c \in \mathcal{M}^* - F[\mathcal{M}']$ then there is c' definable from c with the same property, lying in a semiprojective component of new type, whose defining parameter is in the image of F. Again (c) applies and leads to the maximality clause in the definition of envelope after passing to the canonical projective associated with the given component (one of the sorts which should be added to $\mathcal{M}$ in an adequate expansion).

The last point is that the isomorphism type of a coordinatizing component of $F[\mathcal{M}']$ with a given defining parameter b is constant over all conjugates of b (in $\mathcal{M}^*$) lying in the image. This follows since F is an L-embedding.

So what remains to be checked is the extendability of an approximation (f, U) to the next element a of $\mathcal{M}'$. Let J be the component of $\mathcal{M}'$ in which a lies. Then the L'-type of a is determined either by an explicitly algebraic formula ψ, or a geometric type over part of U. We extend f by letting $f(a)$ be any realization of the corresponding type in $\mathcal{M}^*$. If a is explicitly algebraic then condition (v) implies that $\mathcal{M}^*$, a model of ξ will realize this type. If a is geometric, then $\mathcal{M}^*$, being a Lie coordinatized model in the first place, will realize the appropriate type, using saturation. Let the extension be denoted (f', U'). We claim that the conditions (b, c) are preserved.

Condition (b) is controlled by properties (vi.a, vi.b) of ξ. Note here that the auxiliary formulas in Σ are existential and hence are preserved by embedding.

The condition (c) is obviously preserved if a is algebraic over U or more generally if $acl\, f[U'] \cap J_{fb} = acl\, f[U] \cap J_{fb}$. So we must consider the case in which a is not algebraic over U but some element of J_{fb} not in $acl\, f[U] \cap J_{fb}$ becomes algebraic over $f[Ua]$. Let S be the support of the section U, and let U^* be the section of $\mathcal{M}^*$ supported by $f[S]$, which contains J_{fb} in particular. Then fa is algebraic over U^* and hence is k-algebraic over some section of $\mathcal{M}^*$ whose support $fS' \subseteq fS$ is of size at most k^*. Accordingly ξ asserts some element a' of the geometry J containing a in $\mathcal{M}$ will be algebraic over the section U' supported by S'. In particular $acl(U)$ meets J. On the other hand $a \notin acl(U')$. Thus J is a new geometry and by orthogonality theory in $\mathcal{M}^*$, $acl\, f[Ua] \cap J_{fb} = acl\, f[U] \cap J_{fb}$.

4.5 ZIEGLER'S FINITENESS CONJECTURE

Proposition 4.5.1. *Let a skeletal type and corresponding skeletal language $L_{\rm sk}$ be fixed, and let L be a fixed finite language containing $L_{\rm sk}$. Then there are only finitely many Lie coordinatized structures in the language L having a given skeleton $M_{\rm sk}$, up to isomorphism.*

Proof. It suffices to combine Proposition 4.4.3 with the Compactness Theorem. For this one must check that the class of Lie coordinatized structures in the language L with the specified skeleton is an elementary class. Thus one must review the various conditions involved in Lie coordinatization.

Note that the skeleton fixes the language of the individual geometries. In particular, the notion of canonical embedding is first order, as is the notion of orientability.

One must also express the condition of stable embedding for the geometries. We can use Lemma 2.3.3. Thus it suffices to bound the size of $acl(a) \cap J$ uniformly. But $|\, acl(a) \cap J|$ has dimension at most the height of the skeleton times the length of a.

Thus compactness applies.

Definition 4.5.2. *Let $\mathcal{M}$ be a structure.*

1. *A* cover *of $\mathcal{M}$ is a structure $\mathcal{N}$ and a map $\pi : N \to M$ such that the equivalence relation E_π given by "$\pi x = \pi y$" is 0-definable in $\mathcal{N}$, and the set of E_π-invariant 0-definable relations on $\mathcal{N}$ coincides with the set of pullbacks along π of the 0-definable relations in $\mathcal{M}$.*
2. *Two covers $\pi_1 : \mathcal{N}_1 \to \mathcal{M}$, $\pi_2 : \mathcal{N}_2 \to \mathcal{M}$ are* equivalent *if there is a bijection $\iota : N_1 \leftrightarrow N_2$ compatible with π_1, π_2 which carries the 0-definable relations of $\mathcal{N}_1$ onto those of $\mathcal{N}_2$.*
3. *If $\pi : \mathcal{N} \to \mathcal{M}$ is a cover, then $Aut(\mathcal{N}/\mathcal{M})$ is the group of automorphisms of $\mathcal{N}$ which act trivially on the quotient $\mathcal{M}$. Thus $Aut(\mathcal{N}/\mathcal{M}) \leq \prod_{a\in M} Aut_{\mathcal{N}}(C_a)$ where $C_a = \pi^{-1}(a)$ and $Aut_{\mathcal{N}}(C_a)$ is the permutation group induced by the setwise stabilizer of C_a in $Aut\,\mathcal{N}$.*

The problem of the theory of covers is to classify or at least restrict the possible covers with given quotient and specified fiber; that is, typically the structures $(C_a, Aut_{\mathcal{N}}(C_a))$ are specified in advance and are essentially independent of a. Any automorphism group will be a closed subgroup of the symmetric group (in the topology of pointwise convergence with the discrete topology on the underlying set); by the finiteness of language, in the Lie coordinatized case it is even k-closed for some finite k: any permutation which agrees on every set of k elements with an automorphism is itself an automorphism. in the $\aleph_0$-categorical context, furthermore, $Aut\,\mathcal{N}$ induces $Aut\,\mathcal{M}$; in particular, if the automorphism group of the fibers is abelian, then $Aut(\mathcal{N}/\mathcal{M})$ is an $Aut(\mathcal{M})$-invariant subgroup of the product.

Proposition 4.5.3. *Let $\mathcal{M}$ be a fixed Lie coordinatized structure and let J be a fixed geometry or a finite structure. Then there are only finitely many covers $\pi : \mathcal{N} \to \mathcal{M}$ up to equivalence which have fiber J and a given relative automorphism group $Aut(\mathcal{N}/\mathcal{M}) \leq \prod_{N/E} Aut\,J$.*

Proof. We apply Proposition 4.5.1. The skeleton N_{sk} of $\mathcal{N}$ is determined by the given data and thus it suffices to find a single finite language L adequate for all such covers $\mathcal{N}$. Thus it suffices to bound the arity k of L and the number of k-types occurring in $\mathcal{N}$.

We deal first with the arity, using the language of permutation groups. We must find a bound k so that $Aut(\mathcal{N})$ is a k-closed group, for all suitable covers $\mathcal{N}$. $Aut(\mathcal{M})$ is $k_\circ$-closed for some $k_\circ$. If we restrict attention to $k \geq k_\circ$, then $Aut(\mathcal{N})$ is k-closed if and only if $Aut(\mathcal{N}/\mathcal{M})$ is k-closed, as is easily checked. (Note that $Aut\,\mathcal{N}$ induces $Aut\,\mathcal{M}$ by $\aleph_0$-categoricity.)

Thus for $k \geq k_\circ$ the choice of k is independent of the cover, as long as the relative automorphism group is fixed in advance.

Now with k fixed, consider the number of k-types available in $\mathcal{N}$. If the fiber is finite of order m, then each k-type of $\mathcal{M}$ corresponds to at most m^k k-types of $\mathcal{N}$, so we have the desired bound in this case.

If the fiber is a geometry, to bound the number of k-types we proceed by induction, bounding the number of 1-types over a set A of size j for $j < k$. The 1-type of an element a of the geometry J_b over A is determined by its restriction to the algebraic closure of A in a limited part of J_b^{eq}, e.g. in the affine case the linear version must also be taken. It suffices therefore to bound the dimension of $acl(a) \cap J$ for geometries J associated to J_b. As $rk(Aa/\pi[A]a) \leq j$, the space $acl(\pi[A]a)$ has codimension at most j in $acl(Aa) \cap J$ and thus the desired bound for $\mathcal{N}$ can be given in terms of the data for $\mathcal{M}$.

Remarks 4.5.4

In cohomological terms, if $Aut\,J$ is abelian this may be expressed by:

$$H^1_c(Aut\,\mathcal{M}, (\textstyle\prod_{\mathcal{M}} Aut\,J)/K) \text{ is finite}$$

for $K \leq \prod_{\mathcal{M}} Aut\,J$ closed and $(Aut\,\mathcal{M})$-invariant. Cf. [HoPi].

For a more algebraic approach to this type of problem, due to David Evans, see the paper [Ev].