5

Geometric Stability Generalized

5.1 TYPE AMALGAMATION

Definition 5.1.1. *Let $\mathcal{M}$ be a structure.*

1. *An* amalgamation problem *(for types) of length n is given by the following data:*
   - (i) *A base set, A;*
   - (ii) *Types $p_i(x_i)$ over A for $1 \le i \le n$;*
   - (iii) *Types $r_{ij}(x_i, x_j)$ over A for $1 \le i < j \le n$;*

   subject to the conditions:
   - (iv) *r_{ij} contains $p_i(x_i) \cup p_j(x_j)$;*
   - (v) *$r_{ij}(x_i, x_j)$ implies the independence of x_i from x_j.*
2. *A* solution *to such an amalgamation problem is a type r of an independent n-tuple $x_1, \ldots, x_n$ such that the restrictions of r coincide with the given types.*

Definition 5.1.2. *A structure $\mathcal{M}$ has the* type amalgamation property *if whenever $(p_i; r_{ij})$ is an amalgamation problem defined over an algebraically closed base set in $\mathcal{M}^{\text{eq}}$, then the amalgamation problem has a solution.*

Our goal here is to prove that Lie coordinatized structures have the type amalgamation property. By absorbing the base set A into the language we may suppose it coincides with $acl(\emptyset)$ and we will do so whenever it is notationally convenient. Our usual notation for an amalgamation problem will be either $(p_i; r_{ij})$ or just (r_{ij}), assuming the length n is known. Occasionally we will take note of generalized amalgamation problems where other restrictions are placed on the desired type r.

We build up to the general result via a series of special cases, beginning with types in a single geometry. The general result does not follow directly from the case of a single geometry, but reflects more specific properties of the geometries, as is seen in the proof of Lemma 5.1.13.

Lemma 5.1.3. *Let J be a Lie geometry, and $(p_i; r_{ij})$ an amalgamation problem of length n in which the p_i are types of sequences of elements of J over $acl(\emptyset)$. Then the amalgamation problem has a solution.*

We will leave the details to the reader, but we make a few remarks. This statement essentially comes down to the fact that inner products and quadratic forms can be prescribed arbitrarily on a basis, subject to the restrictions associated with the various types of inner product.

It may be more instructive to take note of some counterexamples to plausible strengthenings of this property. We give two examples where the solution sought is not unique, and one example of an amalgamation property incorporating a bit more data which fails to have a solution.

Example 5.1.4. *Let (V, V^*) be a polar geometry, and A an affine space over V^*. Consider independent triples (a_1, a_2, a_3) with $a_1 \in V$ and $a_2, a_3 \in A$. The relevant types r_{ij} are then determined but the type of the triple depends on the value of $(a_1, a_2 - a_3)$, which is arbitrary.*

Example 5.1.5. *In a projective space $\hat{V}$ associated with a unitary geometry V over a field K of order q^2, consider the 2-type r of a pair $\hat{x}, \hat{y}$ of independent elements of $\hat{V}$ for which $(x, y) \neq 0$ and $(x, x) = (y, y) = 0$. This defines a complete type over $acl(\emptyset)$. We consider the amalgamation problem of length 3 with all r_{ij} equal to r. For an independent triple $(\hat{x}, \hat{y}, \hat{z})$ whose restrictions realize the type r, the quantity $(x, y)(y, z)(z, x)/(y, x)(z, y)(x, z)$ is a projective invariant taking on $q + 1$ possible values α/α^σ ($\alpha \in K^*$, σ an involutory automorphism of K).*

Example 5.1.6. *We will give a generalized amalgamation problem of length 4, determined by a compatible family of 3-types r_{ijk} over $acl(\emptyset)$ of independent triples, which has no solution. Let V be a symplectic space, A affine over V, and consider the type of a quadruple x_1, x_2, x_3, x_4 with $x_1 \in V$ and the remaining x_i affine. Let the types r_{1ij} all contain the requirement: $(x_1, x_i - x_j) = 1$. These requirements are incompatible.*

Lemma 5.1.7. *Let $\mathcal{M}$ be a structure, and suppose that every amalgamation problem of length 3 in $\mathcal{M}$ over an algebraically closed subset has a solution. Then every amalgamation problem in $\mathcal{M}$ has a solution.*

Proof. This is a straightforward induction. Collapse the last two variables $x_{n-1}x_n$ to one variable y_n and define a new amalgamation problem (r'_{ij}) of length $n - 1$. The only point requiring attention is the choice of the types $r'_{i,n-1}$, which are 3-types when written in terms of the x_i. These are taken to be solutions to the amalgamation problem

$(r_{i,n-1}, r_{i,n}, r_{n-1,n})$ of length 3.

In the next lemma we find it convenient to deal with a variant form of amalgamation problem incorporating some additional information.

Lemma 5.1.8. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure, and J a geometry of $\mathcal{M}$. Suppose that $(p_i; r_{1,i}, r_{2,\dots,n})$ is a generalized amalgamation problem over $acl(\emptyset)$ in which p_1 is the type of some element of J and $r_{2,\dots,n}$ is the type of an independent $(n-1)$-tuple, with the types r extending the corresponding types p appropriately. Then this generalized amalgamation problem has a solution.*

Proof. We fix a realization $(c_2, \dots, c_n)$ of $r_{2,\dots,n}$, we set $C_i = acl(c_i) \cap J$, and we choose $c_1^i c_i$ satisfying r_{1i} for $2 \le i \le n$. We define an auxiliary generalized amalgamation problem in J by setting $r'_{1i} = tp(c_1^i C_i)$, $r'_{2,\dots,n} = tp(C_2, \dots, C_n)$. By inspection of the geometries, this type of problem has a solution r'. We may choose c'_1 so that $c'_1 C_2 \dots C_n$ realizes the type r'. As any c_i-definable subset of J is C_i-definable, we find that $tp(c'_1 c_i) = tp(c_1 c_i)$ and the sequence $c_1, c_2, \dots, c_n$ is independent.

Roughly speaking our goal is now to treat the general amalgamation problem of length 3 by reduction to the case in which the type p_1 has rank 1. More specifically we deal with the following notion.

Definition 5.1.9. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure and J one of its geometries.*

A semigeometric *1-type relative to J is the type over $acl(\emptyset)$ of some pair (a, b) with $a \in J$ and b algebraic over a. The* multiplicity *of such a type is the multiplicity of b over a.*

Lemma 5.1.10. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure and suppose that every amalgamation problem $(p_i; r_{ij})$ of length 3 with p_1 semigeometric has a solution. Then every amalgamation problem of length 3 has a solution.*

Proof. If we can solve amalgamation problems with p_1 semigeometric, then by compactness we can solve amalgamation problems in which p_1 is a type in infinitely many variables, representing the full algebraic closure in $\mathcal{M}^{\mathrm{eq}}$ of an element of a geometry of $\mathcal{M}$.

We now argue by induction on the rank of p_1, which we may take to be at least 1. Let c_1 realize p_1 and let $a_1 \in acl(c_1)$ belong to a coordinatizing geometry J of $\mathcal{M}$. Let A be $acl(a_1)$ in $\mathcal{M}^{\mathrm{eq}}$ and $p'_1 = tp(A)$.

Take c_2, c_3 independent and such that $c_1 c_i$ realizes the type r_{1i} for $i = 2, 3$. Let $r'_{1i} = tp(Ac_i / acl(\emptyset))$ and $r'_{23} = r_{23}$. Then (r'_{ij}) gives an amalgamation problem of length 3 of the type referred to at the outset. Let r' be a solution to this problem. We may suppose that Ac_2c_3 satisfies

r'.

Now we will work over A with $p''_i = tp(c_i/A)$ for $i = 1, 2, 3$ and $r''_{ij} = tp(c_i c_j/A)$. By the choice of r' this is an amalgamation problem, and the rank of p'_1 is less than the rank of p_1, so we conclude by induction.

Before treating the general amalgamation problem of length 3 with p_1 semigeometric, we will deal with the case in which $r_{12} = r_{13}$ up to a change of variable. We begin with some technical considerations.

Definition 5.1.11. *Let $\mathcal{M}$ be a structure, E a definable binary relation, D a definable set, and a, b elements of $\mathcal{M}$.*

1. *E is a* generic equivalence relation *on D if it is generically symmetric and transitive: for any independent triple a, b, c in its domain, $E(a, b)$ and $E(b, c)$ imply $E(b, a)$ and $E(a, c)$.*
2. *An indiscernible sequence I is 2-*independent *if $acl(a) \cap acl(b) = acl(\emptyset)$ for $a, b \in I$ distinct.*
3. *$E_2(x, y)$ is the smallest equivalence relation containing all pairs belonging to infinite 2-independent indiscernible sequences.*

Lemma 5.1.12. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and E a generic equivalence relation defined on the locus of a complete type p over $acl(\emptyset)$. Then*

1. *E agrees with a definable equivalence relation E^* on independent pairs from p.*
2. *If every pair of elements belonging to an infinite 2-independent indiscernible sequence belongs to E, then any pair of independent realizations of p belongs to E.*

Proof. Ad 1. Define $E^*(x, y)$ by "$p(x)$ and $p(y)$ hold and either $x = y$ or there is a z which realizes p and is independent from x, y such that $E(x, z)$ and $E(y, z)$ both hold." This is easily seen to agree with E on independent pairs, and is reflexive and symmetric. We check transitivity.

Assume $E^*(a, b)$ and $E^*(b, c)$ hold, specifically

$$E(a, d_1),\ E(b, d_1),\ E(b, d_2),\ E(c, d_2)$$

with d_1 independent from a, b and d_2 independent from b, c; we may assume, in fact, that d_2 is independent from a, b, c, d_1. Then a, d_1, d_2 and b, d_1, d_2 are independent triples and thus $E(d_1, d_2)$ and $E(a, d_2)$ hold. Thus $E^*(a, c)$ holds.

Ad 2. In view of the preceding and the hypotheses, we may assume that E is a definable equivalence relation containing E_2. It suffices now to show that any two elements of $\mathcal{M}$ with the same type over $acl(\emptyset)$ are E_2-equivalent. We show in fact that M/E_2 is finite, and hence is part of $acl(\emptyset)$ in $\mathcal{M}^{\text{eq}}$, yielding the claim.

Suppose toward a contradiction that M/E_2 is infinite. We will choose realizations a_i of p inductively, distinct modulo E_2, so that

$$acl(a_n) \cap \bigcup_{i<n} acl(a_i)) = acl(\emptyset).$$

Then we may suppose that the sequence $I = (a_i)$ is also indiscernible, and we have a blatant contradiction to the definition of E_2.

For the choice of a_n given a_i $(i < n)$ we first choose a new E_2-class C outside $acl(\emptyset)$ independent from $a_1, \ldots, a_{n-1}$ and then choose $a \in C$ independent from $a_1, \ldots, a_{n-1}$ over C.

Lemma 5.1.13. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure. Let $(p_i; r_{ij})$ be an amalgamation problem of length 3 over $acl(\emptyset)$ with p_1 semigeometric and with $r_{12} = r_{13}$ up to a change of variable; in particular $p_2 = p_3$. Then the amalgamation problem has a solution.*

Proof. As a matter of notation, take $p_1 = p_1(xy)$, $p_i = p_i(z_i)$ for $i = 2, 3$. Let J be the geometry in which the first coordinates of realizations of p_1 are found, and let C be the set defined by p_2 or p_3. We make a preliminary adjustment to ensure that for $c \in C$ we have

$$(*) \qquad r_{12}(xy, c) \text{ isolates a type over } acl(c).$$

We may replace c by some $c' \in acl(c)$ such that $c \in dcl(c')$ and $r_{12}(xy, c')$ isolates a type r'_{12} over $acl(c) = acl(c')$; the condition "$c \in dcl(c')$" means that c' can be thought of as being an extension cc'' of c. We then replace the given amalgamation problem by a problem (r'_{ij}) in which $r'_{23}(z'_1 z'_2)$ is any complete type over $acl(\emptyset)$ extending $r_{23}(z'_1 z'_2) \cup p'(z_1) \cup p'(z_2)$ where p' is the type of c' and the connection between the variables z_i and z'_i reflects the relation $c \in dcl(c')$; one may even suppose that z_i is an initial segment of z'_i. After these adjustments $(*)$ holds.

Now for $a \in J$ satisfying p_1, $c, c' \in C$ we consider the set $B(a, c) = \{y : r_{12}(ay, c)\}$ and the sets $J(c) = \{a \in J : B(a, c) \neq \emptyset\}$, $J(c, c') = \{a \in J : B(a, c) = B(a, c') \neq \emptyset\}$. In particular $J(c, c') \subseteq J(c) \cap J(c')$. We define a relation E on C as follows: $E(c, c')$ if and only if $J(c, c')$ is infinite. Using our understanding of J we will show that E is a generic equivalence relation extending E_2, and hence by the preceding lemma that $E(c_2, c_3)$ holds for any independent pair c_2, c_3 in C, in particular for a realization of r_{23}. This then allows us to solve the amalgamation problem directly.

We now check that E contains all pairs belonging to an infinite 2-independent indiscernible sequence I. Let μ be the multiplicity of the semigeometric type p_1 and let I' be a subset of I of cardinality 2^{μ}. By Lemma 5.1.8 we can find an element a independent from I' such that

$B(a,c) \neq \emptyset$ for $c \in I'$. As this gives us 2^μ nonempty subsets $B(a,c)$ of $\{b : p_1(a,b)\}$, two of them must coincide, and then by indiscernibility, any two of them must coincide. As there are infinitely many such elements a, $E(c,c')$ holds for pairs in I.

It remains to be seen that E is a generic equivalence relation. We take c, c', c'' independent with $E(c,c')$ and $E(c',c'')$ holding. Thus $J(c,c')$ and $J(c',c'')$ are infinite subsets of $J(c')$, and we claim that $J(c,c'')$ is also infinite; in fact we claim that the intersection $J(c,c') \cap J(c',c'')$ is itself infinite. This involves specific features of the geometry J. We consider two representative cases: an affine space, and a linear space with a quadratic form.

Let A be an affine space corresponding to a linear model V, with V^* the definable dual. Let W_c denote the minimal $acl(c)$-definable subspace of V of finite codimension. Then $J(c)$ contains all but finitely many elements of some coset of W_c in A. Similarly, $J(c,c')$ contains all but finitely many elements of some coset of the minimal $acl(c,c')$-definable subspace $W_{c,c'}$ of finite codimension. Now $W_{c,c'} + W_{c',c''} \leq W_{c'}$ is definable over both $acl(c,c')$ and $acl(c',c'')$, and as c, c', c'' are independent, this space is definable over $acl(c')$. Thus the sum equals $W_{c'}$, which means that any two cosets of $W_{c,c'}$ and $W_{c',c''}$ will intersect; the intersection is then infinite, being a coset of $W_{c,c'} \cap W_{c',c''}$. This completes the proof in the affine case.

If J is linear and carries a quadratic form then the argument is similar, but the sets involved contain almost all elements of a subset of the spaces W_c, $W_{c,c'}$ on which the quadratic form Q takes on a specific value. This set will be infinite on any subspace of J of finite codimension.

Lemma 5.1.14. *Let $\mathcal{M}$ be weakly Lie coordinatized. Let $(p_i; r_{ij})$ be an amalgamation problem of length 3 over $acl(\emptyset)$ with p_1 semigeometric. Then the problem has a solution.*

Proof. We proceed by induction on the multiplicity μ of p_1.

Take realizations $a_1 b_1 c_i$ of r_{1i} for $i = 2, 3$. If the multiplicity of b_i over $a_1 c_i$ is μ for $i = 2, 3$ then we may use Lemma 5.1.8 to choose $a_1 c_2 c_3$ appropriately, and then add b_1.

Accordingly, we may assume

The multiplicity of b_1 over $a_1 c_2$ is less than μ.

In this case the basic idea is to absorb the parameter c_2 into the base of the type and continue by induction. We first expand c_2 to an algebraically closed set C_2 and adjust the amalgamation problem accordingly. We will keep the notation as before apart from writing C_2 for c_2. The types involved now have infinitely many variables but this can be handled using the compactness theorem.

Let C_2c_3 realize r_{23} and suppose $a_1b_1c_3$ realizes r_{13} with a_1b_1 independent from from C_2c_3. Take C_2' with $a_1b_1C_2'$ realizing r_{12} and C_2' independent from $a_1b_1C_2c_3$. We will use C_2' as the basis of a new amalgamation problem.

Let $r'_{13} = tp(a_1b_1/C_2')$, $r'_{23} = tp(C_2c_3/C_2')$. To complete the specification of our auxiliary amalgamation problem, we will require a type $r'_{12}(xy,z)$ over C_2' implying the independence of xy from z and compatible with $tp(a_1b_1/C_2')$, $tp(C_2/C_2')$, and $r_{12}(xy,z)$. If we construe the desired r'_{12} as a type in the variables xy, z, z', with z' replacing C_2', then this is itself an amalgamation problem involving the types $r_{12}(xy,z)$, $r_{12}(xy,z')$, and $tp(C_2,C_2')$. This case is covered by the preceding lemma. Thus we have a new amalgamation problem (r'_{ij}) defined over C_2', containing the original problem. As the multiplicity of the initial 1-type $p'_1 = tp(a_1b_1/C_2')$ is less than μ, we conclude by induction.

Proposition 5.1.15. *Let $\mathcal{M}$ be weakly Lie coordinatized. Then $\mathcal{M}$ has the type amalgamation property.*

The following corollary shows that the Shelah degree is bounded by the rank.

Corollary 5.1.16. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure, or more generally an $\aleph_0$-categorical structure of finite rank with the type amalgamation property. Let I be an independent set, $p(x)$ a complete type over $acl(\emptyset)$, and $\varphi_a(a,x)$ $(a \in I)$ a collection of formulas for which $\varphi_a \& p$ is consistent of rank $rk\, p$. Then $\bigwedge_I \varphi_a \& p$ is consistent of rank $rk\, p$.*

Proof. We may assume first that I is finite and then that $|I| = 2$, as the statement is iterable. So we consider $\varphi_1(a_1,a_3)\&\varphi_2(a_2,a_3)\&p(a_3)$, with a_1, a_2 independent. This can be converted into an amalgamation problem of the type covered by the preceding proposition.

We now concern ourselves with the number of types of various sorts existing over finite sets of a given order.

Lemma 5.1.17. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure, and $\varphi(x,y)$ an unstable formula. Then for each n there is a set I of size n over which there are 2^n distinct φ-types. In particular φ has the independence property.*

Proof. The instability of φ means that there is an infinite sequence I of parameters (a_i, b_i) such that $\varphi(a_i, b_j)$ will hold if and only if $i < j$. We may take I to be indiscernible. I is independent over a finite set B and we may take it to be indiscernible over B, which we absorb into the language. Let $p = tp(b_i/\, acl(\emptyset)$. The formulas $\varphi(a_i, x)$ and $\neg\varphi(a_i, x)$

are consistent with p and of maximal rank, so the same applies to their various conjunctions by the preceding corollary.

Lemma 5.1.18. *Let $\mathcal{M}$ be Lie coordinatized with finitely many sorts, and J a 0-definable geometry of $\mathcal{M}$. Then for $X \subseteq M$ finite, and $b \in M$, we have the following estimate, uniformly:*

$$| \mathit{acl}(Xb) \cap J| = O(| \mathit{acl}(X) \cap J|).$$

Proof. Let $J(X) = \mathit{acl}(X) \cap J$, $J(Xb) = \mathit{acl}(Xb) \cap J$. It suffices to show that $\dim(J(Xb)/J(X) = \mathit{rk}\, b$. As J is stably embedded with weak elimination of imaginaries, a basis B for $J(Xb)$ modulo $J(X)$ will be independent from X over $J(X)$. Thus $\dim(J(Xb)/J(X)) = \mathit{rk}(B/X) \leq \mathit{rk}(b/X) \leq \mathit{rk}\, b$.

Lemma 5.1.19. *Let $\mathcal{M}$ be a Lie coordinatized structure with finitely many sorts, J a b-definable Lie geometry. Then for X varying over algebraically closed subsets of $\mathcal{M}$ we have*

$$| \mathit{acl}(Xb) \cap J| = O(|X|).$$

Proof. All cases are controlled by the projective case, so we assume that J is projective. Let J' be a canonical projective geometry nonorthogonal to J, with defining parameter $b' \in \mathit{dcl}(b)$.

If $b' \in \mathit{acl}(X)$, then $\mathit{acl}(Xb') \cap J' \subseteq X$ and otherwise, $\mathit{acl}(Xb') \cap J' = \emptyset$, so in any case $| \mathit{acl}(Xb') \cap J'| \leq |X|$. Thus by the previous lemma

$$| \mathit{acl}(Xb) \cap J| \leq |J' \cap \mathit{acl}(Xb)| = O(| \mathit{acl}(b'X) \cap J'|) = O(|X|)$$

Proposition 5.1.20. *Let $\mathcal{M}$ be Lie coordinatizable, $D \subseteq \mathcal{M}$ 0-definable of rank k. Then the number of types of elements of D over an algebraically closed set of order n in $\mathcal{M}$ is $O(n^k)$.*

Proof. Suppose first that $D = J$ is a coordinatizing geometry of $\mathcal{M}$. For algebraically closed X the types under consideration are determined by their restrictions to $X \cap J$. Thus we may assume $\mathcal{M} = J$ in this case. The statement is then clear by inspection. For example, in the presence of a quadratic form, the behavior of the the form on an extension of a subspace by a single point is determined by its value on the additional point and an induced linear function defined on the subspace. If the geometry is affine the situation remains much the same.

We turn to the general case. We may assume that D is the locus of a single type. Take $c \in D$ of rank k and $b \in \mathit{acl}(c)$ of rank $k-1$ supporting a coordinate geometry J_b, with $a \in J_b$ such that $c \in \mathit{acl}(ba)$. Let D', D'', and D''' be the loci of the types of b, ba, and bac respectively. Inductively, the number of types of elements of D' over an algebraically closed subset X of order n is $O(n^{k-1})$. By Lemma 5.1.19 for $b \in D'$ we have $|\,acl(Xb) \cap J| = O(|X|)$ and thus the number of types in J over $acl(Xb)$ is also $O(|X|)$. Thus the number of types in D'' over X is $O(n^k)$. As D''' is a finite cover of D'' the number of types of elements in D''' is also $O(n^k)$ and as the types of elements of D lift to types of elements of D''' this bound applies to D'''.

Definition 5.1.21. *For D a definable set let $s(D,n)$ denote the minimum number of types of elements of D existing over a subset of D of order n.*

Observe, for example, that in one of the standard geometries this will be $O(n)$, with the optimal subset being as close to a subspace as possible.

The following corollary depends on estimates for the sizes of envelopes to be given shortly.

Corollary 5.1.22. *Let $\mathcal{M}$ be Lie coordinatized with finitely many sorts, D a 0-definable subset of $\mathcal{M}$. Then $s(D,n)$ is polynomially bounded.*

Proof. We show in Proposition 5.2.2 below that the size of D in an envelope E is given by a polynomial function of certain quantities q^d, q being approximately the size of the base field and d varying over the dimensions of E. Varying just one of these dimensions, we can find envelopes in which the size of D is asymptotically a constant times q^d for some d. Thus for m large we can find envelopes E in which the size of D is comparable to m; that is, $m \le |D| \le (q+\epsilon)m$. Thus taking X to be a subset of $D \cap E$ of order m and applying the previous result, we get the desired bound.

We mention two problems. The first relates to the amalgamation of types.

Problem 1. *Find independent elements a_1, a_2, a_3 such that there is no B independent from $a_1a_2a_3$ for which:*

$$tp(a_1a_2/B) \cup tp(a_1a_3/B) \cup tp(a_2a_3/B) \text{ determines } tp(a_1a_2a_3/B).$$

Problem 2. *Are types over envelopes uniformly definable?*

5.2 THE SIZES OF ENVELOPES

We deal here with the computation of the size of an envelope as a function of its dimensions, and also with the sizes of the automorphism groups. We wish to express the sizes of envelopes as polynomial functions of the relevant data, and to do so it will be convenient to work with square roots of the sizes of the associated fields.

Notation 5.2.1. *Let $\mathcal{M}$ be Lie coordinatized and p a canonical projective geometry. For an envelope E we let $d_E(p)$ be the corresponding dimension (or cardinality in the degenerate case) and we let $d^*_E(p) = (-\sqrt{q})^{d_E(p)}$, where q is the size of the base field; in the degenerate case we set $d^*(p) = \sqrt{d(p)}$. When E is understood we write $d(p)$ and $d^*(p)$.*

Proposition 5.2.2. *Let $\mathcal{E}$ be a family of envelopes for the Lie coordinatized structure $\mathcal{M}$ such that for each dimension p corresponding to an orthogonal space, the signature and the parity of the dimension is constant on the family. Then there is a polynomial ρ in several variables such that for every E in $\mathcal{E}$, $|E| = \rho(d^*(E))$, where $d^*(E)$ is the vector $(d^*_E(p))$. The total degree of ρ is $2\,rk(\mathcal{M})$ and all leading coefficients are positive. If $\mathcal{M}$ is the locus of a single type (with the coordinatization in $\mathcal{M}^{\mathrm{eq}}$), then ρ is a product of polynomials in one variable.*

Proof. We show that for any definable set D_a of $\mathcal{M}$, there is a polynomial of the type described giving the cardinality of D_a in any $E \in \mathcal{E}$ which contains the parameter a. We may suppose that D_a is the locus of a single type over a. We will proceed by induction on $rk(D_a)$.

Take $d \in D_a$ and $c \in acl(ad)$ lying in an a-definable geometry J, which we may take to be degenerate, linear, or affine, with associated canonical projective p. Let D'_{ac} be the set of realizations of $tp(d/ac)$. Then we may take $\rho_{D_a} = \rho_J \rho'_{D_{ac}} / \mathit{Mult}(c/ad)$. This reduces to the case $D = J$.

If J is affine or quadratic, add a parameter to reduce to a basic linear geometry J. Then the dimension of J in E is $d_E(p)$ minus a constant depending on the type of a. Thus it suffices to find a polynomial giving the number of realizations of a type in J in terms of $d^*_E(p)$ or equivalently in terms of the corresponding expression $(\pm\sqrt{q})^{\dim J}$. The essential point is to compute the sizes of sets defined by equations $Q(x) = \alpha$ with Q a quadratic or unitary form. Let $n(d, \alpha)$ be this cardinality as a function of the dimension and α, depending also the type of the geometry. These are straightforward computations. We give details.

In the orthogonal case we can break up the space as the orthogonal sum of a $2i$-dimensional space H with a standard form $Q(\bar{\alpha}, \bar{\beta}) = \sum \alpha_i \beta_i$ and a complement of dimension $j \leq 2$. So on H we have $n(2i, 0) = (q^i - 1)q^{i-1} + q^i$ and $n(2i, \alpha) = (q^{2i} - n(2i, 0))/(q - 1)$ for $\alpha \neq 0$. Thus on the whole space

$$n(2i + j, \alpha) = n(2i, 0)n(j, \alpha) + [(q^{2i} - n(2i, 0)/(q - 1)](q^j - n(j, \alpha))$$

where the parameter n is computed with respect to the corresponding induced form. This simplifies to

$$n(2i + j, \alpha) = q^i n(i, \alpha) + q^{j-1}(q^{2i} - q^i)$$

and for small i $n(i, \alpha)$ is treated as a constant, corresponding to the particular form used.

In the unitary case $n(d, \alpha)$ is independent of α for α nonzero and thus it suffices to compute $n(d, 0)$. Using an orthonormal basis and proceeding inductively one gets $n(d, 0) = q^{d-1}(\sqrt{q} + 1) - n(d - 1, 0)\sqrt{q}$ and then $n(d, 0) = q^d/\sqrt{q} + (-\sqrt{q})^{d-1}(1 - \sqrt{q})$.

Remarks 5.2.3. *If we are working with graphs, for example, the number of edges is given by a polynomial. The polynomials ρ can be determined given a sufficiently large envelope in which the subenvelopes are known.*

We now discuss the chief factors of automorphism group of an envelope, which are the successive quotients in a maximal chain of normal subgroups of this group.

Lemma 5.2.4. *Let G be the automorphism group of the envelope $E(d)$ in a Lie coordinatized structure $\mathcal{M}$. Then the number of chief factors of G is bounded, independently of d, and each chief factor is of one of the following kinds:*

1. *abelian;*
2. *$H^{\rho(d)}$, where H is a fixed finite group and ρ is one of the functions described in the preceding proposition;*
3. *$K^{\rho(d)}$, with $\rho(d)$ as in the preceding proposition and K a classical group $PSL(d_i, q_i)$, $PSp(d_i, q_i)$, $\mathrm{P}\Omega^{\pm}(d_i, q_i)$, $PSU(d_i, q_i)$, or $Alt(d_i)$ as appropriate to the ith dimension.*

Proof. Once the dimensions are sufficiently large, the socle of the automorphism group of one layer of the coordinate tree over the previous layer is of the form (3) or abelian, unless the geometry is finite (in $\mathcal{M}$), with the number of factors corresponding to the size of a definable set modulo an equivalence relation. The remainder of the automorphism group at that layer is solvable. If the layer consists of copies of a finite geometry, consider a chief factor H/K with H, K $Aut(E)$-invariant subgroups acting trivially on the previous layer. Let A be the automorphism group of the finite geometry involved, and let L be the part of E lying in the previous level of the coordinate tree, so that H, K lie in A^L. If H/K is nonabelian then it is a product of a certain number of copies of a single isomorphism type of finite simple group S. The number of factors is the order of L modulo the equivalence relation: $a \sim b$ if the projection of H/K onto $A_a \times A_b$ is a diagonal subgroup isomorphic to S. This relation is $Aut(E)$-invariant and hence definable. Thus the number ρ of factors involved is equal to the size of a definable set in an envelope (a definable quotient of L).

Corollary 5.2.5. *Let $\mathcal{M}$ be a Lie coordinatized structure. Then for the dimension function d large enough, $Aut(E(d))$ determines d up to a permutation of the coordinates and up to orientation in the odd-dimensional orthogonal case.*

Proof. Let f be a bound on the size of the chief factors of the second type above. Let d be large enough that the chief factors of the third type are all of order greater than f. Then these chief factors can be recovered from the automorphism group unambiguously and the data d can be read off.

Lemma 5.2.6. *Let $\mathcal{M}$ be a Lie coordinatized structure and D a definable subset. Then the following are equivalent:*

1. $rk(D) < rk(\mathcal{M})$.
2. $\lim_{E \to \mathcal{M}} |D[E]|/|E|) = 0$.

Here the limit is taken over envelopes whose dimensions all go to infinity, and $D[E]$ means D taken in E, which for large enough E is $D \cap E$. The convergence is exponentially rapid if all geometries are nondegenerate.

Proof. We compare the polynomials ρ_D, ρ_E giving the sizes of D and E.

If the ranks are equal, then both polynomials have positive leading coefficients and total degree $2\,rk(\mathcal{M})$. For each dimension d_i, ρ_D, ρ_E involve the parameter $d_i^* = \alpha_i^{d_i}$ for an appropriate α_i (read this expression as d_i in the degenerate case). Let the dimensions d_i be taken momentarily as arbitrary real numbers going jointly to infinity along the curve $d_1^* = d_2^* = \ldots$, so that the polynomials ρ_D, ρ_E reduce to one variable polynomials converging to a positive γ. After a slight perturbation we may suppose that $d_1, d_2, \ldots$ are rational, that ρ_D/ρ_E approaches γ, and that the terms of total degree less than $2\,rk(\mathcal{M})$ make a negligible contribution. After rescaling by a common denominator, the "dimensions" are integers, the ratio of the highest order parts of ρ_D and ρ_E goes to γ, and the lower-order terms are even more negligible. Thus we have a sequence of dimension assignments tending jointly to infinity on which the quotient ρ_D/ρ_E will not go to zero.

Now assume that $rk(D) < rk(\mathcal{M})$. We may take D, E to be realizations of single types, so that ρ_D and ρ_E factor as products of polynomials in one variable $\rho_{D,i}$, $\rho_{E,i}$. The ratios $\rho_{D,i}/\rho_{E,i}$ are bounded, as otherwise varying only the one relevant dimension we would get a proper subset with more elements than the whole set E. On the other hand at least one of the $\rho_{D,i}$ has degree less than the degree of $\rho_{E,i}$ so the limit goes to 0 (rapidly, if the geometry is nondegenerate).

We now prove a finitary Löwenheim–Skolem principle.

Lemma 5.2.7. *Let $\mathcal{M}$ be Lie coordinatized. For any subset X of $\mathcal{M}$ there is an envelope E of $\mathcal{M}$ containing X, in which each dimension is at most $2\,rk(X) \leq 2\,rk(\mathcal{M}) \cdot |X|$.*

Proof. Let $J_1, \ldots, J_n$ be the $acl(\emptyset)$-definable dimensions, and $E_i = acl(X) \cap J_i$. The dimension of E_i is at most $rk(X)$. If the geometry J_i carries a form then increase E_i to a nondegenerate subspace, of dimension at most $2\,rk(X)$. Let $\mathcal{M}'$ be a maximal algebraically closed subset of $\mathcal{M}$ containing X, and such that $M' \cap J_i = E_i$. Then $\mathcal{M}'$ is Lie coordinatized and has smaller rank, unless these geometries are finite, in which case iteration of the process will eventually lower the rank or the height of the coordinatizing tree. By induction on rank we may suppose that in $\mathcal{M}'$ there is an envelope E with the desired properties. This will then be an envelope in $\mathcal{M}$, with the desired properties.

Remark 5.2.8. *The existence of indiscernible sets of order n in all large finite structures with a fixed number of 5-types is proved in [CL]. In particular, an infinite quasifinite structure contains an infinite set of indiscernibles. Conversely, from the latter result it follows that there is a constant c such that for large n, a pseudofinite structure with at least c^n elements contains a sequence of indiscernibles of length n. This follows from the last lemma using the bounds on the sizes of envelopes, since the ranks involved can be bounded in terms of the number of 4-types. It is possible that an explicit bound of this kind can also be extracted by tracing through the arguments in [CL].*

Problem 3. *Do the abelian chief factors of automorphism groups of envelopes have orders $p^{\sigma(d,d^*)}$ with σ a polynomial similar to ρ—in particular, a product of polynomials in one variable (i.e., depending on one dimension)?*

One can treat the case of affine covers by dualization, reducing to finite covers. Then by results in [EH] the problem reduces to the following: if J is a definable combinatorial geometry on a definable set D of a Lie structure $\mathcal{M}$, which is subordinate to algebraic closure, show that the dimension of J in an envelope of $\mathcal{M}$ is given by a polynomial in d, d^*.

5.3 NONMULTIDIMENSIONAL EXPANSIONS

We show here that Lie coordinatizable structures have "nonmultidimensional" expansions, lifting [HrTC, §3] to the present context. As in that earlier case, the difficulty lies in the interaction of orthogonal geometries, which means that the outer automorphism groups may be related even if the simple parts of the groups are not.

Definition 5.3.1. *A Lie coordinatized structure is said to be* nonmultidimensional *if it has only finitely many dimensions, or equivalently (and more explicitly) if all canonical projectives are definable over* $acl(\emptyset)$.

Proposition 5.3.2. *Every Lie coordinatized structure can be expanded to a nonmultidimensional Lie coordinatized structure.*

Proof. We use a locally transitive coordinatizing tree, meaning that the type of a point at a given level depends only on the level. We also allow the introduction of a finite number of additional sorts, each carrying a single basic geometry.

Let M_i be the coordinatizing tree up to level i together with the elements of the special sorts, and let Δ be the set of indices i for which the geometries J_a associated to points at level i are orthogonal to M_i. We proceed by induction on M_i, the case $\Delta = \emptyset$ being the nonmultidimensional case. So we take Δ nonempty.

Now let $n \in \Delta$ be maximal. Let T_n be the set of elements lying at level n in the coordinatizing tree. For $a \in T_n$ let $P_{a'}$ be the canonical projective geometry associated with P_a and let q be the type of a'. Let $V_{a'}$ be the corresponding linear geometry. If these linear geometries are not actually present in the structure, we may attach them freely to the canonical projectives. (In the degenerate case, the geometry is considered to be both linear and projective.) The isomorphism type of $V_{a'}$ is independent of a', but there will not be any system of identifications present between the various $V_{a'}$.

Suppose for definiteness that $V_{a'}$ is of orthogonal type in odd characteristic, with base field $K_{a'}$, and bilinear form $B_{a'} : V_{a'} \times V_{a'} \to L_{a'}$, a 1-dimensional $K_{a'}$-space. Fix a copy K of the base field, and a 1-dimensional space L over K. Fix a 2-dimensional space $U_\circ$ over K and a nondegenerate bilinear form $(\ )_\circ : U_\circ \times U_\circ \to L$ which takes the value 0 at some nonzero point. The pair $(U_\circ, (\ )_\circ)$ is unique up to an isomorphism fixing K and L.

Now let U_1, Q_1 be an infinite dimensional nondegenerate orthogonal space over the prime field $F \leq K$ and set $U = U_1 \otimes U_\circ$ as a K-space. The forms $(,)_\circ$ and $(,)_1$ induce a bilinear form $(,)$ on U satisfying $(a_1 \otimes$

$a_\circ, b_1 \otimes b_\circ) = (a_1, b_1)_1 \cdot (a_\circ, b_\circ)_\circ$. This makes sense by the universal property of tensor products. Let Γ be the family $\{a \otimes U_\circ : a \in U_1\}$. Then

(1) Any automorphism h of (K, L) extends to an automorphism of U fixing Γ pointwise.

The uniqueness of $U_\circ$ signifies that h extends to $U_\circ$. To extend to U fix U_1 pointwise. Then Γ is fixed pointwise.

Add U as a new sort. For b satisfying q pick isomorphisms $h_b : U \to V_b$, and let $\Gamma_b = h_b[\Gamma]$. Let $\mathcal{M}'$ be $\mathcal{M}$ expanded by the sort U and a family of maps $f_b : \Gamma \to \Gamma_b$ for b satisfying q. f_b is to be coded by a ternary relation on $q \times U \times \bigcup_b V_b$. h_b is not part of the structure but the sets Γ and Γ_b can be recovered from f_b in $(\mathcal{M}')^{\text{eq}}$. We claim that $\mathcal{M}'$ remains 4-quasifinite and that Δ is reduced by 1.

By a *normal* subset of $\mathcal{M}^{\text{eq}}$ we mean a union of 0-definable sets. The restriction of a normal subset to a finite number of sorts is then 0-definable. We consider normal subsets S satisfying the additional condition:

For b satisfying q, V_b is orthogonal to S.

This means that any basic geometry corresponding to V_b (with $acl(b)$ fixed) is orthogonal to S. Let Q be a maximal normal subset of this type containing T_n. Then Q contains the locus of q and is algebraically closed. We claim that Q is also stably embedded in $\mathcal{M}$, since for any projective or affine geometry in Q, if the dual exists in $\mathcal{M}$, then it is contained in Q.

We claim now:

(2) For any automorphisms α of Q and β of U, the map $\alpha \cup \beta$ is induced by an automorphism of $\mathcal{M}'$.

Let $Q_1 = Q \cup \bigcup_b V_b$. Then Q_1, like Q, is stably embedded in $\mathcal{M}$. We first extend $\alpha \cup \beta$ to Q_1. For b satisfying q, α induces maps $K_b \to K_{\sigma b}$ and L_b to $L_{\sigma b}$. By (1) these maps are induced by a linear isomorphism $\theta_b : V_b \to V_{\sigma b}$ compatible with $f_{\sigma b} \beta f_b^{-1}$. Using the orthogonality condition, $\alpha \cup \beta \cup \bigcup_b \theta_b$ is elementary and extends to an automorphism of $\mathcal{M}'$.

It remains to be seen that apart from the introduction of U, the rest of the coordinatization of $\mathcal{M}$ is unaffected; specifically, if J_c is a canonical projective geometry of $\mathcal{M}$ orthogonal to the geometries V_b, then

J_c has no extra structure as a subset of $\mathcal{M}'$;
If J_c is stably embedded in $\mathcal{M}$, then it remains stably embedded in $\mathcal{M}'$

We may assume that J_c is stably embedded in $\mathcal{M}$. If J_c is contained in Q this follows from (2), and otherwise any automorphism of J_c fixing

$acl(c)$ extends to an automorphism of $\mathcal{M}$ fixing Q_1 pointwise. This is then elementary in $\mathcal{N}'$.

This completes the orthogonal case in odd characteristic. The linear, symplectic, and unitary cases are similar, with the auxiliary space $U_\circ$ 1-dimensional in the unitary case. In the orthogonal case in characteristic 2, the orthogonal geometry is an enrichment of a symplectic geometry and we may suppose that the pure symplectic space occurs as well, and that the quadratic form used occurs also as a point in an associated quadratic geometry. Then we can switch to the symplectic case. Similarly, in the case of a polar geometry (V, V^*) reduce the scalars to the prime field and introduce linear isomorphisms $\iota_V : V \to V^*$. This can be done without destroying outer automorphisms and brings us back to the symplectic case.

Proposition 5.3.3. *For $\mathcal{M}$ quasifinite the following are equivalent:*

1. *$\mathcal{M}$ is stable.*
2. *$\mathcal{M}$ is $\aleph_0$-stable.*
3. *$\mathcal{M}$ does not interpret a polar space.*

Proof. We must show that (3) implies (2). So assume (3). In particular none of the canonical geometries for $\mathcal{M}$ involve bilinear forms. The geometries occurring are therefore all strongly minimal and stably embedded. Morley rank is subadditive in the $\aleph_0$-categorical setting, for stably embedded definable subsets (cf. [HrTC]), so using the coordinatization, $\mathcal{M}$ has finite Morley rank.

Remarks 5.3.4

As the class of *stable* polar spaces is the class of *finite* polar spaces, which is not an elementary class, the notion of a stable quasifinite structure in a given language is not an elementary notion. On the other hand, for a fixed finite language L, the class of stable homogeneous L-structures is elementary [CL]. This can be seen fairly directly as follows. By a result of Macpherson [Mp1] in a finitely homogeneous structure, no infinite group is interpretable. In particular for finitely homogeneous structures, quasifiniteness and stability are equivalent. But for finitely homogeneous structures quasifiniteness is elementary.

Although we work outside the stable context, we still require the analysis of [CL] for primitive groups with nonabelian socle, which enters via [KLM].

5.4 CANONICAL BASES

We do not have a theory of canonical bases as such, but the following result serves as a partial substitute.

Proposition 5.4.1. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Suppose that a_1, a_2, a_3 is a triple of elements which are independent over a_1, over a_2, and over a_3. Then a_1, a_2, a_3 are independent over the intersection of $acl(a_i)$, $i = 1, 2, 3$, in $\mathcal{M}^{\text{eq}}$.*

We begin with a few lemmas.

Lemma 5.4.2. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and let R be a 0-definable symmetric binary relation satisfying*

> *Whenever $R(a, b)$, $R(b, c)$ hold with a, c independent over b, then $R(a, c)$ holds and b, c are independent over a.*

Then there is a 0-definable equivalence relation E such that

> *$R(a, b)$ implies the following:*
> *$E(a, b)$ holds and a, b are independent over $a/E = b/E$.*

Proof. We define $E(a, b)$ as follows: For some c independent from a over b and from b over a, $R(a, c)$ and $R(b, c)$ holds.

We check first that R implies E. If $R(a, b)$ holds, choose c independent from a over b such that $R(c, b)$ holds. Then by $(*)$ $R(a, c)$ holds and c is independent from b over a. Thus E holds. The domain of E is the same as the domain of R and E is clearly reflexive and symmetric on this domain. We now check transitivity.

Suppose $E(a_1, a_2)$ and $E(a_2, a_3)$ hold and let a_{12}, a_{23} be witnesses. Thus we have $R(a_i, a_{ij})$; $R(a_j, a_{ij})$; and a_{ij} is independent from a_i over a_j and from a_j over a_i. As a_{12} is independent from a_1 over a_2, we may take it independent from $a_1a_2a_3$ over a_2; and similarly for a_{23}. Furthermore, we may take a_{12}, a_{23} independent over a_1, a_2, a_3 and hence over a_2. From $R(a_2, a_{12})$ and $R(a_2, a_{23})$ we then deduce $R(a_{12}, a_{23})$.

Pick c independent from $a_1a_2a_3a_{23}$ over a_{12} such that $R(a_{12}, c)$ holds. We claim then:

(1) $R(a_i, c)$ holds for all i, and
c is independent from a_{ij} over a_i and over a_j.

First, since c is independent from a_{23} over a_{12} we get $R(a_{23}, c)$ and c is independent from a_{12} over a_{23}; the latter implies that c is independent from $a_1a_2a_3a_{12}$ over a_{23}. So c is independent from a_1 or a_2 over c_1, and from a_2 or a_3 over c_2. By another application of $(*)$ the relation (1) follows.

Now using (1) we get c independent from $a_1a_2a_3a_{12}a_{23}$ over each a_i and, in particular, c is independent from a_3 over a_1 and from a_1 over a_3; so $E(a_1, a_3)$ is witnessed by c. Thus E is transitive.

Finally, we must show that if $R(a, b)$ holds and $c = a/E = b/E$, then a, b are independent over c. Let a' realize the type of a over c with a' independent from a over c. We will show then that a and b are independent over a' and thus a and b are independent over c.

As $E(a, a')$ holds, there is d satisfying

$R(a, d)$, $R(a', d)$, and d is independent
from a over a' and from a' over a.

We will take a', d independent from b over a. In particular we have a' independent from b over ad, and b independent from d over a; the latter, with $(*)$, gives b independent from a over d and then combined with the former, we get aa' independent from b over d, hence a independent from b over $a'd$. As a is independent from d over c we get finally a independent from b over a'.

Definition 5.4.3. *Let $a_1, \ldots, a_n$ be a sequence of elements in a structure of finite rank.*

1. *The sequence is said to be* 1-locally independent *if it is independent over any of its elements.*
2. *We set $\delta(a_1, \ldots, a_n) = \sum_i rk\, a_i - rk(a_1 \ldots a_n)$.*

Lemma 5.4.4. *Let $\mathcal{M}$ be a structure of finite rank, $\mathbf{a} = a_1, \ldots, a_n$ a sequence of elements. Then the sequence $\mathbf{a}$ is 1-locally independent if and only if:*

The quantity $\delta = \delta(a_ia_j)$ is independent of i, j (distinct);
and $\delta(\mathbf{a}) = (n-1)\delta$.

Proof. We have in general for any fixed index k, writing $\sum'$ for a sum excluding the index k:

$$\begin{aligned} \delta(\mathbf{a}) &= \sum_i rk(a_i) - (rk(\mathbf{a}/a_k) + rk(a_k)) \\ &= {\sum_i}' rk(a_i) - rk(\mathbf{a}/a_k) \\ &\geq {\sum}' rk(a_i) - {\sum}' rk(a_i/a_k) = {\sum}' \delta(a_i, a_k) \end{aligned}$$

with equality if and only if $\mathbf{a}$ is independent over a_k. Thus if $\delta = \delta(a_i, a_j)$ is constant and $\delta(\mathbf{a}) = (n-1)\delta$, then we have equality regardless of the choice of k and the sequence is 1-locally independent, while, conversely, if the sequence is 1-locally independent, then $\delta(\mathbf{a}) = \sum' \delta(a_i a_k)$ for any k and it suffices to check that the $\delta(a_i a_j)$ are independent of i, j. But the restriction of $\mathbf{a}$ to any three terms $a_i, a_{i'}, a_{i''}$ remains 1-locally independent, and applying our equation to a sequence of length 3 with $k = i'$ or $k = i''$ yields $\delta(i, i') = \delta(i, i'')$, from which it follows that δ is constant.

Lemma 5.4.5. *Let $\mathcal{M}$ be a structure of finite rank.*

1. *Suppose that $\mathbf{a} = a_1, a_2, a_3, a_4$ is a sequence with a_1, a_2, a_3, and a_2, a_3, a_4 1-locally independent. If a_1 and a_4 are independent over $a_2 a_3$, then $\mathbf{a}$ is 1-locally independent.*
2. *If $\mathbf{a} = a_1 a_2 b_1 b_2 c_1 c_2$ is a sequence whose first four and last four terms are 1-locally independent, and $a_1 a_2$ is independent from $c_1 c_2$ over $b_1 b_2$, then $\mathbf{a}$ is 1-locally independent.*

Proof. Ad 1. We have $\delta(a_i a_j) = \delta$ constant with the possible exception of the pair a_1, a_4, and repeating the calculation of the previous lemma over $a_2 a_3$ rather than a_k, using $rk(a_1 a_2 a_3 a_4 / a_2 a_3) = rk(a_1 / a_2 a_3) + rk(a_4 / a_2 a_3)$, we get $\delta(\mathbf{a}) = 3\delta$. Thus it remains only to be checked that $\delta(a_1 a_4) = \delta$. We may show easily that $\mathbf{a}$ is independent over a_2 or over a_3, starting from the independence of $a_1 a_2 a_3$ from a_4 over $a_2 a_3$. Thus

$$\begin{aligned} rk\, a_2 - \delta &= rk(a_2/a_1) \geq rk(a_2/a_1 a_4) \geq rk(a_2/a_1 a_3 a_4) \\ &= rk(a_2/a_3) = rk(a_2) - \delta \end{aligned}$$

and, in particular, we have the equation $rk(a_2/a_1 a_4) = rk(a_2) - \delta$. Now

$$\begin{aligned} rk(\mathbf{a}) &= rk(a_1 a_4) + rk(a_2/a_1 a_4) + rk(a_3/a_1 a_2 a_4) \\ &= rk(a_1 a_4) + (rk(a_2) - \delta) + rk(a_3) - \delta \end{aligned}$$

and thus

$$3\delta = \sum rk(a_i) - rk(\mathbf{a}) = \delta(a_1 a_4) + 2\delta$$

and $\delta(a_1 a_4) = \delta$.

Ad 2. It is straightforward that $\mathbf{a}$ is independent over b_1 or over b_2 and by symmetry it will be sufficient to prove that $\mathbf{a}$ is independent over a_1.

We have by assumption $c_1 c_2$ independent from $a_1 a_2 b_1 b_2$ over $b_1 b_2$ and thus c_1 is independent from $a_1 a_2 b_1 b_2$ over $b_1 b_2 c_2$, but also c_1 is assumed independent from $b_1 b_2 c_2$ over c_2, and thus

$$c_1 \text{ is independent from } a_1 a_2 b_1 b_2 c_2 \text{ over } c_2.$$

In particular, c_1c_2 is independent from $a_1b_1b_2$ over a_1c_2. By Case 1 $a_1b_1b_2c_2$ is 1-locally independent and is, in particular, independent over a_1, so from the previous relation we derive the independence of c_1c_2 from b_1b_2 over a_1. Combining this with the independence of c_1c_2 from $a_1a_2b_1b_2$ over b_1b_2, we find that c_1c_2 is independent from $a_1a_2b_1b_2$ over a_1. Now c_1 is independent from c_2 over b_1b_2 and c_1c_2 is independent from a_1 over b_1b_2 so c_1 is independent from c_2 over $a_1b_1b_2$, and hence, by transitivity, over a_1. Thus $a_1a_2b_1b_2$ is independent over a_1, c_1c_2 is independent from $a_1a_2b_1b_2$ over a_1, and c_1 is independent from c_2 over a_1. Thus $\mathbf{a}$ is independent over a_1.

Proof of Proposition 5.4.1. We have a_1, a_2, a_3 1-locally independent. Let X be the set of pairs $x = (x_1, x_2)$ such that each coordinate x_1 or x_2 realizes the type of one of the three elements a_i, and define a relation R on X by: $R(x, y)$ if and only if x_1, x_2, y_1, y_2 is a 1-locally independent quadruple. We will apply Lemma 5.4.2 to R. Note first that if $R(x, y)$ and $R(y, z)$ hold with x and z independent over y then the 6-tuple (x, y, z) satisfies the conditions of case 2 of the previous lemma, and thus the six coordinates form a 1-locally independent sequence. Thus Lemma 1 applies and there is a 0-definable equivalence relation E such that

$$R(x, y) \text{ implies: } E(x, y), \text{ and } x, y \text{ are independent over } x/E.$$

Now consider the 1-locally independent triple (a_1, a_2, a_3). We extend it by two further elements a_4, a_5 satisfying the following conditions: $tp(a_i/a_2a_3) = tp(a_1/a_2a_3)$, for $i = 4, 5$; a_4 independent from a_1 over a_2a_3; and a_5 is independent from a_1, a_4 over a_2, a_3. We claim that any 4-tuple from a_1, a_2, a_3, a_4, a_5 is 1-locally independent. This follows from Lemma 5.4.5, part (1), for $a_1a_2a_3a_4$, $a_1a_2a_3a_5$, or $a_2a_3a_4a_5$. In the remaining two cases, $a_1a_2a_4a_5$ and $a_1a_3a_4a_5$, we need to check that a_5 is independent from a_4 over a_1a_2 or a_1a_3. But a_5 is independent from a_4 over $a_1a_2a_3$ and from $a_1a_2a_3$ over a_2 or a_3. Thus all of these 4-tuples are 1-locally independent, and hence any two disjoint pairs are E-equivalent; and by transitivity any two pairs are E-equivalent. Let e be the common E-class of these pairs. Then a_1a_2 is independent from a_3a_4 over e and a_1a_3 is independent from a_2a_4 over e. In particular, working over e we have a_3 independent from a_1a_2, and a_1 independent from a_2, and thus $a_1a_2a_3$ is an independent set over e. It remains only to be checked that e is algebraic over each a_i. Certainly $e \in acl(a_1a_2)$ and $acl(a_3a_4)$, and as these pairs are independent over any a_i, we have $e \in acl(a_i)$ for all i.

5.5 MODULARITY

Definition 5.5.1. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. $\mathcal{M}$ is* modular *if whenever A_1, A_2 are algebraically closed sets in $\mathcal{M}^{\rm eq}$, they are independent over their intersection.*

By convention *acl* will always be taken to operate in $\mathcal{M}^{\rm eq}$. This point may be reemphasized occasionally.

Modularity, as defined here, is called "local modularity" in the literature dealing with the case of finite Morley rank, where the term "modular" is applied only to strongly minimal sets D which in addition to the stated property have "geometric elimination of imaginaries": for $a \in D^{\rm eq}$, there is $A \subseteq D$ with $acl(e) = acl(A)$.

As a matter of notation we will use the symbol $\perp$ for *independence*, a symbol which is more often used for model theoretic *orthogonality*; but the latter concept does not really call for any special notation in our present development.

Lemma 5.5.2. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Then $\mathcal{M}$ is modular if and only if the lattice of algebraically closed subsets of $\mathcal{M}^{\rm eq}$ satisfies the modular law:*

$$a \wedge (b \vee c) = b \vee (a \wedge c) \text{ for } b \leq a.$$

Proof. Suppose $\mathcal{M}$ is modular, and A, B, C are algebraically closed subsets of $\mathcal{M}^{\rm eq}$ with $B \subseteq A$. Our claim is

$$A \cap (acl(BC)) = acl(B \cup (A \cap C))$$

the modular law. From modularity applied to A, C, as $B \subseteq A$ we deduce easily that $A \perp BC$ over $B \cup (A \cap C)$. Thus $A \cap acl(BC) = acl(B \cup (A \cap C))$.

In the converse direction, assume the modular law in $\mathcal{M}^{\rm eq}$, but A, B are algebraically closed and dependent over their intersection. Minimize $rk(A/B)$ and, subject to this constraint, $rk(A)$. We may suppose $A \cap B = acl(\emptyset)$, as the modular law holds in the corresponding sublattice (i.e., above $A \cap B$). We adopt the notation $0 = acl(\emptyset)$ for the present. After these reductions, we claim that A is a lattice atom: a minimal nontrivial algebraically closed set.

Suppose $0 < A' \leq A$ with A' algebraically closed. As $A' > A \cap B$, $rk(A'/B)$ is positive and $rk(A/A'B) < rk(A/B)$, so by minimality

$$A \perp A'B \text{ over } A \cap acl(A'B).$$

If $A \cap acl(AB')$ is independent from B over $A \cap acl(AB') \cap B = 0$, then $A \perp B$ over 0, a contradiction. Thus A may be replaced by $A \cap acl(A'B)$, and by the minimality of $rk\, A$ we find $A \subseteq acl(A'B)$. By the modular law

$$A = A \cap acl(A'B) = acl(A' \cup (A \cap B)) = A'$$

as claimed.

Now consider a conjugate B' of B over A independent from B over A. Note that

$$acl(AB) \cap B' = 0$$

since $acl(AB) \cap B' \subseteq A \cap B' = 0$. If the triple A, B, B' is 1-locally independent, then it is independent over the intersection 0 by Proposition 5.4.1, a contradiction. If it is not 1-locally independent, then either A, B are dependent over B', or A, B' are dependent over B, and in any case $rk(A/BB') < rk(A/B)$. Thus by the minimality of $rk(A/B)$, we have independence of A from BB' over $A_\circ = A \cap acl(BB')$. As A is an atom, we have either $A_\circ = 0$, contradicting the choice of A, or $A \subseteq acl(BB')$. In the latter case, applying the modular law to $acl(A, B)$, B, and B' we get $A \subseteq acl(AB) \cap acl(BB') = acl(B, acl(AB) \cap B') = B$, which is absurd.

Proposition 5.5.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Then the following are equivalent.*

1. *$\mathcal{M}$ is modular.*
2. *For all finite A_1, A_2 in $\mathcal{M}$, A_1 and A_2 are independent over the intersection of their algebraic closures.*
3. *For all finite A_1, A_2 in $\mathcal{M}$, there is a finite C independent from A_1, A_2 such that A_1, A_2 are independent over the intersection of the algebraic closures of $A_1 \cup C$ and $A_2 \cup C$.*
4. *The lattice of algebraically closed subset of $\mathcal{M}^{\mathrm{eq}}$ is a modular lattice.*

Proof. The equivalence of (1) and (2) is clear and the equivalence of (1) and (4) is the previous lemma, so we concern ourselves with the implication "(3) implies (2)." We actually show that each instance of (3) implies the corresponding instance of (2).

Let A_1, A_2 be the algebraic closures of two finite subsets of $\mathcal{M}^{\mathrm{eq}}$. We must work with sets generated by subsets of $\mathcal{M}$ rather than $\mathcal{M}^{\mathrm{eq}}$, so take A_1^*, A_2^* finite subsets of $\mathcal{M}$ such that $A_i \subseteq acl\, A_i^*$ and, in addition,

(3.1) $$A_1^* \perp A_2 \text{ over } A_1$$

(3.2) $$A_2^* \perp A_1^* \text{ over } A_2$$

This ensures $acl(A_1^*) \cap acl(A_2^*) = acl(A_1) \cap acl(A_2)$ by applying first (3.2) and then (3.1). Accordingly, the problem is reduced to the following:

$$A_1^* \perp A_2^* \text{ over } acl(A_1^*) \cap acl(A_2^*).$$

By (3), we have a finite set C independent from $A_1^* A_2^*$ for which

$$A_1^* \perp A_2^* \text{ over } acl(A_1^* \cup C) \cap acl(A_2^* \cup C).$$

Let $A = acl(A_1^* \cup C) \cap acl(A_2^* \cup C)$ and take A_3^* conjugate to A_1^* over $acl(A_2^* \cup C)$, and independent from A_1^* over $A_2^* C$. Then A_3^* is independent from $A_1^* A_2^*$ over A since

$$\begin{array}{rcccl} rk(A_3^*/A_1^*A_2^*A) & \leq & rk(A_3^*/A_1^*A_2^*C) & = & rk(A_3^*/A_2^*C) \\ & = & rk(A_3^*/A_2^*A) & = & rk(A_3^*/A) \end{array}$$

As A_3^* is independent from $A_1^* A_2^*$ over A and A_1^*, A_2^* are independent over A, A_3^*, A_1^*, A_2^* is an independent triple over A. As A_1^* and A_3^* are conjugate over $acl(A_2^* C)$, they are conjugate over A, and thus $A \subseteq acl(A_3^* C)$. Thus $C \subseteq A \subseteq acl(A_i^* C)$ for all i. For any permutation i, j, k of $1, 2, 3$, we have: $A_i^* \perp A_j^*$ over AA_k^*, hence $A_i^* \perp A_j^*$ over CA_k^*, and thus $A_i^* \perp A_J^*$ over A_k^*. By Proposition 5.4.1 the triple A_1^*, A_2^*, A_3^* is independent over the intersection of their algebraic closures, and in particular A_1^*, A_2^* are independent over the intersection of their algebraic closures.

Proposition 5.5.4 (Fundamental Rank Inequality, cf. [CHL]) *Let $\mathcal{M}$ be $\aleph_0$-categorical, of finite rank, modular, and with the type amalgamation property (cf. §5.1). Let D, D' be 0-definable sets with D' parametrizing a family of definable subsets D_b of D of constant rank r for $b \in D'$. Suppose that E is a 0-definable equivalence relation on D' such that for inequivalent $b, b' \in D'$ we have*

$$rk(D_b) \cap rk(D_{b'}) < r.$$

Then $rk(D'/E) + r \leq rk\, D$.

Proof. We may assume that both D and D' each realize a unique type over the empty set. Take $b \in D'$ and $a \in D_b$ with $rk(a/b) = r$. Let $C = acl(a) \cap acl(b)$. Thus $a \perp b$ over C by modularity, and $rk(a/C) = rk(a/b) = r$. We will show

$$(*) \qquad b/E \in C.$$

Thus $rk(D'/E) \leq rk\, C = rk(aC) - rk(a/C) = rk(a) - r$ as claimed. So we turn to $(*)$.

Let b'/E be a conjugate of b/E over C distinct from b/E, with b' independent from b over C. We seek an element b'' of D' satisfying

$$tp(b''b/C) = tp(b'b/C); tp(b'', a/C) = tp(b, a/C)$$

with a, b, b'' independent over C. This amounts to an amalgamation problem for three compatible 2-types: $tp(ba/C)$, $tp(b'b/C)$, $tp(ba/C)$. By the type amalgamation property, this can be done.

In particular, $a \in D_b \cap D_{b''}$ and thus $rk(a/bb'') < r$; but $rk(a/bb'') = rk(a/C) = r$, a contradiction. Thus there is no such conjugate b' and $b \in dcl(C) = C$.

Corollary 5.5.5. *With the hypotheses above, $\mathcal{M}$ interprets no Lachlan pseudoplane.*

Remark 5.5.6. *This refers to a combinatorial geometry $(P, L; I)$ of points and lines such that each point is incident with infinitely many lines, two points are incident with only finitely many lines, and dually. The relevance of these structures to the behavior of $\aleph_0$-categorical stable structures was shown in [LaPP], and the corollary settles a question raised in [KLM].*

Proof. If $(P, L; I)$ is such a pseudoplane, then after dualizing if necessary we may take $n = rk(L) \geq rk\, P$. We apply the fundamental rank inequality with $D = P$, D_l is the set of points incident with the line l as l varies over a subset D' of L of rank n on which $r = rk\, D_l$ is constant, with E the equality relation. By the axioms for pseudoplanes, the previous proposition applies and yields $rk\, D' + r \leq rk\, P \leq rk\, L = rk\, D'$ and thus $r = 0$, a contradiction.

We give a more precise version of the fundamental rank inequality.

Proposition 5.5.7. *Let D, D' be the loci of single types over the empty set, and D_b a uniformly b-definable family of rank r subsets of D parametrized by D'. Then there is a finite cover $^{-} : D'' \to D'$ and an equivalence relation E on D'' such that*

1. *$rk(D''/E) = rk\, D - r$;*
2. *For b, b' E-equivalent in D'', we have $rk(D_{\bar{b}} \cap D_{\bar{b}'}) = r$.*

Proof. We work with a, b, c as in the proof of Proposition 5.5.4, but with c finite rather than algebraically closed: so we require $c \in acl(a) \cap acl(b)$ finite, $a \perp b$ over c. Let D'' be the locus of bc over the empty set, with $\overline{b_1c_1} = b_1$, and with $E(b_1c_1, b_2c_2)$ if and only if $c_1 = c_2$ and the types of b_1 over $acl(c_1)$ and of b_2 over $acl(c_2)$ coincide. Then the amalgamation argument yields (2), and $rk(D'')/E = rk(c) = rk(a) - rk(a/c) = rk(D) - rk(a/b) = rk\, D - r$.

5.6 LOCAL CHARACTERIZATION OF MODULARITY

We show in this section that Lie coordinatized structures are modular by reducing the global property of modularity to local properties of the coordinatizing structures.

Definition 5.6.1. *Let $\mathcal{M}$ be a structure.*

1. *A definable subset D of $\mathcal{M}$ is* modular *if for every finite subset A of $\mathcal{M}$, the structure with universe D and relations the A-definable relations of $\mathcal{M}$ restricted to D, is modular.*
2. *Let $\mathcal{F}$ be a collection of definable subsets of $\mathcal{M}$. Then $\mathcal{M}$ is* eventually coordinatized *by $\mathcal{F}$ if for any $a \in M$ and finite $B \subseteq M$, with $a \notin acl(B)$, there is $B' \supseteq B$ independent from a over B and a B'-definable member D of $\mathcal{F}$ for which $D \cap acl(aB')$ contains an element not algebraic over B'.*

Lemma 5.6.2. *If $\mathcal{M}$ is eventually coordinatized by a family of modular definable sets, then it is eventually coordinatized by a family of modular definable sets of rank 1.*

Proof. Replace each modular definable set by its definable subsets of rank 1. If $a \in M$ and B is a finite set, take $B' \supseteq B$ independent from a over B and take D definable and modular such that $D \cap acl(aB')$ contains an element b not algebraic over B'.

Take $B_1 \supseteq B'$ such that $rk(b/B_1) = 1$. We may suppose that B_1 is independent from a over B. Let $B_2 = acl(bB') \cap acl(B_1)$. Then $B' \subseteq B_2$, B_2 is independent from a over B, and by modularity of D, b is independent from B_1 over B_2, so $rk(b/B_2) = 1$. Let b' be finite, with $B' \subseteq b' \subseteq B_2$, such that $rk(b/b') = 1$, and let D'_b be the locus of $tp(b/b')$. Then $D'_b \subseteq D$ is rank 1, and is modular since D is. Furthermore, $b \in D_{b'} \cap acl(ab') \setminus acl(b')$, and b' is independent from a over B.

Proposition 5.6.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. If $\mathcal{M}$ is eventually coordinatized by modular definable sets, then $\mathcal{M}$ is modular.*

Proof. By the preceding lemma we may take the coordinatization to be in terms of rank 1 modular sets.

Suppose $\mathcal{M}$ is not modular. Then there are elements a, b and a set E such that $acl(a, E) \cap acl(b, E) = E$, with a and b dependent over E. Take a, b, E with $rk(a/E) + rk(b/E)$ minimal. Then as noted in the proof of Proposition 5.5.3, for any $E' \supseteq E$, independent from a, b over E, a and b remain dependent over $acl(a, E') \cap acl(b, E')$. Thus after applying the eventual coordinatization we may assume in addition that $acl(a, E)$ and $acl(b, E)$ contain elements a_1, b_1 of rank 1 over E, lying in rank 1 modular definable sets D_1, D_2 respectively, defined over E.

For the argument below some further expansion of E may be necessary. Specifically, we will assume that E satisfies the following condition:

1. If it is possible to expand E to E' independent from ab over E so that $acl(a, E')$ contains an element a_2 of rank 1 over E' independent from a_1 over E, then the same occurs already over the base E; and similarly for b.

We will also want to assume the following condition for a finite number of elements $a' \in acl(a)$ of rank 1 over E, to be determined below:

2. If there exists E' as described in (1) and $a'' \in D_1$ with $acl(a', E') = acl(a'', E')$, then there is $a^* \in D_1$ for which $acl(a', E) = acl(a^*, E)$; and similarly for b.

After these preliminaries we may add constants and take $E = acl(\emptyset)$. We will write $0 = acl(\emptyset) = E$. We will show now that $a \subseteq acl(a_1 b)$ and $b \subseteq acl(b_1 a)$.

We have $acl(a) \cap acl(b) = 0$, and a, b are dependent. Furthermore, $a_1 \in acl(a)$ has rank 1 and $acl(a_1) \cap acl(b) = 0$, so a_1 and b are independent. As $rk(a/a_1) < rk\, a$, by minimality we have a and b independent over $A = acl(a) \cap acl(a_1, b)$. Since a and b are not independent, A and b are not independent. But $A \subseteq acl(a)$ and hence by minimality of total rank (applied to a finite subset of A, and b) we get $rk(A) = rk\, a$, so $a \subseteq A$. Thus $a \subseteq acl(a_1 b)$; similarly $b \subseteq acl(b_1, a)$.

Now we claim there is a_2 with

$$a_2 \in acl(a); \quad rk(a_2) = 1; \quad a_2 \perp a_1$$

Take b', b'_1 conjugates of b, b_1 over a, and independent from b, b_1 over a. Thus $a \subseteq acl(a_1 b')$, and b'_1 is independent from a, b. As b depends on a and b_1 does not, we have $rk\, b > rk\, b_1$ and hence we may choose E' containing b'_1, independent from a, b, b' over b'_1, and some $b'_2 \in acl(b', E')$, so that $rk(b'_2/E') = 1$. Now E' is independent from a, b' and $b'_2 \in acl(b', E') \subseteq acl\, acl(a, b'_1, E') = acl(a, E')$, with a_1 independent from b'_2 over E', so the same holds for some conjugate of E' independent from a, b, and then by condition (1) the same holds over 0 for some a_2 in place of b'_2.

Now $a_2 \in acl(a_1 b)$ and thus $a_1 a_2$ depends on b, but $a_1 a_2 \in acl(a)$, so by minimality $a = acl(a_1 a_2)$. Similarly, we get $b = acl(b_1 b_2)$ with b_2 of rank 1. Here no $a_i \in acl(b)$ and no $b_i \in acl(a)$, but any one of a_1, a_2, b_1, b_2 is algebraic over the remainder, and $a_1 \in D_1$. Consider the base set $F = \{a_2, b_2\}$. Then F is independent from b_1 and D_1 contains an element $x = a_1$ such that $acl(x, F) = acl(b_1, F)$. Taking a conjugate E' of F over b_1 free from a, b, (2) applies and yields an element of D_1 that may replace b_1. In the same fashion we may assume $b_2 \in D_1$, and then after reversing the argument, that $a_2 \in D_1$. Then the pair (a_1a_2, b_1b_2) violates modularity in D_1.

Corollary 5.6.4. *If $\mathcal{M}$ is Lie coordinatized then $\mathcal{M}$ is modular.*

Proof. The embedded linear and projective geometries are seen to be modular using the last criterion in Proposition 5.5.3, as arbitrary parameters from $\mathcal{M}$ may be replaced by parameters in the geometry. Thus it suffices to show that these geometries eventually coordinatize $\mathcal{M}$.

Let $a \in M$, B a finite subset of M, and $a \notin acl(B)$. One may find $c \in acl(a, B) - acl(B)$ lying in a B-definable coordinatizing projective or affine geometry J. If the geometry is affine, then expand B to $B' = B \cup \{c_\circ\}$, adding a generic point of J, and replace c by $c - c_\circ$ in the corresponding linear geometry.

Thus the previous proposition applies.

Definition 5.6.5. *Let a, b be elements of a structure of finite rank. Then b is* filtered *over a if there is a sequence $\mathbf{b} = b_1, \ldots, b_n$ with $rk(b_i/ab_1 \ldots b_{i-1}) = 1$ and $acl(a\mathbf{b}) = acl(ab)$.*

The following was essentially invoked above, and will be applied again subsequently.

Lemma 5.6.6. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular. Then for any a, b in $\mathcal{M}'$, b is filtered over a in $\mathcal{M}'^{\text{eq}}$.*

Proof. Adding constants we may work over the empty set in place of a. We use induction on $n = rk(b)$ and we may suppose $n \geq 1$. We take $b' \in \mathcal{M}'^{\text{eq}}$ with $rk(b/b') = 1$. In particular, b is filtered over b' by b itself, and hence by the previous lemma is independent from b' over $B = acl(b) \cap acl(b')$. Thus $rk(b/B) = rk(b/b') = 1$ and $rk(B) = n - 1$, so by induction after replacing B by a finite set b'' we have a filtration for b' to which we may append b.

5.7 REDUCTS OF MODULAR STRUCTURES

In this section we prove the following theorem on reducts of modular structures:

Proposition 5.7.1. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and modular. Then every structure $\mathcal{M}'$ interpretable in $\mathcal{M}$ inherits these properties.*

As we will to some extent have both $\mathcal{M}$ and $\mathcal{M}'$ in view throughout the analysis, we adopt the convention that when not otherwise specified, model theoretic notions like rank and algebraic closure that depend on the ambient model will be taken to refer to $\mathcal{M}'$. In any case $\mathcal{M}'$ inherits the $\aleph_0$-categoricity and finite rank. The latter point would however be dubious in general for other notions of rank such as S_1-rank. Furthermore, we cannot assume that the notions of independence in $\mathcal{M}$ and $\mathcal{M}'$ stand in any close relationship.

The main case is that of reducts. In fact, as we can add some parameters and work in $\mathcal{M}^{\mathrm{eq}}$, we may suppose that $\mathcal{M}'$ has as its universe a 0-definable subset of $\mathcal{M}$, and that the structure present on $\mathcal{M}'$ is a reduct of the full structure induced from $\mathcal{M}$. We will refer to this situation as a reduct in (not "of") $\mathcal{M}$.

Lemma 5.7.2. *Let $\mathcal{M}$ be $\aleph_0$-categorical, $\mathcal{M}'$ a reduct, and $\mathbf{a}$ a finite sequence which is algebraically independent in the naive sense: none of its entries is algebraic in $\mathcal{M}'$ over the remainder. Then there is a realization $\mathbf{b}$ of the type of $\mathbf{a}$ in $\mathcal{M}'$, which is algebraically independent in $\mathcal{M}$.*

Proof. Let $\mathbf{b}$ be a realization of the specified type with $acl_{\mathcal{M}}(\mathbf{b})$ as large as possible. If $\mathbf{b}$ contains an entry b which is algebraic over the remainder in $\mathcal{M}$, $\mathbf{b}'$, note that in $\mathcal{M}'$ $b \notin acl(\mathbf{b}')$ and hence there is another realization of the type consisting of $\mathbf{b}'$ extended by some $c \notin acl_{\mathcal{M}}(\mathbf{b}')$. But then $|\, acl_{\mathcal{M}}(\mathbf{b})| = |\, acl_{\mathcal{M}}(\mathbf{b}'b)| < |\, acl_{\mathcal{M}}(\mathbf{b}'c)|$, a contradiction.

Lemma 5.7.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular, $\mathcal{M}'$ a reduct in $\mathcal{M}$, and a, b elements of $\mathcal{M}'$ with $rk(b/a) = 1$. Then a is independent from b over $acl(a) \cap acl(b)$.*

We emphasize that our convention applies here, to the effect that the notions used are those of $\mathcal{M}'$ rather than $\mathcal{M}$.

Proof. We will proceed by induction on the rank of a. We may suppose that a and b are algebraically independent, since if $a \in acl(b)$ our claim becomes trivial. By the preceding lemma we may even suppose that they are algebraically independent in $\mathcal{M}$.

Now in $\mathcal{M}$ let $I = (c_1, c_2, \ldots)$ be an infinite $\mathcal{M}$-independent and $\mathcal{M}$-indiscernible sequence over a, with $tp_{\mathcal{M}}(c_i/a) = tp_{\mathcal{M}}(b/a)$. We claim that the sequence I is $\mathcal{M}'$-independent over a. For example, $rk(c_{n+1}/ac_1, \ldots, c_n) = 1$ since $rk(c_{n+1}/a) = 1$ and c_{n+1} is not algebraic over $ac_1, \ldots, c_n$ in $\mathcal{M}$, hence certainly not in $\mathcal{M}'$.

The quantity $rk(a/c_1 \ldots c_i)$ as a function of i is eventually constant, say from $i = m$ onward. Let $d = (c_1, \ldots, c_m)$ and $d' = (c_{m+1}, \ldots, c_{2m})$. $rk(a/d) = rk(a/d') = rk(a/dd')$, the latter equality by the choice of m. Thus in $\mathcal{M}'$ we have $a \perp d$ over d', $a \perp d'$ over d, and also $d \perp d'$ over a as checked above. By Proposition 5.4.1, which is applicable to $\mathcal{M}'$, the triple a, d, d' is independent over $A = acl(a) \cap acl(d) \cap acl(d')$. In particular a, c_1 are independent over A.

We now apply the modularity of $\mathcal{M}$. Let $A^* = acl_{\mathcal{M}}(a) \cap acl_{\mathcal{M}}(c_1)$. Since $a \notin acl_{\mathcal{M}}(b)$, also $a \notin acl_{\mathcal{M}}(c_1)$ and thus $a \notin A^*$. By modularity $a \perp_{\mathcal{M}} c_1$ over A^* and by indiscernibility $a \perp_{\mathcal{M}} c_k$ over A^*. As $ac_1 \ldots c_{i-1}$ is $\mathcal{M}$-independent from c_i over a, we find that $a, c_1, c_2, \ldots$ are $\mathcal{M}$-independent over A^*. Hence $a \notin acl_{\mathcal{M}}(c_1, c_2, \ldots)$ and in $\mathcal{M}'$ we have $a \notin acl(d)$, $a \notin A$, and $rk(A) < rk(a)$. Thus by induction $A \perp c_1$ over $A' = A \cap acl(c_1)$, and hence $A \perp c_1$ over A'. Since $tp(ac_1) = tp(ab)$ we have a, b independent over $acl(a) \cap acl(b)$.

Lemma 5.7.4. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and modular, and let $\mathcal{M}'$ be a reduct in $\mathcal{M}$. Then every rank 1 subset D of $\mathcal{M}'$ is modular.*

Proof. After absorbing an arbitrary finite set of parameters into the language our claim is that if $\mathbf{a}$, $\mathbf{b}$ are two algebraically independent sequences in D with $acl(\mathbf{a}) \cap acl(\mathbf{b}) = acl(\emptyset)$ in $\mathcal{M}'^{\mathrm{eq}}$, then $\mathbf{a}$ and $\mathbf{b}$ are independent. This claim reduces inductively (after further absorption of parameters) to the case in which $\mathbf{a}$ and $\mathbf{b}$ have length 2. In this case if they are not independent, we have $rk(\mathbf{b}/\mathbf{a}) = 1$, and this case was handled in the previous lemma.

Proof of Proposition 5.7.1. It suffices to show that $\mathcal{M}'$ is eventually coordinatized by its rank 1 subsets, since these are modular; we then apply Proposition 5.6.3.

So take $a \notin acl(B)$ with B finite. Let $n = rk(a/B)$. We may find a', c with $a' \in acl(aBc) - acl(Bc)$ and $rk(a/a'Bc) = n - 1$ (cf. Lemma 2.2.3). As $rk(aa'/Bc) = rk(a/Bc)$ this yields

$$rk(a/Bc) = (n-1) + rk(a'/Bc) \geq rk(a/B)$$

and thus a and c are independent over B and a' has rank 1 over Bc. This shows that $\mathcal{M}'$ is eventually coordinatized by rank 1 subsets.