7

Reducts

7.1 RECOGNIZING GEOMETRIES

Our main objective in the present section is to characterize coordinatizing geometries as follows.

Proposition 7.1.1. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and let A, A^* be rank 1 groups equipped with vector space structures over a finite field F, and a definable F-bilinear pairing into F, with everything 0-definably interpreted in $\mathcal{M}$. Assume the following properties:*

- *L1 Every $\mathcal{M}$-definable F-linear map $A \longrightarrow F$ is represented by some element of A^*, and dually.*
- *L2 Algebraic closure and linear dependence coincide on A and on A^*.*
- *L3 A and A^* have no nontrivial proper 0-definable subspaces.*
- *L4 Every definable subset of A or of A^* is a boolean combination of 0-definable subsets and cosets of definable subgroups.*
- *L5 If D is the locus of a complete type in A over $\operatorname{acl}(\emptyset)$ and $a_1', \ldots, a_n' \in A^*$ are F-linearly independent, then there is an element d of D with (d, a_i') prescribed arbitrarily.*

Then the pair (A, A^*) is a linear Lie geometry, possibly weak, which is stably embedded in $\mathcal{M}$.

The proof will require a number of preliminary lemmas. We remark that in view of hypothesis $(L3)$, either one of the groups A, A^* vanishes (in which case we might as well assume $A^* = (0)$), or the pairing is nondegenerate on both sides. In the latter case the notation A^* is justified by hypothesis $(L1)$.

We will continue to label the various hypotheses as in the statement of Proposition 7.1.1.

Lemma 7.1.2. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and let A, A^* be rank 1 groups equipped with vector space structures over a finite field F, and a definable F-bilinear pairing into F, with everything 0-definably interpreted in $\mathcal{M}$. Assume:*

- *L2 Algebraic closure and linear dependence coincide on A and on A^*.*

L3 A and A^* have no nontrivial proper 0-definable subspaces.

Then either A and A^* are algebraically independent, or there is a 0-definable bijection between their projectivizations P and P^*.

Proof. This is the standard nonorthogonality result. We assume an algebraic relation between A and A^*, specifically $\mathrm{rk}\,(\mathbf{a}) = k$, $\mathrm{rk}\,(\mathbf{a}^*) = k^*$, $\mathrm{rk}\,(\mathbf{a}\mathbf{a}^*) < k + k^*$ with $\mathbf{a} \in A$ and $\mathbf{a}^* \in A^*$. We will first find an element of A algebraic over $\mathbf{a}^*$. Suppose $\mathbf{a}$ is not itself algebraic over $\mathbf{a}^*$. Then we take independent conjugates $\mathbf{a}_i$ of $\mathbf{a}$ over $\mathrm{acl}\,(\mathbf{a}^*)$ and find $\mathrm{rk}\,(\mathbf{a}_1, \ldots, \mathbf{a}_n) < nk$ for n large. By the dimension law in projective space there is then $a \in A - (0)$ in $\mathrm{acl}\,(\mathbf{a}_1, \ldots, \mathbf{a}_i) \cap \mathrm{acl}\,(\mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$ and hence algebraic over $\mathbf{a}^*$.

Switching sides, we may then find $a^* \in A^* - (0)$ algebraic over a. Then $\mathrm{acl}\,(a) = \mathrm{acl}\,(a^*)$ and this gives a bijection between a subset of P and a subset of P^*. Furthermore the argument shows that the domain and range of the bijection are algebraically closed, and thus correspond to 0-definable subspaces of A and A^*. By hypothesis ($L3$) the bijection is total.

Lemma 7.1.3. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and let A, A^* be rank 1 groups equipped with vector space structures over a finite field F, and a definable F-bilinear pairing into F, with everything 0-definably interpreted in $\mathcal{M}$. Assume:*

L1 Every $\mathcal{M}$-definable F-linear map $A \longrightarrow F$ is represented by some element of A^*, and dually.

L2 Algebraic closure and linear dependence coincide on A and on A^*.

Assume in addition that the projectivizations P, P^* of A and A^* correspond by a 0-definable bijection. Then there is an identification of A with A^* according to which the given pairing $A \times A^* \longrightarrow F$ is symplectic, unitary, or orthogonal.

Proof. As P and P^* are definably isomorphic, there is a semilinear isomorphism of A with A^*, which gives rise to a self-pairing $A \times A \longrightarrow F$ which is linear in the first variable and satisfies $(x, \alpha y) = \alpha^\sigma (x, y)$ with an automorphism σ on the right. In particular the map $\lambda_x : A \longrightarrow A$ defined by $(x, y)^{\sigma^{-1}}$ is F-linear and hence by hypothesis is given by a unique element x^*: $(y, x^*)^\sigma = (x, y)$. As x^* is definable from x, we have $x^* = \alpha x$ for some $\alpha = \alpha(x) \in F$ possibly dependent on x.

We have

$$\begin{aligned} (y, (\beta x)^*)^\sigma &= (\beta x, y) = \beta(x, y) = \beta(y, x^*)^\sigma = (\beta^{\sigma^{-1}}(y, x^*))^\sigma \\ &= (y, \beta^{\sigma^{-2}} x^*)^\sigma \end{aligned}$$

and thus $(\beta x)^* = \beta^{\sigma^{-2}} x^*$. Now for x_1, x_2 linearly independent with $\alpha(x_1) = \alpha(x_2) = \alpha_\circ$ we have $(x_1 + \beta x_2)^* = \alpha_\circ(x_1 + \beta^{\sigma^{-2}} x_2)$, and as the latter is a scalar multiple of $x_1 + \beta x_2$, we find that σ^2 is the identity and x^* is a linear function of x. The same computation shows that for x_1, x_2 linearly independent, $\alpha(x_1) = \alpha(x_2)$, and thus $\alpha(x)$ is independent of x; so $x^* = \alpha x$ for a fixed α:

$$(x, y) = \alpha(y, x)^\sigma$$

Applying this law twice, $(x, y) = \alpha\alpha^\sigma(x, y)$ and

$$\alpha\alpha^\sigma = 1$$

If σ is the identity, then $\alpha = \pm 1$ and the form (x, y) is either symmetric or symplectic. In characteristic 2 we conclude only that it is symmetric, but in this case the form (x, x) is the square of a linear functional and vanishes on a subspace of codimension at most 1. If we exclude 0-definable proper subspaces of finite codimension we may conclude that in characteristic 2 the form is symplectic.

When σ is nontrivial we have in any case the norm of α equal to 1 and thus $\alpha = \gamma^\sigma/\gamma$ for some $\gamma \in F$. Then one checks that $\gamma(x, y)$ is a unitary form on A.

Definition 7.1.4. *The* geometric language *for* (A, A^*) *consists of the F-space structure, the pairing, an identification of A with A^* as above, if available, and all* $\operatorname{acl}(\emptyset)$*-definable subsets of A and A^*. Vector space operations and the identification, if present, are taken as functions, rather than being encoded by relations.*

We are working over $\operatorname{acl}(\emptyset)$ here. The identification between A and A^* depends in the unitary case on a parameter from the fixed field of the automorphism, but is algebraic over $\operatorname{acl}(\emptyset)$.

Lemma 7.1.5. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and let A, A^* be rank 1 groups equipped with vector space structures over a finite field F, and a definable F-bilinear pairing into F, with everything 0-definably interpreted in $\mathcal{M}$. Assume:*

L1 Every $\mathcal{M}$-definable F-linear map $A \longrightarrow F$ is represented by some element of A^*, and dually.

L2 Algebraic closure and linear dependence coincide on A and on A^*.

L3 A and A^* have no nontrivial proper 0-definable subspaces.

L4 Every definable subset of A or of A^* is a boolean combination of 0*-definable subsets and cosets of definable subgroups.*

L5 If D is the locus of a complete type in A over $\operatorname{acl}(\emptyset)$ *and* $a'_1, \ldots, a'_n$ *are F-linearly independent, then there is an element d of D with* (d, a'_i) *prescribed arbitrarily.*

Then the induced structure on (A, A^*) *admits quantifier elimination in the geometric language.*

Proof. This may seem obvious; but condition (L4) is rather vague as to the provenance of the parameters involved.

We show by induction on n that the quantifier-free type of $a_1, \ldots, a_n$ determines its full type. If A and A^* are identified we work in A exclusively. By hypothesis $(L2)$ we may suppose the a_i are algebraically independent.

We will establish the following for any finite set C and any C-definable subset D of A:

> D is a boolean combination of 0-definable sets, a finite subset of $\operatorname{acl}(C)$, and cosets of the form
> $H_\alpha = \{x \in A : (x, c) = \alpha\}$ with $c \in A^*$ algebraic over C.

Assuming the claim, let C be $\operatorname{acl}(a_1, \ldots, a_{n-1}) = \operatorname{dcl}(a_1, \ldots, a_{n-1})$. By our induction hypothesis the type of C is known. By $(*)$ $\operatorname{tp}(a_n/C)$ is determined by its atomic type over C and hence over $a_1, \ldots, a_{n-1}$ since C is generated by functions over $a_1, \ldots, a_{n-1}$.

It remains to establish $(*)$. We may suppose that the set D is the locus of a complete nonalgebraic type over $\operatorname{acl}(C) = \operatorname{dcl}(C)$. Let D' be the minimal $\operatorname{acl}(\emptyset)$-definable set containing D. We note first that in hypothesis $(L4)$ we may take the definable subgroups involved to be subspaces of finite codimension. Indeed if B is an infinite definable subgroup of A then it has finite index in A and the intersection of αB for $\alpha \in F^\times$ is a definable subspace of finite codimension contained in B. Thus modulo the ideal of finite sets, D is the intersection with D' of a boolean combination D_1 of translates of definable subspaces of finite codimension. There is a definable linear map θ from A to a finite dimensional space F^n, and a subset X of F^n, such that $D_1 = \theta^{-1}[X]$. Minimize n. We may represent θ as $(a^*_1, \ldots, a^*_n)$ for some $a^*_i \in A^*$. We claim the a^*_i lie in $\operatorname{acl}(C)$. We may in any case assume $a_i \in \operatorname{acl}(C)$ for $i \le n_\circ$ and the remaining a_i are algebraically independent over $\operatorname{acl}(C)$. If $n_\circ < n$ then let $a'_{n_\circ+1}, \ldots, a'_n$ be conjugate to $a^*_{n_\circ+1}, \ldots, a^*_n$ over C and linearly independent from $a^*_1, \ldots, a^*_n$. As n has been minimized we can find $\alpha \in F^{n_\circ}$ and $\beta, \beta' \in F^{n-n_\circ}$ with $(\alpha, \beta) \in X$, $(\alpha, \beta') \notin X$. Applying $(L5)$, we may find infinitely many elements $d \in D'$ satisfying

$$(d, a^*_i) = \alpha_i; (d, a^*_{n_\circ+i}) = \beta_i; (d, a'_{n_\circ+i}) = \beta'_i$$

Off a finite set this yields $d \in D$ and $d \notin D$, a contradiction. Thus the a_i^* are algebraic over C. Finally, the finite set involved is the difference of two sets defined over $\operatorname{acl}(C)$ and hence lies in $\operatorname{acl}(C)$.

Proof of Proposition 7.1.1:
In view of Lemma 7.1.5, to complete the analysis of (A, A^*), we must determine the 0-definable subsets of A (and similarly, A^*) more or less explicitly. Let P be the set of types of nonzero elements of A over $\operatorname{acl}(\emptyset)$. For $a \in A$ set $q(a) = \operatorname{tp}(a/\operatorname{acl}(\emptyset))$. Note that these types have rank 1 with the exception of $\operatorname{tp}(0/\operatorname{acl}(\emptyset))$. By the proof of the previous lemma, if a and b are algebraically independent elements of A then the type of $a+b$ over $\operatorname{acl}(\emptyset)$ is determined by: $q(a)$, $q(b)$, and $(a,b) \in F$. (When there is no identification of A with A^*, let the form (a,b) be identically 0 on A.) Thus $q(a+b) = f(q(a), q(b), (a,b))$ for some function $f : P \times P \times F \longrightarrow P$.

Consider $+ : P^2 \longrightarrow P$ defined by $p_1 + p_2 = f(p_1, p_2, 0)$. We claim that $+$ is an abelian group operation on P. This operation is clearly commutative. For associativity, let $p_1, p_2, p_3 \in P$. We may assume they are all nonzero. By type amalgamation and the hypothesis $(L5)$ we can find a_1, a_2, a_3 independent with the prescribed types and with $(a_i, a_j) = 0$ for distinct i, j. Then $p_1 + p_2 + p_3$ computed in either possible way will give $q(a+b+c)$. Finally we check cancellation. Suppose $p_\circ + p_1 = p_\circ + p_2$. We may then choose independent $a_\circ, a_1, a_2$ realizing the prescribed types, with $(a_\circ, a_1) = (a_\circ, a_2) = 0$. We have $q(a_\circ + a_1) = q(a_\circ + a_2)$ and $(-a_\circ, a_\circ + a_1) = (-a_\circ, a_\circ + a_2) = -(a,a)$. Thus $q(a_1) = f(q(-a), p_\circ + p_1, -(a,a)) = q(a_2)$, as claimed.

Thus P is a finite abelian group. Let the zero element of P be denoted $p_\circ$, and let D be the locus of this type in A.

We now dispose of the polar case, in which there is no identification of A with A^*. Then $q : A \longrightarrow P$ is generically a homomorphism and hence extends to a homomorphism by sending 0 to 0. As A has no proper 0-definable subspace of finite codimension, it has no proper 0-definable subgroup of finite index, and thus the homomorphism is trivial, and $A - (0)$ realizes a unique type over $\operatorname{acl}(\emptyset)$. This completes the analysis of the polar case.

For the remainder of the argument we may suppose that A and A^* have been identified, or in other words that A carries a symmetric, symplectic, or unitary form. If P consists of a single type then this form is symplectic and the types are entirely known. We may assume therefore that P contains more than one type. It is of course still possible that the form is symplectic.

D is infinite, and is the locus of a type over $\operatorname{acl}(\emptyset)$, and hence generates A. The group $\operatorname{Stab}(D)$ has rank 1 and hence coincides with

A. Thus a generic element of A belongs to $\mathrm{Stab}_{\circ}(D)$ and can therefore be written $a+b$ with $a, b \in D$ independent. As the type of $a+b$ for $a, b \in D$ independent is determined by the value of (a, b), we get a function $f^* : F \longrightarrow P$.

For independent $a, b, c \in D$ with $(a, b) = 0$ we have $q(a+b) = q(a) + q(b) = p_\circ$ and thus $a + b \in D$, and as $(a+b, c) = (a, c) + (b, c)$ it follows that f^* is an additive homomorphism. We define a map $\nu : F^\times \longrightarrow \mathrm{End}\,(P)$ by $\nu(\alpha) \cdot q(a) = q(\alpha a)$. This is clearly a well defined multiplicative homomorphism into $\mathrm{End}\,(P)$. In particular $p_\circ$ is fixed by $\nu[F^\times]$, and thus D is invariant under nonzero scalar multiplication. Thus we may make the following computation with $a, b \in D$ independent, $(a, b) = \alpha$:

$$(*) \qquad f^*(\beta\beta^\sigma\alpha) = q(\beta a + \beta b) = \nu(\beta) q(a+b) = \nu(\beta) f^*(\alpha)$$

Now let K be the kernel of f^*, and $F_\circ$ the fixed field of σ (which may be all of F). We will show that $K = \ker \mathrm{Tr}$ with Tr the trace from F to $F_\circ$, which will allow us to identify P and $F_\circ$.

By $(*)$ K is invariant under multiplication by elements $\beta\beta^\sigma$, that is by norms or squares according as σ is nontrivial or trivial, and therefore is an $F_\circ$-subspace of F in all cases. Furthermore $K < F$ since P has more than one element. Thus if σ is the identity and $F_\circ = F$ we have only the possibility $K = (0)$, which is the claim in this case. Suppose now that σ is nontrivial so that F is a quadratic extension of $F_\circ$. As $q(x+y) = q(y+x)$ we get $f^*(\alpha) = f^*(\alpha^\sigma)$ so K contains the kernel $\{\alpha - \alpha^\sigma : \alpha \in F\}$ of the trace, which is of codimension 1 in F. Thus K coincides with this kernel.

Accordingly we now identify P with $F_\circ$ and f^* with the trace. The formula $(*)$ then states that ν is the norm if σ is nontrivial, and the squaring map otherwise. In particular there are $|F_\circ|$ nontrivial types over $\mathrm{acl}\,(\emptyset)$. These types must therefore be determined by the function (x, x), unless the form is symplectic.

Suppose finally that the form is symplectic; we still suppose that $|P| = |F_\circ|$. Take x, y independent and orthogonal. Then $(x-y, y) = 0$ and thus $q(x) = q(x-y) + q(y) = q(x) + q(-y) + q(y)$, that is $q(-y) = -q(y)$. On the other hand by $(*)$ $q(-y) = q(y)$ and thus the characteristic is 2. Our final objective is to show that q is a quadratic form, so that A is an orthogonal space in characteristic 2. In any case $(*)$ says that $q(\alpha x) = \alpha^2 q(x)$ and it remains to study $q(x+y)$.

Take x_1, x_2, y_1, y_2 in D independent with x_i orthogonal to y_i for $i = 1, 2$, and let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$. Let $z_i = x_i + y_i$; then $z_i \in D$ and $q(z_1 + z_2) = (z_1, z_2) = \alpha + \beta + (x_1, y_2) + (x_2, y_1)$. Let $x = x_1 + x_2$, $y = y_1 + y_2$. Then x, y are independent; $q(x) = \alpha$ and $q(y) = \beta$; and $(x, y) = (x_1, y_2) + (x_2, y_1)$. As $x + y = z_1 + z_2$ we have $q(x+y) = q(x) +$

$q(y) + (x, y)$. This argument applies to x, y independent and nonzero. When x, y are dependent they are linearly dependent and it follows easily that this formula holds in general. Thus q is a quadratic form associated to the given symplectic form. This determines the structure of A in this last case.

Lemma 7.1.6. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let A, A^* be 0-definably interpreted rank 1 vector spaces over a finite field F with a definable F-bilinear pairing satisfying:*

L1 Every $\mathcal{M}$-definable F-linear map $A \longrightarrow F$ is represented by some element of A^*, and dually.

L2 Algebraic closure and linear dependence coincide on A and on A^*.

L3 A and A^* have no nontrivial proper 0-definable subspaces.

Suppose that over $\operatorname{acl}(\emptyset)$, A, A^* are part of a linear Lie geometry stably embedded in $\mathcal{M}$. Then A, A^* are part of a linear Lie geometry stably embedded in $\mathcal{M}$.

Proof. We have to show that if A carries a bilinear form or quadratic form defined over $\operatorname{acl}(\emptyset)$ then the set of scalar multiples of the form is 0-definable, and similarly if A, A^* are part of a quadratic geometry in characteristic 2.

Note that any $\operatorname{acl}(\emptyset)$-definable linear automorphism of A acts trivially on the projective space PA, by $(L2)$, and hence is given by a scalar multiplication. As A^* contains all definable linear forms on A, any two nondegenerate bilinear forms differ by a definable automorphism of A, hence differ by a scalar. In odd characteristic this disposes of all cases since quadratic forms correspond to inner products.

Consider now the case of a symplectic space in characteristic 2, where the form is known up to a scalar multiple. With the form fixed, the set of quadratic forms compatible with it and definable over $\operatorname{acl}(\emptyset)$ corresponds to $A^* \cap \operatorname{acl}(\emptyset)$. By $(L3)$ this is (0). Thus if there are quadratic forms definable over $\operatorname{acl}(\emptyset)$, they are the scalar multiples of a single form.

Suppose finally that there are no $\operatorname{acl}(\emptyset)$-definable quadratic forms but that there is an $\operatorname{acl}(\emptyset)$-definable quadratic geometry. In this case the set of $\operatorname{acl}(\emptyset)$-definable quadratic forms compatible with one of the bilinear forms carries a regular action by A^* and hence this is the standard quadratic geometry over $\emptyset$, corresponding to a form known up to a scalar multiple. Note that the pairing is known but the identification of A with A^* is only known up to a scalar multiple.

Proposition 7.1.7. *Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let A, A^* be 0-definably interpreted rank 1 vector spaces over a finite field F*

with a definable F-bilinear pairing satisfying:

L1 Every $\mathcal{M}$-definable F-linear map $A \longrightarrow F$ is represented by some element of A^*, and dually.

L3 A and A^* have no nontrivial proper 0-definable subspaces.

Let $c \in \mathcal{M}$, with $\operatorname{acl}(c) \cap (A, A^*) = \operatorname{dcl}(c) \cap (A, A^*)$ nondegenerate, and set $(A', A'^*) = [\operatorname{acl}(c) \cap (A, A^*)]^{\perp}$. Assume that relative to a possibly larger field F', in $\mathcal{M}' = \mathcal{M}$ with c added as a constant, (L1,L3) hold for A', A'^* as well as:

L2 Algebraic closure (over c and linear dependence (over the extended scalar field) coincide on A' and on A'^*.

L4′ Every definable subset of A' or of A'^* is a boolean combination of c-definable subsets and cosets of definable subgroups.

L5′ If D is the locus of a complete type in A' over $\operatorname{acl}(c)$ and $a'_1, \ldots, a'_n$ are F-linearly independent, then there is an element d of D with (d, a'_i) prescribed arbitrarily.

Then there is a 0-definable sort Q in $\mathcal{M}$ such that (A, A^*, Q) form a weak linear Lie geometry, stably embedded in $\mathcal{M}$.

Proof. We will work over $\operatorname{acl}(\emptyset)$. We let Q be $\emptyset$ unless A carries an $\operatorname{acl}(\emptyset)$-definable symplectic bilinear form in characteristic 2, in which case we let Q be the set of all definable quadratic forms compatible with one of these symplectic forms on A; each component of this set, corresponding to a particular form, has a regular action by A^* and in particular is uniformly definable. Thus Q is 0-definable. We let $J = (A, A^*, Q)$ with all structure defined over $\operatorname{acl}(\emptyset)$ and we claim that this is stably embedded.

Let $\mathcal{M}'$ be the expansion of $\mathcal{M}$ by the constant c, and J' the geometry A', A'^* with the structure inherited from $\mathcal{M}'$. By Proposition 7.1.1, J' is a stably embedded weak linear geometry. Let $A_\circ = \operatorname{acl}(c) \cap A$. Then $A = A_\circ \oplus A'$, and similarly for A^*, and Q. Thus J is contained in the definable closure of J' in $\mathcal{M}'$. Thus J inherits the following properties:

J is stably embedded in $\mathcal{M}$; J has finite rank and is modular;
J has the type amalgamation property of Proposition 5.1.15

By Proposition 6.2.3, if H is a parametrically definable subgroup of $A \times A$ or $A \times A^*$ in $\mathcal{M}$, then H is commensurable with an $\operatorname{acl}(\emptyset)$-definable subgroup.

Let F' be the ring of endomorphisms of A which are 0-definable in J. By the third hypothesis F' is a field and it must restrict to a subfield of the field of scalars for J'. We claim in fact that F' induces the scalars of J'. Let α be one of the scalar multiplications on J'. The graph of α is commensurable with an $\operatorname{acl}(\emptyset)$-definable subgroup H of $A \times A$. By the third condition H is the graph of a group isomorphism from A to A. Let $\alpha \in F'$ be the element with graph H. As the graphs of α and α' are commensurable $\operatorname{acl}(c)$-definable automorphisms of A', they agree there.

The same sort of argument shows that an isomorphism $A' \longrightarrow A'^*$ is induced by an $\operatorname{acl}(\emptyset)$-definable isomorphism on A of the same type. The same applies to quadratic forms in odd characteristic since they correspond to bilinear forms. In characteristic 2 one can in any case extend quadratic forms to forms on A in $\operatorname{acl}(c)$, taking them to vanish on $\operatorname{acl}(c) \cap A$.

Now let J^- be J reduced to its geometric structure. The structure on J' is known and is defined from this geometric structure by Proposition 7.1.1. As J is interpreted in J', every 0-definable relation in J is definable in J^- from parameters in $\operatorname{acl}(c)$. Let R be 0-definable in J, with canonical parameter $e \in J^-$, and definable in J^- from the parameter a. By weak elimination of imaginaries in J^- we may take $a \in \operatorname{acl}(e)$ in J^-; but $e \in \operatorname{acl}(\emptyset)$ in J, so $a \in \operatorname{acl}_J(\emptyset) \cap J^-$ which is trivial by assumption. Thus R is 0-definable in J^- and $J = J^-$ is a stably embedded Lie geometry.

This argument took place over $\operatorname{acl}(\emptyset)$ (and our last 0-definability claim is blatantly false in general); to remove this, we use the preceding lemma.

Remark 7.1.8. *We are dealing in Proposition 7.1.1 with the rank 1 case of the analysis of settled groups with* $\operatorname{acl}(\emptyset) \cap A = (0)$, $\operatorname{acl}(\emptyset) \cap A^* = (0)$. *It would be interesting to tackle the general case. Two special cases: analyze the case of prime exponent, or the case of rank 2.*

7.2 FORGETTING CONSTANTS

The following is a special case of Proposition 7.5.4 below, for which we will give a proof by a method not depending on the classification of finite simple groups. The proof given here goes via smooth approximation rather than coordinatization and involves [**KLM**], hence the classification of the finite simple groups.

Proposition 7.2.1. *Let $\mathcal{M}$ be a structure and $\mathcal{M}_c$ an expansion of $\mathcal{M}$ by a constant c. If $\mathcal{M}_c$ is smoothly approximable by finite structures, then there is an expansion $\mathcal{M}^\circ$ of $\mathcal{M}$ by an algebraic constant which is smoothly approximable.*

The key example here is due to David Evans: one takes $\mathcal{M}$ to be the reduct of a basic quadratic geometry in which the orientation is forgotten, but the corresponding equivalence relation is remembered. In a finite approximation the two classes are distinguished, so $\mathcal{M}$ is not smoothly approximable by finite structures. The orientation itself is an algebraic constant. It can be shown that this is the only sort of algebraic constant which comes in to Proposition 7.2.1.

Definition 7.2.2. *If $\mathcal{M}$ is Lie coordinatizable and E is an envelope in $\mathcal{M}$ it is said to be* equidimensional *if all the isomorphism types of specified geometries of a given type are the same; that is the dimensions and Witt defects are constant.*

Lemma 7.2.3. *Let $\mathcal{N}$ be smoothly approximable, $c \in \mathcal{N}$, E a finite subset of $\mathcal{N}$ containing c. Then*

1 If E is an envelope of $\mathcal{N}$, it is an envelope of $\mathcal{N}_c$.

2 If E is an equidimensional envelope of $\mathcal{N}_c$, it is an envelope of $\mathcal{N}$, provided that:

1.1 The locus of c over $\emptyset$ is nonmultidimensional;

1.2 For any $\mathrm{acl}(\emptyset)$-definable canonical projective geometry P_b with canonical parameter b, $\mathrm{tp}(b)$ implies $\mathrm{tp}(b/c)$.

Proof. We use the criterion given in the corollary to Lemma 3.2.4. Of the three conditions given there, only the last one is actually sensitive to the presence of the parameter c. In $\mathcal{N}$ this may be phrased as follows:

> If c_1, c_2 are conjugate in $\mathcal{M}$ and D_{c_1}, D_{c_2} are corresponding conjugate definable sets, then $D_{c_1} \cap E$ and $D_{c_2} \cap E$
>
> are conjugate by an elementary automorphism of E.

This condition is certainly inherited "upward", giving the first point. For the second, assuming conditions 1.1 and 1.2, and the conjugacy condition in $\mathcal{N}_c$, it suffices to to show the conjugacy condition for canonical projective geometries D_{c_i}. There are two cases.

Suppose first that $c_i \notin \operatorname{acl}(\emptyset)$. Then D_{c_i} is orthogonal to $\operatorname{tp}(c/c_i)$ as the latter is analyzed by $\operatorname{acl}(\emptyset)$-definable geometries. Hence D_{c_i} remains a projective geometry in $\mathcal{N}_c$. It is also canonical: every proper conjugate in $\mathcal{N}_c$ is in particular a conjugate in $\mathcal{N}$, and hence orthogonal to D_{c_i}. Thus the dimension of D_{c_i} in E is one of the specified dimensions as an envelope in $\mathcal{N}_c$; these are all assumed equal, so D_{c_1} and D_{c_2} have the same dimension and similarly, where applicable, the same Witt defect.

Now suppose $c_i \in \operatorname{acl}(\emptyset)$. Then by 2.2 $\operatorname{tp}(c_1/c) = \operatorname{tp}(c_2/c)$ and thus they are conjugate in E_c, and the $D_{c_i} \cap E$ are conjugate.

We now deal with a special case of Proposition 7.2.1.

Lemma 7.2.4. *Let $\mathcal{M}$ be a structure and $\mathcal{M}_c$ an expansion of $\mathcal{M}$ by a constant c. Assume that the locus P of c in $\mathcal{M}$ is nonmultidimensional in $\mathcal{M}_c$ and that for any $\operatorname{acl}(c)$-definable canonical projective geometry J_b, $\operatorname{tp}(b/c)$ implies $\operatorname{tp}(b/\operatorname{acl}(c))$. If $\mathcal{M}_c$ is smoothly approximable by finite structures, then there is an expansion $\mathcal{M}^\circ$ of $\mathcal{M}$ by an algebraic constant which is smoothly approximable.*

Proof. An envelope in $\mathcal{M}_c$ is determined by a k-tuple of dimensions for some k. Let q be a 2-type realized in P. Define a binary relation R_q between k-tuples of dimensions by: $R_q(d, d')$ if and only if there is a realization (c, c') of q, and a finite subset E of $\mathcal{M}$ which is an envelope of dimension d in $\mathcal{M}_c$ and is an envelope of dimension d' in $\mathcal{M}_{c'}$. We claim that R_q defines a partial function. If (c, c') realizes q, then $\operatorname{tp}(c'/c)$ in $\mathcal{M}_c$ determines $\operatorname{tp}(c'/c)$ in U and hence determines the corresponding dimension d'. We will use function notation, writing $f_q(d) = d'$.

We define an equivalence relation on P by: $E(a, b)$ holds if there is a finite subset $C_\circ$ of P such that for any finite subset C of P containing $C_\circ$, any equidimensional envelope of $\mathcal{M}_C$ is an envelope of $\mathcal{M}_a$ and $\mathcal{M}_b$, with the same dimensions. We claim:

$$\text{If } a, b, b' \in P \text{ and } \operatorname{tp}(ab) = \operatorname{tp}(ab') \text{ then } E(b, b')$$

Given such a, b, b' we let $q = \operatorname{tp}(ab) = \operatorname{tp}(ab')$ and $C_\circ = \{a, b, b'\}$. If C contains a, b, b' and U is an equidimensional envelope of $\mathcal{M}_C$, then U is an equidimensional envelope over a, b, or b'; and the dimension over b or b' is f_q applied to the dimension over a.

Thus the relation E has finitely many equivalence classes. Let $c_\circ = c/E \in \operatorname{acl}(\emptyset)$. We claim that $\mathcal{M}$ is smoothly approximable over $c_\circ$.

Let P be the increasing union of finite subsets C_n with $C_1 = \{c\}$ and let U_n be an n-equidimensional envelope in $\mathcal{M}_{C_n}$ containing U_{n-1}. Let

$\mathcal{L}$ be the canonical language for $\mathcal{M}$ (consisting of complete types over $\emptyset$). Let $\mathcal{F}$ be a nonprincipal ultrafilter on ω and let the term "almost all n" be understood with reference to this ultrafilter. Let $\mathcal{L}^*$ be the set of relations which are 0-definable in $\mathcal{L}(c)$ whose restrictions to U_n are L-definable for almost all n. We will show that $\mathcal{L}^* = L(c_\circ)$ and that $\mathcal{M}$ is smoothly approximable in the language L^*.

$\mathcal{L}^*$ is a sublanguage of $\mathcal{L}(c)$ which contains $\mathcal{L}(c_\circ)$ since the proof that E has finitely many classes also shows $c_\circ$ is definable in U_n from some point on. To see that $\mathcal{M}$ is smoothly approximable in the language $\mathcal{L}^*$, let k be fixed and let $\mathbf{a}, \mathbf{b}$ be k-tuples with the same type in $\mathcal{L}^*$. It suffices to show that for almost all n, two such k-tuples in U_n will be conjugate in U_n. If not, then for almost all n, there is a 0-definable k-ary relation R_n on U which does agree on U_n with any relation in $\mathcal{L}^*$. However it must agree with some c-definable relation restricted to U_n, and there are only finitely many such, so for almost all n R_n agrees with the same c-definable relation on U_n, which means it agrees with a relation of $\mathcal{L}^*$, a contradiction.

It remains to be shown that $\mathcal{L}^* \subseteq \mathcal{L}(c_\circ)$. Let P' be the equivalence class of c with respect to E; this is a subset of P. We claim first:

$$P' \text{ realizes a unique } \mathcal{L}^*\text{-type}$$

Take $c' \in P'$. It suffices to show that for almost all n, and indeed for all sufficiently large n, there is an automorphism of U_n carrying c to c'. For large n, U_n contains c and c' and is an equidimensional envelope with the same dimensions relative to c and to c'. Thus $\mathcal{M}_c$ and $\mathcal{M}_{c'}$ are isomorphic smoothly approximable models and U_n over c or c' is an equidimensional envelope with respect to the same data in both cases; by uniqueness of envelopes, $(U_n, c) \simeq (U_n, c')$.

It follows that any automorphism σ of $\mathcal{M}_{c_\circ}$ preserves $\mathcal{L}^*$: as σ preserves P', by the previous claim we may suppose that σ fixes c, and hence $\mathcal{L}^*$. Thus $\mathcal{L}^* \subseteq \mathcal{L}(c_\circ)$.

Lemma 7.2.5. *Let $\mathcal{M}$ be smoothly approximable, and for $a \in \mathcal{M}$ let $a^{(1)} = \{a' \in \operatorname{acl}(a) : \operatorname{rk}(a') = 1\}$. Define $E(a,b)$ by: $a^{(1)} = b^{(1)}$. Then:*

1 If S is an $\operatorname{acl}(\emptyset)$-definable subset of $\mathcal{M}$ of rank $n > 0$, then each E-class in S has rank less than n.

2 $\mathcal{M}/E$ is nonmultidimensional.

3 If $c \in \mathcal{M}$ and a and b are both independent from c, then $a^{(1)} = b^{(1)}$ if and only if the same relation holds in $\mathcal{M}_c$.

Proof. The first point is the coordinatization theorem, i.e., without loss of generality $\mathcal{M}$ is Lie coordinatized. The second point is clear as the

0-definable closure of $a^{(1)}$ is a set of rank at most 1 over $\emptyset$.

For the final point, write $a_c^{(1)}$ for $a^{(1)}$ computed over c. We wish to show that each of $a^{(1)}$, $a_c^{(1)}$ determines the other. As $a^{(1)} = \{a' \in a_c^{(1)} : a'$ is independent from $c\}$, it suffices to deal with the reverse direction. We claim:

$$a_c^{(1)} = \operatorname{acl}(a^{(1)}, c)$$

In any case the right side is contained in the left. Conversely we must show that if $d \in \operatorname{acl}(a,c)$ has rank at most 1 over c then $d \in \operatorname{acl}(a^{(1)}, c)$. By modularity a and c, d are independent over $a' = \operatorname{acl}(a) \cap \operatorname{acl}(c,d)$. Thus a and d are independent over $a'c$ and therefore $d \in \operatorname{acl}(a'c)$. But $\operatorname{rk}(a'/c) \le 1$ and a, c are independent, so $\operatorname{rk}(a') \le 1$. Thus $a' \in a^{(1)}$ and $d \in \operatorname{acl}(a^{(1)}, c)$.

Proof of Proposition 7.2.1:
We assume $\mathcal{M}_c$ is smoothly approximable and we seek $c_\circ \in \operatorname{acl}(\emptyset)$ with $\mathcal{M}_{c_\circ}$ smoothly approximable. We work over $\operatorname{acl}(\emptyset)$, and we replace c by a finite subset C of $\operatorname{acl}(c)$ such that for P_b an $\operatorname{acl}(c)$-definable canonical projective geometry, $\operatorname{tp}(b/C)$ implies $\operatorname{tp}(b/\operatorname{acl}(c))$. We again write c rather than C. After these adjustments, if the locus P of c is nonmultidimensional, then Lemma 7.2.4 applies. We treat the general case by induction on $\operatorname{rk} c$.

If there is $c_1 \in \operatorname{acl}(c)$ with $c \notin \operatorname{acl}(c_1)$ then after expanding c_1 if necessary to a slightly larger subset of $\operatorname{acl}(c_1)$ we may take $\mathcal{M}_{c_1}$ to be smoothly approximable, by induction, as $\operatorname{rk}(c/c_1) < \operatorname{rk}(c)$, and then by a second application of induction, as $\operatorname{rk}(c_1) < \operatorname{rk}(c)$, we reduce to a parameter in $\operatorname{acl}(\emptyset)$. We assume therefore that there is no such element c_1.

We define a relation E on P by: $E(a,b)$ if for some $c \in P$ independent from a, b: $a_c^{(1)} = b_c^{(1)}$, where $a_c^{(1)}$ is $a^{(1)}$ computed over c, as in the previous lemma. We claim that if $c, c' \in P$ are both independent from ab and $a_c^{(1)} = a_c^{(1)}$, then the same applies over c'. Working with an element c'' independent from a, b, c, c', we reduce to the case in which c and c' are independent over a, b, in other words the triple ab, c, c' is independent. As $\mathcal{M}_c$ is smoothly approximable, and ab and c' are independent there, the previous lemma applies and yields $a_c^{(1)} = b_c^{(1)}$ if and only if $a_{c,c'}^{(1)} = b_{c,c'}^{(1)}$; arguing similarly over c', our claim follows. In particular, E is a 0-definable equivalence relation.

Suppose toward a contradiction that is degenerate, i.e. $E = P^2$. Then for $c \in P$ fixed, the relation $E_c(a,b) : a_c^{(1)} = b_c^{(1)}$ has a class of maximal rank. This violates the first clause of the previous lemma. As we are working over $\operatorname{acl}(\emptyset)$, it follows that P/E is infinite. If c_1 is c/E, then

$c_1 \in \operatorname{acl}(c)$ and $\operatorname{rk}(c/c_1) < \operatorname{rk}(c)$. Therefore by our initial assumption $c_1 \in \operatorname{acl}(\emptyset)$, that is: E has finite classes.

Let P have rank n and let $c_1, \ldots, c_{2n+1} \in P$ be independent. Let E_i be the equivalence relation $a^{(1)}_{c_i} = b^{(1)}_{c_i}$, and E' the intersection of the E_i. For any a, b in P, there is an i for which ab is independent from c_i and thus E' refines E, and has finite classes. Now $P/E' \leftrightarrow \prod_i P/E_i$, $\{c_1, \ldots, c_{2n+1}\}$-definably, and the quotients P/E_i are nonmultidimensional. Hence P is nonmultidimensional in $\mathcal{M}_{c_1,\ldots,c_{2n+1}}$. Therefore P is also nonmultidimensional over $\mathcal{M}(c_1)$, since any orthogonality over c_1 would be preserved (after conjugation) over $c_1, \ldots, c_{2n+1}$. As this case is the base of our induction, we are done.

7.3 DEGENERATE GEOMETRIES

Lemma 7.3.1. *Let $\mathcal{M}$ be a structure and D 0-definable in $\mathcal{M}$. Then the following are equivalent:*

(1) D is stable and stably embedded in $\mathcal{M}$.

(2) There is no unstable formula $\phi(x, y)$ with $\phi(x, y) \Longrightarrow (x \in D)$.

(3) There is no unstable formula $\phi(x_1, \ldots, x_n, y)$ for which $\phi(\mathbf{x}, y) \Longrightarrow (x_i \in D)$, all i.

Proof. The equivalence of (2) and (3) is [**Sh**] [II:2.13 (3,4), p. 36]. We check the equivalence of (1) and (3).

Suppose first that (1) fails. If D is unstable then relativization to D produces a suitable ϕ. If D is not stably embedded and $\phi(x, c)$ defines a subset of D which is not D-definable, one can find a countable set of parameters in D over which there are $2^{\aleph_0}$ ϕ^*-types (ϕ^* being ϕ with the variables interchanged). Indeed, for any finite set $A \subseteq D$ and any ϕ^*-type p over A realized by a conjugate of c, there are conjugates of c realizing contradictory ϕ-types over a larger finite subset of D; for this, we may suppose that p is satisfied by c, and take a 1-type over A in D which is split by $\phi(x, c)$; then we have $\phi(d_1, c)$ and $\neg\phi(d_2, c)$ with d_1 conjugate to d_2 over A, and after identifying d_1 with d_2 we have realizations c, c' of contradictory ϕ-types by elements conjugate to c.

It follows that ϕ^* is unstable [**Sh**] [II:2.2 (1,2), pp. 30-31].

Now suppose (1) holds. Let A be a countable subset of D and $\mathcal{M}^*$ an elementary extension of $\mathcal{M}$. As D is stably embedded, any ϕ-type over D realized in $\mathcal{M}^*$ is definable with a parameter e in $D[\mathcal{M}^*]$, and since D is stable $\mathrm{tp}(e/A)$ is definable. Thus the types over A are definable and (3) follows [**Sh**] [II:2.2 (1,8), pp. 30-31].

Lemma 7.3.2. *Let $\mathcal{M}$ be an $\aleph_0$-categorical structure which does not interpret a Lachlan pseudoplane. If $a, b \in \mathcal{M}$ with neither algebraic over the other, then there is a conjugate b' of b over a distinct from b for which $a \notin \mathrm{acl}(b, b')$.*

Proof. Write down a theory asserting that $a_1, a_2, \ldots$ are distinct solutions to the conditions $\mathrm{tp}(xb) = \mathrm{tp}(xb') = \mathrm{tp}(ab)$, with $b \neq b'$. Our claim is that this theory is consistent.

Suppose that this theory is inconsistent. Then for some n, b is definable from any n distinct conjugates $a_1, \ldots, a_n$ of a over b, by the conjunction of the formulas:

$$(*) \qquad \mathrm{tp}(a_i, y) = \mathrm{tp}(ab)$$

With n minimized (and at least 2) let $\mathbf{a} = \{a_1, \ldots, a_{n-1}\}$ be a set (unordered) of conjugates of a over b chosen so that $b \notin \mathrm{acl}(\mathbf{a})$. By assumption none of the a_i is algebraic over b.

We claim:

(1) $\mathbf{a} \notin \operatorname{acl}(b)$;

(2) $b \notin \operatorname{acl}(\mathbf{a})$;

(3) b is definable from any two distinct conjugates of $\mathbf{a}$ over b;

(4) $\mathbf{a}$ is definable from any two distinct conjugates of b over $\mathbf{a}$.

Granted this, we have a Lachlan pseudoplane with points conjugate to $\mathbf{a}$, lines conjugate to b, and incidence relation given by $\operatorname{tp}(\mathbf{a}b)$.

Now (1) is clear, (2) holds by the choice of $\mathbf{a}$ (and n), and for (3) observe that any two conjugates of $\mathbf{a}$ over b will involve at least n distinct conjugates of a over b. Finally for (4), if b and b' have the same type over $\mathbf{a}$ and $\mathbf{a}, \mathbf{a}'$ are distinct and have the same type over bb', then b is definable from $\mathbf{aa}'$ in the manner of $(*)$ above, as is b', so $b = b'$.

Definition 7.3.3. *A subset D of a structure $\mathcal{M}$ is* algebraically irreducible *if for $b \in D$ we have: $a \in \operatorname{acl}(b) - \operatorname{acl}(\emptyset)$ implies* $[b \in \operatorname{acl}(a)]$.

Lemma 7.3.4. *Let $\mathcal{M}$ be $\aleph_0$-categorical, let D be the locus of a 1-type over $\emptyset$ in $\mathcal{M}$, and suppose that D is algebraically irreducible and $\mathcal{M}$ does not interpret a pseudoplane. If there is a definable strongly minimal subset D_b of D with defining parameter b, then finitely many conjugates of D_b cover D.*

Proof. Let Q be the locus of b over $\emptyset$. Define an equivalence relation $E(b, b')$ on Q by: D_b and $D_{b'}$ differ by a finite set. By Lachlan's normalization lemma [**LaPP**] for each $b \in Q$ there is a $D_{b/E}$-definable set agreeing with D_b up to a finite set. Thus we may factor out E and assume that distinct conjugates of D_b have finite intersection. Then the previous lemma applies to $a \in D_b - \operatorname{acl}(b)$ and b, and as the conclusion fails, we find that for such pairs a, b we have $b \in \operatorname{acl}(a)$. Now by the algebraic irreducibility of D it follows that $b \in \operatorname{acl}(\emptyset)$. This yields our claim.

Lemma 7.3.5. *Let $\mathcal{M}^-$ be a reduct of the smoothly approximable structure $\mathcal{M}$. Let D be a rank 1 0-definable set in $\mathcal{M}^-$, and suppose that for any finite subset B of $\mathcal{M}^-$ and any a_1, a_2 in D: $\operatorname{acl}(Ba_1a_2) = \operatorname{acl}(Ba_1) \cup \operatorname{acl}(Ba_2)$ where the algebraic closure is taken in D, and in the sense of $\mathcal{M}^-$. Then D is stable and is stably embedded in $\mathcal{M}^-$.*

Proof. Model theoretic notions are to be understood in $\mathcal{M}^-$ except where otherwise noted. The proof of (1) will proceed by induction on the rank r of D in $\mathcal{M}$. By Lemma 7.3.1 the class of stable and stably embedded 0-definable subsets of $\mathcal{M}^-$ is closed under finite unions. Thus we may suppose that D realizes a single type over $\emptyset$.

We show first

Any infinite subset of D which is definable in $\mathcal{M}^-$ has rank r in $\mathcal{M}$.

Suppose on the contrary D' is of lower rank in $\mathcal{M}$. Then by induction D' is stable and is stably embedded in $\mathcal{M}$ relative to a defining parameter for D'. From $\mathcal{M}$ D' inherits the following properties: it is $\aleph_0$-categorical, and does not interpret a pseudoplane. By Lachlan's theorem [**LaPP**] it is $\aleph_0$-stable and in particular contains a definable strongly minimal subset D'_b definable in $\mathcal{M}^-$. Then by the previous lemma finitely many conjugates of D'_b in $\mathcal{M}^-$ cover D and thus D is stable and stably embedded in $\mathcal{M}^-$.

From this it follows that for any sequence $a_1, a_2, \ldots$ in D which is algebraically independent in $\mathcal{M}^-$, there is a conjugate sequence which is independent in $\mathcal{M}$. Indeed choosing the conjugates inductively, at stage n we have to realize the type of a_n over $a_1, \ldots, a_{n-1}$ in $\mathcal{M}^-$ (or more exactly a conjugate type) by an element independent from $a_1, \ldots, a_{n-1}$ in $\mathcal{M}$. The locus of this type is an infinite set defined in $\mathcal{M}^-$ and hence of full rank r in $\mathcal{M}$, so this is possible.

Now suppose we do not have D stable and stably embedded in $\mathcal{M}^-$, or equivalently that we have an unstable formula $\phi(x, y)$ which implies $(x \in D)$. We then find a finite set B and types p, q over $\operatorname{acl}(B)$ such that both $p(x), q(y), \phi(x, y)$ and $p(x), q(y), \neg\phi(x, y)$ have solutions with x, y independent over B. For this it suffices to take an indiscernible sequence (a_i, b_i) such that $\phi(a_i, b_j)$ holds if and only if $i < j$, letting B be an initial segment over which the sequence is independent.

Now fix realizations b_{-1}, b_1 of q independent over B and set $B' = B \cup \{b_{-1}, b_1\}$. Let $D' = \{x \in D : \phi(x, b_1) \& \neg\phi(x, b_{-1})\}$. As $\mathcal{M}^-$ inherits the type amalgamation property from $\mathcal{M}$, by the corollary to Proposition 5.1.15 the set D' is infinite. Let $D'' \subseteq D'$ be the locus of a complete nonalgebraic type over B' in $\mathcal{M}^-$.

Now let $a_1, \ldots, a_n$ be elements of D'', pairwise algebraically independent over B'. We will show that there are 2^n ϕ-types over $a_1, \ldots, a_n$. By our basic assumption on D the set $A = \{a_1, \ldots, a_n\}$ is algebraically independent over $B'$and after conjugation we may suppose that these elements are independent in $\mathcal{M}$ over B'. For each i both $\phi(a_i, y) \& q(y)$ and $\neg\phi(a_i, y) \& q(y)$ are consistent, with rank equal to $\operatorname{rk}(q)$, so by the corollary to type amalgamation the same applies to any combination of these properties as i varies. This produces the desired 2^n types.

Now let k be the size of $\operatorname{acl}(B'a) \cap D$ in $\mathcal{M}^-$ for $a \in D''$. Then any set of n elements of D'' contains $[n/k]$ pairwise independent elements and hence allows $2^{[n/k]}$ ϕ-types. This is greater than the bound allowed by the corollary to Proposition 5.1.20. So we have a contradiction.

Corollary 7.3.6. *With the hypotheses and notation of Lemma 7.3.5, if D carries no nontrivial 0-definable equivalence relation, then there is no induced structure on D beyond the equality relation.*

Proof. The additional hypothesis implies that $\operatorname{acl}(a) = a$ for $a \in D$ and hence $\operatorname{acl}(X) = X$ for $x \subseteq D$.

As we remarked in the previous proof, once we know D is stable, we know that it is $\aleph_0$-stable, and of Morley rank 1. By the Finite Equivalence Relation Theorem, the Morley degree is 1, that is D is strongly minimal. As acl is trivial on D, the claim follows.

7.4 REDUCTS WITH GROUPS

Lemma 7.4.1. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a locally definable abelian group of bounded exponent n in $\mathcal{M}^-$. Then*

1 *For any definable subset S of A, the subgroup generated by S is definable.*

2 *If A is 0-definable in $\mathcal{M}^-$ of exponent p then the dual A^* and the pairing $A \times A^* \longrightarrow F_p$ are interpretable in $\mathcal{M}^-$. If A has no nontrivial proper 0-definable subgroups in $\mathcal{M}^-$ then either A^* is trivial or the pairing is a perfect pairing.*

3 *If A is 0-definable and carries a 0-definable vector space structure over a finite field K, then A^* (the definable F_p-dual) allows a 0-definable K-bilinear pairing $\mu : A \times A^* \longrightarrow K$ with $\mathrm{Tr} \circ \mu(a, f) = f(a)$.*

Proof. These statements were proved in the Lie coordinatizable context as Lemma 6.1.8, Proposition 6.3.2, and Lemma 6.3.4.

The first statement is inherited from $\mathcal{M}$. The subgroup generated by S is definable in $\mathcal{M}$ if and only if it is generated in a finite number of steps, and this is equivalent to its definability in $\mathcal{M}^-$. Thus this first property passes to reducts.

For the second statement we have a definable dual $\hat{A}$ in $\mathcal{M}$, which in particular involves only finitely many sorts of $\mathcal{M}$, and we are interested in the subgroup A^* of $\mathcal{M}^-$-definable elements. Let A^*_n be the subset of $\mathcal{M}^-$-definable elements which are definable from at most n parameters. This generates a 0-definable subgroup of $\hat{A}$ and hence for large n is all of A^* in the sense of $\mathcal{M}^-$.

The proof of the third property is purely formal, given the second.

Lemma 7.4.2. *Let $\mathcal{M}$ be a structure, and A a 0-definable abelian group in $\mathcal{M}^-$. Let H_i $(i = 1, \ldots, n)$ be a finite set of subgroups of A, and let D be a finite union of cosets of the H_i, such that:*

(1) *$[H_i : H_i \cap H_j]$ is infinite for i, j distinct;*

(2) *D contains a coset of each H_i and if D_i is the union of the cosets of H_i which are contained in D, there is no group $T > H_i$ commensurable with H_i for which D_i is the union of cosets of T.*

Then the groups H_i are $\mathrm{acl}(\emptyset)$-definable in $(A; D)$.

Proof. This is an application of Beth's definability theorem applied to the set $\{H_1, \ldots, H_n\}$, which we claim is implicitly definable. Let n_i be the number of cosets of H_i contained in D_i and let $\mathcal{T}$ be the theory of (A, D) expanded by axioms ϕ for the H_i: they are subgroups with the stated properties, for which D_i is the union of exactly n_i cosets. Suppose we have two models of the form $(A, D, \bar{H})$ and $(A, D, \bar{H}')$ with the same (A, D). For each i, as some coset of H_i is covered by cosets of the H'_j, by Neumann's lemma we have $[H_i : H_i \cap H'_j] < \infty$ for some j. Similarly for each j we can find a corresponding i; by the hypothesis on the H_i, these two correspondences are reciprocal, and after rearrangement this means that H_i is commensurable with H'_i for all i. Then for each i D_i is the same set in both models and is a union of cosets of both H_i and H'_i, hence of $H_i + H'_i$; if this group extends H_i or H'_i properly, we contradict (2); but (2) can be included in ϕ since there is a bound on the possible index $[H_i + H'_i : H_i]$. Thus $H_i = H'_i$.

Lemma 7.4.3. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, and A a 0-definable abelian group in $\mathcal{M}^-$. Suppose that A has no definable subgroups in $\mathcal{M}^-$ of $\mathcal{M}$-rank strictly between* 0 *and* $\mathrm{rk}_{\mathcal{M}}(A)$. *Then in $\mathcal{M}^-$, A has rank 1, and every infinite $\mathcal{M}^-$-definable subset of A has full rank in $\mathcal{M}$.*

Proof. The first statement follows from the second.

Suppose the second statement fails, and D is $\mathcal{M}^-$-definable in A with $0 < \mathrm{rk}_{\mathcal{M}}(D) < \mathrm{rk}_{\mathcal{M}}(A)$. Let $r = \mathrm{rk}_{\mathcal{M}}(D)$ be minimal. By Lemma 6.2.5 in $\mathcal{M}$, D is contained in a finite union of cosets C_i of subgroups H_i of A definable in $\mathcal{M}$ with $\mathrm{rk}\, H_i = r$, and a set of rank less than r. Let D be chosen to minimize the number n of distinct subgroups involved. Then the indices $[H_i : H_i \cap H_j]$ are infinite for i, j distinct.

We show that $n = 1$. By Lemma 6.2.5 $S_1 = \mathrm{Stab}\,(D \cap C_1)$ has rank r and evidently $S_1 \leq H_1$; but $\mathrm{rk}\, H_1 = r$, so $[H_1 : S_1] < \infty$. Let a be a generic point of $\mathrm{Stab}_\circ(D \cap C_1)$. Then $a \in H_1$ and $a \notin H_j$ for $H_j \neq H_1$, and furthermore $[a + C_j] \cap C_k = \emptyset$ for j, k distinct. Let $D' = D \cap (D + a)$; then $\mathrm{rk}\, D' = r$ and up to a set of rank r D' is contained in the union of the $C_i \cap (C_j + a)$, which up to a set of rank less than r is the union of the cosets C_i for $H_i = H_1$. By the choice of D, the same applies to D and all $H_i = H$ coincide.

For $a \in A$ the set $D \cap (D + a)$ is $\mathcal{M}^-$-definable and hence is of rank r or finite. Thus $S_\circ = \{a \in A : \mathrm{rk}\,(D \cap (D + a)) = r\}$ is definable in $\mathcal{M}^-$. Decompose D into loci of types D_i over $\mathrm{acl}\,(\emptyset)$ in $\mathcal{M}$. Then $S_\circ = \bigcup_{ij} S_{ij}$ with $S_{ij} = \{a \in A : \mathrm{rk}\,(D_i \cap (D_j + a)) = r\}$. By Lemma 6.2.5 each nonempty S_{ij} is contained in a coset C_{ij} of a subgroup T_{ij} of rank r, with $C_{ij} - S_{ij}$ of rank less than r. As D is contained in a finite union of cosets of H, also of rank r, H and the T_{ij} are commensurable.

Thus for some subgroup T of finite index in H, $S_\circ$ is a union of sets A_k contained in cosets of T and differing from these cosets by sets of rank less than r. Take $a_k \in A_k$ for each k, and let $Y_{kl} = (A_k - a_k) \cap (A_l - a_l)$. Then Y_{kl} is generically closed under addition and inverse, and applying Lemma 6.1.3, $A_k - A_l$ is a coset of a subgroup of T which differs from

T by a set of smaller rank; so $A_k - A_l$ is a coset of T. From all of this it follows that $S_\circ - S_\circ$ is itself a finite union of cosets of T. As the set $S_\circ - S_\circ$ is definable in $\mathcal{M}^-$, the preceding lemma implies that some subgroup commensurable with T is also definable in $\mathcal{M}^-$. This contradicts our assumption on A.

Lemma 7.4.4. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, and A a rank 1 0-definable abelian group of prime exponent p in $\mathcal{M}^-$. Let A^* be the dual in $\mathcal{M}^-$ and let $\hat{A}$ be the dual in $\mathcal{M}$. Then*

1. *In $\mathcal{M}^-$, A^* has rank at most 1.*
2. *If in $\mathcal{M}^-$ we have $\operatorname{acl}(\emptyset) \cap A = (0)$, $\operatorname{acl}(\emptyset) \cap A^* = (0)$, and $A^* \neq (0)$, then $A^* = \hat{A}$.*

Proof. *Ad* 1. We apply the preceding lemma. Suppose A^* has a definable subgroup B in $\mathcal{M}^-$ with B and A^*/B infinite. Let $B^\perp$ be the annihilator of B in A. Then A^*/B acts faithfully on $B^\perp$, so $B^\perp$ is infinite. Similarly $(A/B^\perp, B)$ form a nondegenerate pair, so $A/B^\perp$ is infinite. This is a contradiction.

Ad 2. Let B be the annihilator in A of A^*. By hypothesis $B < A$ and hence $B = (0)$. Thus in $\mathcal{M}$ we have two perfect pairings (A, A^*) and $(A, \hat{A})$, and by the pseudofiniteness of $\mathcal{M}$ these dual groups coincide.

Lemma 7.4.5. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable abelian group of prime exponent p in $\mathcal{M}^-$, and D an infinite 0-definable subset of A. Then for generic independent $a_1^*, \ldots, a_n^*$ in A^* there is $d \in D$ with (d, a_i^*) prescribed arbitrarily.*

Proof. By the last two lemmas every infinite $\mathcal{M}^-$-definable subset of A^* has full rank and thus the sequence $a_1^*, \ldots, a_n^*$ is conjugate in $\mathcal{M}$ to a generic independent sequence in A^*. Apply Lemma 6.4.1 in $\mathcal{M}$.

Lemma 7.4.6. *Let $\mathcal{M}$ be a Lie coordinatizable structure, A a definable group abelian of rank r, and D a definable subset of A of rank r whose complement is also of rank r. Then there is a coset C of a definable subgroup of finite index in A, and an intersection D' of finitely many translates of D, so that*

$$\operatorname{rk}(D') = r; \quad \operatorname{rk}(D' \cap C) < r.$$

Proof. We may assume that A is settled over the empty set and that D is 0-definable. Let P be the locus of a 1-type over $\operatorname{acl}(\emptyset)$. Then every definable subset of P is the intersection of P with a boolean combination of definable cosets of A of finite index, and of sets of rank less than $r = \operatorname{rk}(P)$ (Lemma 6.6.2).

We may find a generic element $g \in A$ for which the rank of $P \setminus (D+g)$ is r: take $a \in A \setminus D$ generic, $b \in P$ generic with a, b independent, and

$g = b - a$. There is a coset C of a definable subgroup of finite index in A, for which $C \cap P$ is contained in $P \setminus (D+g)$ up to a set of lower rank, or in other words $(D+g) \cap C \cap P$ has rank less than r. Furthermore as A is settled over $\operatorname{acl}(g)$, we may take C to be $\operatorname{acl}(g)$-definable.

For each 1-type P over $\operatorname{acl}(\emptyset)$ choose g_P and C_P as in the foregoing paragraph so that $\bigcap_P (D+g_P) \cap \bigcap_P C_P$ has rank less than r. Taking the g_P independent over the empty set, both intersections $\bigcap_P (D+g_P)$ and $\bigcap_P C_P$ will have rank r, and the latter is a coset of a definable subgroup of A of finite index. This proves the claim.

Lemma 7.4.7. *Let $\mathcal{M}$ be a Lie coordinatizable structure, A a definable abelian group of rank r, and D a definable subset of A of rank r whose complement is also of rank r. Then there is an intersection D' of finitely many translates of D, which has rank r and is contained in a proper subgroup of finite index in A. In particular, the subgroup generated by D' will be a proper subgroup of finite index in A, which is definable in the structure (A, D).*

Proof. We apply the previous lemma to find a definable subgroup H of finite index in A, a coset C of H, and a finite intersection D' of finitely many translates of D, so that $D' \cap C$ has rank less than r. Take D' such an intersection so that the number of cosets of H which meet D' in a set of rank r is minimized, subject to the constraint that $\operatorname{rk} D' = r$. We may suppose that $D = D'$: so if D meets $D+g$ in a set of rank r, then D and $D+g$ meet the same cosets of H in a set of rank r.

Let $X \subseteq A/H$ be the set of cosets which meet D in a set of rank r. We may suppose $H \in X$.

We claim that X is a subgroup of A/H. We may take D and H to be 0-definable. Take $C \in X$ and choose a representative g for C as follows. Fix a 1-type over $\operatorname{acl}(\emptyset)$ whose locus P is contained in $D \cap H$, and let Q be the locus of a 1-type over $\operatorname{acl}(\emptyset)$ which is contained in $D \cap C$. Take $(a,b) \in P \times Q$ generic; then $g = b - a$ is generic, and $g + H = C$. Furthermore, $(g+D) \cap D \cap Q$ contains $(g+P) \cap Q$ (in particular, a) and hence has full rank. Thus $g + D$ also meets all the cosets in X in sets of rank r, in other words $X - g = X$. Thus X is a group.

Let $X = B/H$ with $H \leq B \leq A$. As $C \notin X$, we have $B < A$. In addition, by our construction $D \setminus B$ has rank less than r. Let $S = D \setminus B$. As $\operatorname{rk} S < r$, for any $r+1$ independent generic elements $h_1, \ldots, h_{r+1}$ in A we will have $\bigcap_i (S+h_i) = \emptyset$; if c lies in the intersection and is independent from h_i, then $\operatorname{rk}(h_i/c) = r$, and $c - h_i \in S$, a contradiction.

Thus if we replace D by the intersection D' of its translates by $r+1$ independent generic elements of B, we will retain $\operatorname{rk} D' = r$, while now $D' \subseteq B$.

Proposition 7.4.8. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$. If $A^* = (0)$ in $\mathcal{M}^-$ then A is strongly minimal and stably embedded in $\mathcal{M}^-$.*

Proof. Supposing the contrary, there is a subset D of A which in $\mathcal{M}^-$ is definable (from parameters in $\mathcal{M}^-$), infinite, and with infinite complement. By Lemma 7.4.3, both D and its complement have full rank in A. By Lemma 7.4.7 there is a proper subgroup of finite index in A which is definable in $\mathcal{M}^-$; so A^* is nontrivial in $\mathcal{M}^-$.

Proposition 7.4.9. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$. Suppose $\operatorname{acl}(\emptyset) \cap A = (0)$, and $\operatorname{acl}(\emptyset) \cap A^* = (0)$. Then there is a finite field F and an $\operatorname{acl}(\emptyset)$-definable F-space structure on A for which algebraic closure on A and F-linear span coincide.*

Proof. We let F be the ring of $\operatorname{acl}(\emptyset)$-definable group endomorphisms of A, which is a division ring and is finite by $\aleph_0$-categoricity; thus it is a finite field.

We show by induction on n that any n F-linearly independent elements of A are independent. Assuming the claim for n, suppose that $a \in \operatorname{acl}(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n$ independent. We claim that a is a linear combination of the a_i. Taking a conjugate of $a_1, \ldots, a_n$ in $\mathcal{M}^-$ we may suppose that the elements $a_1, \ldots, a_n$ are independent of maximal rank in $\mathcal{M}$.

Consider the locus D of $a_1, \ldots, a_n, a$ over $\operatorname{acl}(\emptyset)$ in $\mathcal{M}$, and let $S = \operatorname{Stab}(D)$. By Lemma 6.2.5 $\operatorname{rk} S = \operatorname{rk} D = n{\cdot}\operatorname{rk}_{\mathcal{M}}(A)$, and D is contained in a coset of S. Let T be the projection of S onto the first n coordinates. Then the projection of D is contained in a coset of T and thus $\operatorname{rk} T = \operatorname{rk} S$. Therefore the kernel is finite, and T has finite index in A^n. We claim:

$$\text{Some subgroup } S' \text{ of } A^n \text{ commensurable with } S \text{ is } \operatorname{acl}(\emptyset)\text{-definable in } \mathcal{M}^-. \qquad (*)$$

For any $\mathcal{M}^-$-definable subset X of A^n one sees easily by induction on n that $\operatorname{rk}_{\mathcal{M}} X = \operatorname{rk} X \cdot \operatorname{rk}_{\mathcal{M}} A$. Accordingly $\operatorname{Stab}_\circ(D)$ in the sense of $\mathcal{M}$ is definable in $\mathcal{M}^-$. One then continues as in the final paragraph of the proof of Lemma 7.4.3. Thus $(*)$ holds.

In $\mathcal{M}$, $S' \cap S$ is also $\operatorname{acl}(\emptyset)$-definable and induces an equivalence relation on D with finitely many classes. As D is complete over $\operatorname{acl}(\emptyset)$ in $\mathcal{M}$, it is contained in a single coset of $S \cap S'$ and thus $S \leq S'$ with $[S' : S] < \infty$. The kernel of the projection of S' to the first n coordinates is also finite, hence trivial by our hypotheses, and the image is of finite index in A^n, hence the projection is surjective. It follows that S'

represents a linear function $s(\mathbf{x}) = \sum_i \alpha_i x_i$ with coefficients in F. As D lies in a coset of S, it lies in a coset of S', and the function $y - s(\mathbf{x})$ is constant on D, hence in $\operatorname{acl}(\emptyset)$ in $\mathcal{M}^-$, hence 0. Thus $h(a) = \sum_i \alpha_i a_i$.

Lemma 7.4.10. *Let $\mathcal{M}$ be $\aleph_0$-categorical and modular of finite rank, $\mathcal{M}^-$ a reduct of $\mathcal{M}$ with $\operatorname{acl}_{\mathcal{M}}(\emptyset) = \operatorname{acl}_{\mathcal{M}^-}(\emptyset)$. If X, Y are sets which are independent in $\mathcal{M}$ then they are independent in $\mathcal{M}^-$.*

Proof. If X, Y are dependent in $\mathcal{M}^-$ then in $\mathcal{M}^-$ by inherited modularity there is $a \in \operatorname{acl}(X) \cap \operatorname{acl}(Y) - \operatorname{acl}(\emptyset)$ and by our hypothesis this holds also in $\mathcal{M}$.

Proposition 7.4.11. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$, with $\operatorname{acl}_{\mathcal{M}}(\emptyset) \cap (\mathcal{M}^-)^{\mathrm{eq}} = \operatorname{dcl}_{\mathcal{M}^-}(\emptyset)$. If A is settled over $\emptyset$ in $\mathcal{M}$ then it is settled over $\emptyset$ in $\mathcal{M}^-$ and thus every definable subset in $\mathcal{M}^-$ is a boolean combination of 0-definable subsets, a finite set, and cosets of definable subgroups.*

Proof. We must show in $\mathcal{M}^-$ that for a independent from an algebraically closed set c,

$$\operatorname{tp}(a) \cup \operatorname{gtp}(a/c \cap A^*) \Longrightarrow^* \operatorname{tp}(a/c)$$

(all types are computed in $\mathcal{M}^-$). We will show in fact that for *any* c there is a linearly independent k-tuple $\mathbf{b} \in A^*$ for some k for which:

$$\begin{gathered}\operatorname{tp}_{\mathcal{M}^-}(a) \cup \operatorname{gtp}(a/\mathbf{b}) \Longrightarrow \operatorname{tp}_{\mathcal{M}^-}(a/c)\\ \text{for any } a \in A \text{ not algebraic over } b, c.\end{gathered} \tag{$*$}$$

After absorbing those parameters in $\mathbf{b}$ which are algebraic over c into c, the rest are independent over c and are conjugate in $\mathcal{M}^-$ to parameters independent in $\mathcal{M}$ over c. For a independent from c in $\mathcal{M}^-$ with $\operatorname{gtp}(a/c \cap A^*)$ as specified, we can conjugate a over c to an independent element in $\mathcal{M}$, then by type amalgamation complete a, c to $a, \mathbf{b}, c$ with the same 2-types $\operatorname{tp}(a\mathbf{b})$ and $\operatorname{tp}(\mathbf{b}c)$ as in the original triple $a, \mathbf{b}, c$ (that is, the version in which $\mathbf{b}$ is independent from c). This then determines $\operatorname{tp}(a/c)$. Note that in the course of the argument a portion of $\operatorname{acl}(c \cap A^*)$ was absorbed into c.

We now begin the lengthy verification of $(*)$.

Let C be the locus of the type of c over $\emptyset$ in $\mathcal{M}^-$ and let k be the maximum dimension of $\operatorname{acl}_{\mathcal{M}}(c) \cap A^*$ for $c \in C$. Let B_k be the set of linearly independent k-tuples in A^*. We introduce the notation $\operatorname{cl} c$ for $\{a \in A : \operatorname{rk}_{\mathcal{M}}(a/c) < \operatorname{rk}_{\mathcal{M}}(A)\}$.

We define two relations E^-, E on pairs from $B_k \times C$ as follows. $E^-((\mathbf{b}, c), (\mathbf{b}', c'))$ holds if and only if (b, c) is independent from (b', c') in $\mathcal{M}^-$ and for $a \in A - \operatorname{acl}(b, b', c, b')$, $\operatorname{gtp}(a/b) = \operatorname{gtp}(a/b')$ implies

$\mathrm{tp}(a/c) = \mathrm{tp}(a/c')$; $E((\mathbf{b}, c), (\mathbf{b}', c'))$ holds if and only if (b, c) is independent from (b', c') in $\mathcal{M}$ and for $a \in A - \mathrm{cl}\,(b, b', c, b')$, $\mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a/b')$ implies $\mathrm{tp}(a/c) = \mathrm{tp}(a/c')$.

Then easily E holds if and only if E^- holds and the pairs involved are independent in $\mathcal{M}$. Now we show that E is a generic equivalence relation in the sense of §5.1. So take an independent triple $x = (b, c); x' = (b', c'); x'' = (b'', c'')$ in $\mathcal{M}$ with $E(x, x')$ and $E(x', x'')$. We must show $E(x, x'')$.

Take $a \in A - \mathrm{cl}\,(b, b'', c, c'')$ with $\mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a/b'')$. We claim $\mathrm{tp}(a/c) = \mathrm{tp}(a/c'')$. Let $q = \mathrm{tp}(a)$, $r = \mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a/b'')$. By Lemma 6.4.1 $q(x) \cup r(x/b')$ is consistent, of rank $\mathrm{rk}\, q$. By the corollary to type amalgamation (Proposition 5.1.15), so is $q(x) \cup r(x/b') \cup \mathrm{tp}_{\mathcal{M}}(a/bb''cc'')$,

Take $a' \in A - \mathrm{cl}_{\mathcal{M}}(bb'b''cc'c'')$ realizing this type. From $E(x, x')$, $E(x', x'')$ we get in $\mathcal{M}^-$: $\mathrm{tp}(a'c) = \mathrm{tp}(a'c') = \mathrm{tp}(a'c'')$ and thus $\mathrm{tp}(ac) = \mathrm{tp}(ac'')$.

Now we claim that E^- is also a generic equivalence relation. Let x, x', x'' be independent in $\mathcal{M}^-$ with $E^-(x, x')$ and $E^-(x', x'')$. We can conjugate x, x', x'' in $\mathcal{M}^-$ to an independent triple in $\mathcal{M}$ and reduce to the case of E.

Accordingly by Lemma 5.1.12 there is a 0-definable equivalence relation E' in $\mathcal{M}^-$ that agrees with E^- on independent pairs in $B_k \times C$. Then E' also agrees with E on $\mathcal{M}$-independent elements of $B_k \times C$. The domain of the relation E' is $D =$:

$$\{x \in B_k \times C \quad : \quad \text{There is } x' \in B_k \times C \text{ independent from } x \text{ so that } E^-(x, x')\}$$

Note that in this definition we may take independence in the sense either of $\mathcal{M}$ or of $\mathcal{M}^-$ since these notions agree up to conjugation in $\mathcal{M}^-$.

We consider also the following set, which will turn out to coincide with D:

$$D_1 = \quad \{(b, c) \in B_k \times C : \quad \text{For } a \in A - \mathrm{cl}\,(b, c), \ \mathrm{tp}_{\mathcal{M}}(a) \cup \mathrm{gtp}\,(a/b) \quad \text{determines } \mathrm{tp}_{\mathcal{M}^-}(a/c)\}$$

Note that if $\mathbf{b}$ includes a basis for $\mathrm{acl}\,(c) \cap A^*$ then as A is settled in $\mathcal{M}$, $(b, c) \in D_1$. Thus D_1 projects onto C. Furthermore E has finitely many classes on D_1 since for $x \in D_1$, the class of x/E' is determined by information in $\mathrm{tp}_{\mathcal{M}}(x')$. (This is clear first for independent pairs x, x' using the definition of E^- and then for general pairs.)

We will show shortly that $D = D_1$. First we check that D projects onto C. Take $c \in C$, and $\mathbf{b}$ linearly independent containing a basis for $\mathrm{acl}_{\mathcal{M}}(c) \cap A^*$. Take a conjugate $(\mathbf{b}', c')$ in $\mathcal{M}$ independent from $(\mathbf{b}, c)$

in $\mathcal{M}$. Then easily $E((\mathbf{b}, c), (\mathbf{b}', c'))$ and thus $(\mathbf{b}, c) \in D$. By the same argument $D_1 \subseteq D$.

We will now show $D \subseteq D_1$. Let $x \in D$, and x' independent from x in $\mathcal{M}$, with $E(x, x')$. With $x = (\mathbf{b}, c)$ we must show that $\mathrm{tp}_{\mathcal{M}}(a) \cup \mathrm{gtp}\,(a/b)$ determines $\mathrm{tp}_{\mathcal{M}^-}(a/c)$ for $a \in A - \mathrm{cl}\,(x)$. Let $a, a' \in A - \mathrm{cl}\,(x)$ satisfy $\mathrm{tp}_{\mathcal{M}}(a) = \mathrm{tp}_{\mathcal{M}}(a') = q$ and $\mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a'/b) = r(x/b)$. By type amalgamation we may choose a, a' so that the triple $a; a'; bb'cc'$ is independent in $\mathcal{M}$ and a, a' satisfy the same type over $b'c'$. This then yields $\mathrm{tp}_{\mathcal{M}^-}(a/c) = \mathrm{tp}_{\mathcal{M}^-}(a/c') = \mathrm{tp}_{\mathcal{M}^-}(a'/c') = \mathrm{tp}_{\mathcal{M}^-}(a'/c)$. Thus $(\mathbf{b}, c) \in D_1$.

Finally we prove $(*)$. The relation E' has finitely many classes on $D_1 = D$. As $\mathrm{acl}\,(\emptyset) = \mathrm{dcl}\,(\emptyset)$ any such class $D_\circ$ is 0-definable in $\mathcal{M}^-$. Let $(\mathbf{b}, c) \in D_\circ$ and suppose that our claim fails for $(\mathbf{b}, c)$. Fix $a, a' \in A - \mathrm{acl}\,(\mathbf{b}, c)$, with equal types in $\mathcal{M}^-$ and with $\mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a'/b)$ but with $\mathrm{tp}_{\mathcal{M}^-}(a/c) \neq \mathrm{tp}_{\mathcal{M}^-}(a'/c)$. Let σ be an automorphism carrying a to a'. Then $\mathrm{gtp}\,(a/\mathbf{b}) = \mathrm{gtp}\,(a/\sigma\mathbf{b})$ but $\mathrm{tp}_{\mathcal{M}^-}(a/c) \neq \mathrm{tp}_{\mathcal{M}^-}(a/c')$.

Take $(\mathbf{b}', c')$ conjugate to $(\mathbf{b}, c)$ over a in $\mathcal{M}^-$ and independent from $a, \mathbf{b}, c, \sigma\mathbf{b}, \sigma c$. Then

$$\mathrm{gtp}\,(a/\sigma b) = \mathrm{gtp}\,(a/b) = \mathrm{gtp}\,(a/b')$$

and $\mathrm{tp}_{\mathcal{M}^-}(a/c') \neq \mathrm{tp}_{\mathcal{M}^-}(a/\sigma c)$. As $(\sigma\mathbf{b}, \sigma c)$ and $(b'c')$ are independent, this shows they are inequivalent with respect to E. However these pairs are conjugate in $\mathcal{M}^-$, a contradiction.

Corollary 7.4.12. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$. If A is settled over $\emptyset$ in $\mathcal{M}$ then it is settled in $\mathcal{M}^-$ over a finite set of $\mathcal{M}$-algebraic constants.*

Proof. By the preceding result A becomes settled over $\mathrm{acl}_{\mathcal{M}}(\emptyset)$ and hence over the collection of definable subsets of A which belong to $\mathrm{acl}_{\mathcal{M}}(\emptyset)$; there are finitely many such.

7.5 REDUCTS

In the present section we show that reducts of Lie coordinatized structures are weakly Lie coordinatized; we may lose the orientation. We must deal mainly with the primitive case (meaning there is no nontrivial 0-definable equivalence relation).

Lemma 7.5.1. *Let $\mathcal{M}$ be a structure realizing finitely many 3-types, and $a \in \mathcal{M}$. Let $\mathrm{acl}(a)$ be computed in $\mathcal{M}^{\mathrm{eq}}$. Then the lattice of algebraically closed subsets of $\mathrm{acl}(a)$ is finite.*

Proof. Let $\mathcal{E}_a$ be the collection of a-definable equivalence relations on $\mathcal{M}$ which have finitely many classes, $\mathcal{C}_a = \bigcup\{\mathcal{M}/E : E \in \mathcal{E}_a\}$, and $\hat{\mathcal{C}}_a$ the collection of subsets of $\mathcal{M}$ which are unions of subsets of $\mathcal{C}_a$. Viewing $\hat{\mathcal{C}}_a$ as a subset of $\mathcal{M}^{\mathrm{eq}}$, we have $\hat{\mathcal{C}}_a \subseteq \mathrm{acl}(a)$, and it suffices to show that for $\alpha \in \mathrm{acl}(a)$ we have

$$(*) \qquad \alpha \in \mathrm{acl}(\mathrm{acl}(\alpha) \cap \hat{\mathcal{C}}_a)$$

Let $\alpha \in \mathrm{acl}(a)$ and let $\phi(x, a)$ be a formula which defines a finite set A containing α. Let $S = \{b \in \mathcal{M} : \phi(\alpha, b)\}$, which we view as an element of $\mathcal{M}^{\mathrm{eq}}$, and let $A_S = \{\beta : \forall x \in S\ \phi(\beta, x)\}$. Then easily $S \in \mathrm{dcl}(\alpha) \cap \hat{\mathcal{C}}_a$, and as $\alpha \in A_S \subseteq A$, we have $\alpha \in \mathrm{acl}(S)$. This proves $(*)$ (and a little more).

Remark 7.5.2. *When $\mathcal{M}$ is $\aleph_0$-categorical, the foregoing lemma applies to any element a of $\mathcal{M}^{\mathrm{eq}}$. (For another approach, see the note at the end of this section.)*

Proposition 7.5.3. *Let $\mathcal{M}$ be a weakly Lie coordinatized structure, $\mathcal{M}^-$ a reduct of $\mathcal{M}$, and D a primitive, rank 1, definable subset of $\mathcal{M}^-$. Then D is a Lie geometry forming part of a Lie geometry stably embedded in $\mathcal{M}^-$; this geometry may be unoriented, and may be affine.*

Proof. As D has rank 1, acl gives a combinatorial geometry on D; the same holds over any finite set.

Suppose first that acl_B gives a degenerate geometry over any finite B, or in other words, that $\mathrm{acl}(A, B) = \bigcup_{a \in A} \mathrm{acl}(a, B)$ in D. In this case, by Lemma 7.3.5, D is a trivial structure, and is stably embedded.

Now we deal with the nondegenerate case. Let $\{D_i\}$ be a set of representatives for the primitive rank 1 $\mathrm{acl}(\emptyset)$-definable sets in D^{eq}, up to 0-definable bijections, with $D_1 = D$, and let $D^\infty = \bigcup_i D_i$. We claim that D^∞, with acl, is a projective space (of infinite dimension) over a field; the field will be finite by the previous lemma, applied as indicated in the subsequent remark.

We show first that some line has more than two points. Take c_1, c_2, c_3 in D and B a finite set so that $c_3 \in \mathrm{acl}(c_1c_2B) - [\mathrm{acl}(c_1B) \cup \mathrm{acl}(c_2B)]$.

By modularity there is $e \in \operatorname{acl}(c_1c_2) \cap \operatorname{acl}(c_3B)$ so that c_1c_2 and c_3B are independent over e. Then $\operatorname{rk}(e) = 1$ and we may take $e \in D^\infty$. As $e \in \operatorname{acl}(c_1, c_2) - [\operatorname{acl}(c_1) \cup \operatorname{acl}(c_2)]$, this suffices.

Now we show that coplanar lines meet. Take a_1, a_2, a_3, a_4 in D^∞ pairwise algebraically independent with $\operatorname{rk}(a_1a_2a_3a_4) = 3$. Take $e \in \operatorname{acl}(a_1a_2) \cap \operatorname{acl}(a_3a_4)$ such that a_1a_2 and a_3a_4 are independent over e. Then again $\operatorname{rk} e = 1$ and e may be taken in D^∞.

Thus D^∞ is an infinite-dimensional projective geometry with finite lines, and there is a vector space model, that is a map $\pi : V - (0) \longrightarrow D^\infty$ in which linear dependence in V corresponds to algebraic independence in D^∞. We do not claim that this vector space is interpreted globally in the model.

Let $V_i = \pi^{-1}[D_i]$, thought of as a new sort for each i. We enrich $\mathcal{M}^-$ by the V_i with the relevant structure, taking π_i to be the restriction of π to V_i, and restricting $+$ and scalar multiplication to a family of relations on the new sorts. The expanded structure will be called $\mathcal{M}^{-*}$; it can be thought of also as a reduct $\mathcal{M}^{*-}$ of an expansion of the original structure $\mathcal{M}$ by the new sorts and relations. Here $\mathcal{M}^*$ is a finite cover of $\mathcal{M}$ by sets of order $q-1$; any automorphism of $\mathcal{M}$ over $\operatorname{acl}(\emptyset)$ extends to an automorphism of $\mathcal{M}^*$. Thus $\mathcal{M}^*$ is weakly Lie coordinatizable.

By Lemma 7.4.1 V_1 lies in a 0-definable rank 1 group A in $\mathcal{M}^{-*}$. We may suppose that A has no 0-definable finite subgroups. Our claim is:

$(*)$ $\qquad$ A is part of a stably embedded Lie geometry in $\mathcal{M}^{-*}$.

Assuming $(*)$, D forms part of an embedded Lie geometry J in $\mathcal{M}^-$; the induced structure may be computed in $\mathcal{M}^{-*}$. Furthermore the geometry in $\mathcal{M}^{-*}$ is algebraic over J; A is algebraic over D and if for example A^* is nontrivial then it is algebraic over its projectivization, which is in J. Thus J is stably embedded in $\mathcal{M}^{-*}$ and *a fortiori* in $\mathcal{M}^-$: for $e \in \mathcal{M}^{-*}$, $\operatorname{tp}(e/A)$ is definable by parameters $a \in A$, whose type over J is algebraic and hence definable. Thus it suffices to prove $(*)$.

Suppose first that A has no 0-definable proper subgroup of finite index. If $A^* = (0)$ in $\mathcal{M}^{-*}$ then Proposition 7.4.8 applies. Otherwise, A^* is the full definable linear dual to A, also in $\mathcal{M}^*$, by Lemma 7.4.4. A and A^* are settled over some parameter c in $\mathcal{M}^*$, hence in $\mathcal{M}^{-*}$ settled over some parameter algebraic in c by the corollary to Proposition 7.4.11. After enlarging c further we may suppose $\operatorname{acl}(c) \cap (A, A^*)$ also carries a nondegenerate pairing and lies in $\operatorname{dcl}(c)$. By Lemma 7.4.5 and Proposition 7.4.9, Proposition 7.1.7 applies.

Now suppose A does have a proper 0-definable subgroup of finite index; let B be the least such. Then by the preceding paragraph B is part of a stably embedded Lie geometry (B, B^*, Q), some components of which may be empty. A is generated by a complete type whose image in A/B must be a single point. Thus the dimension of A/B is 1. Then (A, B, B^*, Q) may be viewed as an affine geometry, by Lemma 2.3.17(1), with $C = \emptyset$.

Below we give another treatment of the degenerate case on somewhat different lines.

Proposition 7.5.4. *Let $\mathcal{M}^-$ be a reduct of a Lie coordinatized structure. Then $\mathcal{M}^-$ is weakly Lie coordinatized.*

Proof. $\mathcal{M}^-$ is $\aleph_0$-categorical, has finite rank, is modular, and satisfies:

If $a, b \in \mathcal{M}^-$, $a \notin \operatorname{acl}(b)$, then there is $a' \in \operatorname{acl}(a)$ of rank 1 over b.

This is contained in Lemma 5.6.6. Thus for any $a \in \mathcal{M}$ we can find a chain of "coordinates" $a_1, \ldots, a_n$ of finite length with a_i belonging to a rank 1 primitive $\operatorname{acl}(a_{i-1})$-definable set D_i and $a_n = a$. By Proposition 7.5.3 D_i is part of a stably embedded Lie geometry and after interposing the algebraic parameters needed to define the D_i we obtain a weak Lie coordinatization.

We now return to the degenerate case, indicating a treatment based on weaker hypotheses. We refer here to the preprint [**HrS1**], which introduced the S_1 *rank* on formulas as the least rank subject to:

(∗) $S_1(\phi) > n$ iff there are $(b_i)_{i \in \mathbb{N}}$ indiscernible over a set of definition for ϕ, and a formula $\phi'(x, y)$, such that

1 $S_1(\phi \& \phi'(x, b_i)) \geq n$ for each i;

2 For some k: $S_1(\phi'(x, b_1) \& \ldots \& \phi'(x, b_k)) < n$.

The independence theorem can be proved for theories of finite S_1 rank by an argument isomorphic to the one which will be given at the end of §8.4.

Lemma 7.5.5. *Let $\mathcal{M}$ be an $\aleph_0$-categorical structure of finite rank with amalgamation of types, not interpreting the generic bipartite graph, and let $\mathcal{M}^-$ be a reduct of $\mathcal{M}$. Let D be a primitive rank one definable subset in $\mathcal{M}^-$ whose geometry is orthogonal to every primitive rank 1 set whose geometry is nondegenerate; in particular D is degenerate over any finite set. Then D is stably embedded and trivial.*

Proof. Any rank 1 subset of $\mathcal{M}^-$ will inherit from $\mathcal{M}$ the property of finite S_1-rank, and hence satisfy the type amalgamation property by [**HrS1**].

To see that D is stably embedded and trivial we will show that for any finite B, D remains primitive over $D - \operatorname{acl}(B)$. For this we may use induction on $\mathrm{rk}\, B$, and thus by analyzing B we may suppose that $B = \{b\}$ has rank 1. Let D' be the locus of b over $\emptyset$, a rank 1 set. Let E be a b-definable equivalence relation on $D - \mathrm{acl}(b)$. As D is degenerate this will not have finite classes, so it will have finitely many infinite classes. Suppose $a_1, a_2 \in D - \mathrm{acl}(b)$ are distinct and equivalent, while a_1', a_2' are inequivalent. Then b, a_1, a_2 are pairwise independent, as are b, a_1', a_2', and hence independent. If a_1, a_2, and a_1' all have the same type over $\mathrm{acl}(b)$ then amalgamating types over $\mathrm{acl}(b)$ we can find $a_1^*, a_2^*, {a_1'}^*$ realizing this type with $\mathrm{tp}(ba_1^*a_2^*) = \mathrm{tp}(ba_1a_2) = \mathrm{tp}(ba_1^*{a_1'}^*)$, and $\mathrm{tp}(ba_2^*{a_1^*}') = \mathrm{tp}(ba_1'a_2')$. Then a_1^* is E-equivalent to a_2^* and ${a_1'}^*$ but they are not E-equivalent to each other, a contradiction.

Thus $D - \mathrm{acl}(b)$ splits into at least two types over $\mathrm{acl}(b)$. In particular D carries a nontrivial equivalence relation definable from the set $\mathrm{acl}(b)$ (or a part of it meeting finitely many sorts), viewed as a single element of $\mathcal{M}^{-\mathrm{eq}}$. This being the case, we may replace D' by a primitive quotient, and the argument of the previous paragraph yields a 0-definable relation $R(x, y)$ on $D' \times D$ so that $R(b, y)$ splits $D - \mathrm{acl}(b)$ for $b \in D'$. We view (D', D) as a bipartite graph with edge relation R. By our hypothesis D' also carries a degenerate geometry.

As $R(b, a)$ and $\neg R(b, a)$ both occur with $a \notin \mathrm{acl}(b)$, by amalgamation of types any two finite subsets of D can be separated by an element of D', and similarly for D' over D. Thus this is the generic bipartite graph, a contradiction.

We now return to Theorem 6 of §1.

Theorem 10 (Theorem 1.6). *The weakly Lie coordinatizable structures $\mathcal{M}$ are characterized by the following nine model theoretic properties.*

LC1 $\aleph_0$*-categoricity.*

LC2 Pseudofiniteness.

LC3 Finite rank.

LC4 Independent type amalgamation.

LC5 Modularity of $\mathcal{M}^{\mathrm{eq}}$.

LC6 The finite basis property for definability in groups.

LC7 Lemma 6.4.1: we call this "general position of large 0*-definable sets".*

LC8 $\mathcal{M}$ *does not interpret the generic bipartite graph.*

LC9 For every vector space V *interpreted in* $\mathcal{M}$*, the definable dual* V^* *(the set of all definable linear maps on* V*) is interpreted in* $\mathcal{M}$.

Proof. One has to check in the first place that these properties hold in weakly Lie coordinatizable structures. These statements have been proved in various earlier sections. Note however that the properties (LC6) and (LC7) were treated in the Lie coordinatizable context. As noted at the outset in §6.3, any group interpreted in a weakly Lie coordinatizable structure is also interpreted in a Lie coordinatizable structure, so these properties also apply in the weakly Lie coordinatizable context.

For the converse, note that we have listed here most of the properties used in the analysis of reducts of Lie coordinatized structures, with the notable exception of aspects of the theory of envelopes. We need to see that the proof of Proposition 7.5.4 can be carried out in this context.

This proposition depends on Proposition 7.5.3 and Lemma 5.6.6; the latter holds in our context, so we need only concern ourselves with Proposition 7.5.3. The use of Lemma 7.3.5 in the proof of that proposition does not fit into the present context, and it must be replaced by Lemma 7.5.5, using hypothesis (LC9) to see that the orthogonality condition in Lemma 7.5.5 will hold for any geometry D which is degenerate over every finite set. In a wider context, it is possible for a set to act as a generic set of linear maps on a vector space, giving a bipartite structure reminiscent of both the generic bipartite graph and the polar geometry; in this case one would have a degenerate geometry nonorthogonal to a linear geometry, and in fact embedded in the definable dual (which, however, would not itself be interpretable.) Condition (LC9) and nonorthogonality imply that over some parameter set, $\mathrm{acl}\,(D)$ contains an infinite definable group; we leave the details of this (involving the definition of orthogonality as well as the nature of the definable sets in a nondegenerate geometry) to the reader.

So it remains to verify that the rest of the proof of Proposition 7.5.3, which makes use of a large body of machinery, is available in the context of properties (LC1-LC9). The ingredients of Proposition 7.5.3, apart from (LC1, LC3, LC5), are: a particular finite covering of $\mathcal{M}^-$; Lemmas 2.3.17 and 6.6.2; Propositions 6.6.1 and 7.1.7; the contents of §7.4.

Properties (LC1-LC5) are inherited directly by the cover. Properties (LC6, LC7) can be deduced by showing that the groups interpreted in the cover are also interpretable in $\mathcal{M}^-$. This is because each sort (V_i) in the cover is interpretable in (part of) the underlying projective geometry: fix two linearly independent vectors v_1, v_2 and associate with any linearly independent v the pair $\langle v - v_1\rangle, \langle v - v_2\rangle$.

Lemma 2.3.17 simply holds, and Lemma 6.6.2 holds for the case needed by (LC7). Proposition 6.6.1 is assumption (LC6) and Proposition 7.1.7 was proved under our assumptions. So it suffices to reexamine §7.4. Lemma 7.4.1 may be replaced by Lemma 6.1.8 in the present context. The remaining lemmas, down to Lemma 7.4.7, are available in our context; note that Lemma 7.4.3 depends on lemmas in §§6.1-6.2 which were proved under sufficiently general hypotheses. Then the proofs of

Propositions 7.4.9 and 7.4.11 can be repeated. We do not need Proposition 7.4.11 since we assume (LC6).

Note. [] The following alternative route to the finiteness statement needed for the proof of Proposition 7.5.3 (Lemma 7.5.1 and the subsequent remark) has its own interest:

Lemma 7.5.6. *If $\mathcal{M}$ is saturated and $a \in \mathcal{M}$, then every algebraically closed subset of $\operatorname{acl}(a)$ is of the form $\operatorname{acl}(a) \cap \operatorname{acl}(a')$ for some conjugate a' of a in $\mathcal{M}$.*

Proof. Let $A \subseteq \operatorname{acl}(a)$ be algebraically closed. We need to check the consistency of the following theory, involving a new constant c and constants for the elements of A:

$$\operatorname{tp}(c/A) = \operatorname{tp}(a/A);\ b \notin \operatorname{acl}(c)\ (\text{for } b \in \operatorname{acl}(a) \setminus A)$$

For this it suffices to check for each finite a-definable subset B of $\operatorname{acl}(a)$ that there is an automorphism α of $\mathcal{M}$ fixing A such that

$$(*) \qquad (B \setminus A) \cap (B \setminus A)^{\alpha} = \emptyset$$

Let $G = \operatorname{Aut}(\mathcal{M})_A$, the pointwise stabilizer of A in $\operatorname{Aut}(\mathcal{M})$. For $b_1, b_2 \in B \setminus A$, let $G(b_1, b_2) = \{\alpha \in G : b_1^{\alpha} = b_2\}$. This is a coset of G_{b_1}, and if G is covered by $G(b_1, b_2)$ as b_1, b_2 vary over $B \setminus A$, then by Neumann's Lemma one of the subgroups G_b ($b \in B \setminus A$) has finite index in G; but this means $b \in \operatorname{acl}(A) = A$, a contradiction. Thus condition $(*)$ can be met.