8

Effectivity

8.1 THE HOMOGENEOUS CASE

If $\mathcal{L}$ is a finite relational language let $\mathcal{L}^\infty$, or more properly $\mathcal{L}^{\infty,\mathrm{eq}}$, be the language augmented by the quantifier $\exists^\infty$ "there exist infinitely many", and expanded so as to apply to imaginary elements.

We consider the following effectivity problems.

Problems.

(A) Given a finite relational language $\mathcal{L}$ and a sentence ϕ in the language $\mathcal{L}^\infty$, is there a stable homogeneous model (of type $\mathcal{L}$) of ϕ?

(B) Given a finite relational language and a finite set of forbidden isomorphism types $\mathcal{C}$, consisting of isomorphism types of finite $\mathcal{L}$-structures, is the corresponding class $\mathcal{A}(\neg\mathcal{C})$ an amalgamation class with stable generic structure? Here $\mathcal{A}(\neg\mathcal{C})$ denotes the class of finite structures omitting the structures of type $\mathcal{C}$.

A restricted version of Problem A was considered by Knight and Lachlan in [**KL**], and treated in the binary case. As there is an *a priori* bound on the rank in this case the question is one of the consistency of a theory in the extended language, hence a negative answer will have a finite verification.

The idea of [**KL**] is to reduce the positive case to Problem B. If $\mathcal{M}$ is a stable homogeneous model satisfying ϕ and $\mathcal{C}$ is the class of minimal isomorphism types of structures omitted by $\mathcal{M}$, then $\mathcal{C}$ is finite, as a consequence of the quasifinite axiomatizability. Thus $\mathcal{C}$ is a finite object witnessing the existence of $\mathcal{M}$, and the problem is to recognize $\mathcal{C}$.

If N bounds the sizes of the constraints in $\mathcal{C}$ then the quantifier $\exists^\infty$ is equivalent to $\exists^{N^*}$ where N^* is so large that every $\mathcal{L}$-structure of size N^* contains an indiscernible sequence of size N. This reduces the problem to the first order case. As $\mathcal{C}$ determines a "quantifier elimination" procedure – where the quotation marks reflect a bad conscience in cases where there is in fact no associated homogeneous structure $\mathcal{M}$ – the question of the truth of ϕ is decidable, modulo the fundamental question stated as Problem B.

The variant of Problem B in which we drop the stability requirement is more general than Problem B and remains open. The problem of amalgamation for relational structures reduces to the case of structures A_1, A_2 extending a common substructure $A_\circ$ by a pair of new points $a_1 \in A_1$ and $a_2 \in A_2$, but this problem remains open except in the binary case, where a direct check produces a finite procedure.

We will give a solution to Problem B. Let $\mathcal{M}$ be the hypothetical structure whose set of constraints $\mathcal{C}$ is specified. The rank of $\mathcal{M}$ is bounded by the number of 2-types and can therefore be computed using quantifier elimination. An inconsistent outcome at this point simply means that $\mathcal{M}$ does not exist. So assume the rank of the still hypothetical structure $\mathcal{M}$ is determined as k. For any definable equivalence relation E on $\mathcal{M}^2$ whose definition involves at most $2k$ parameters, we decide similarly whether or not the quotient is finite, and if it is finite we determine its size. Let μ bound the size of the finite quotients of this type. Then for any formula $\phi(x, y; Z)$ one can bound the rank and multiplicity of $\phi(x, y; B)$ as a function of $\operatorname{tp}(B)$. Do so for $|B| \le 2k$. Let ρ be the arity of $\mathcal{L}$.

Lemma 8.1.1. *Let $\mathcal{M}$ be $\aleph_0$-categorical and $\aleph_0$-stable, and coordinatized by degenerate geometries. Then:*

1 For $a \in \mathcal{M}$, $A \subseteq B \subseteq M$, if $\operatorname{rk}(a/B) < \operatorname{rk}(a/A)$ then for some $b \in B$, $\operatorname{rk}(a/Ab) < \operatorname{rk}(a/A)$.

2 For all $a \in \mathcal{M}$, $A \subseteq M$ there is $A_1 \subseteq A$ with $\operatorname{rk}(a/A_1) = \operatorname{rk}(a/A)$, $|A_1| \le \operatorname{rk}\mathcal{M}$.

Proof. Evidently it suffices to deal with the first point, and we may suppose $B - A$ is finite. We will proceed by induction on $\operatorname{rk}(B/A)$. Clearly $\operatorname{rk}(B/A) > 0$.

For $b \in B$, if $b \notin \operatorname{acl}(A)$ then choose $b' \in \operatorname{acl}(b)$ with $\operatorname{rk}(b'/A) = 1$, and otherwise $b' = b$. Set $B' = \{b' : b \in B - \operatorname{acl}(A)\}$. As the geometries are degenerate, if $\operatorname{rk}(a/B') < \operatorname{rk}(a/A)$, then there is $b \in B$ with $\operatorname{rk}(a/Ab') < \operatorname{rk}(a/A)$ and this yields the claim. If $\operatorname{rk}(a/B') = \operatorname{rk}(a/A)$ then $\operatorname{rk}(a/B) < \operatorname{rk}(a/B')$ and $\operatorname{rk}(B'/A) < \operatorname{rk}(B/A)$, so induction applies, yielding:

$$\operatorname{rk}(a/B'b) < \operatorname{rk}(a/B')$$

for some $b \in B$. Let $b'_1, \ldots, b'_n$ be a maximal subset of B' which is independent from b over A. We are assuming a is independent from $b'_1, \ldots, b'_n$ over A, but not from $b'_1, \ldots, b'_n, b$. By the degeneracy of the geometries $\operatorname{rk}(a/Ab) < \operatorname{rk} A$, as desired.

Lemma 8.1.2. *Let $\mathcal{M}$ be stable, finitely homogeneous, for a language of arity ρ. Let $a, b \in \mathcal{M}$, $A_1 \subseteq A \subseteq \mathcal{M}$, with $\operatorname{rk}(ab/A_1) = \operatorname{rk}(ab/A)$.*

There there is $A_2 \subseteq A$ containing A_1, with $|A_2 - A_1| \leq \rho \cdot \mathrm{Mult}\,(ab/A_1)$, so that $\mathrm{Mult}\,(ab/A_2) = \mathrm{Mult}\,(ab/A)$.

Proof. We proceed by induction on $\mathrm{Mult}\,(ab/A_1)$. We may suppose that $\mathrm{Mult}\,(ab/A) < \mathrm{Mult}\,(ab/A_1)$. Take two distinct types over A extending $\mathrm{tp}(ab/A_1)$ and a set C of size at most ρ over which they are distinct. Working over A_1C we conclude by induction.

Definition 8.1.3 *1 We consider amalgamation problems of the form $(A; b_1, b_2)$ signifying that a finite relational language $\mathcal{L}$ is specified, $A' = Ab_1$ and $A'' = Ab_2$ are specified finite $\mathcal{L}$-structures agreeing on A, and we seek an amalgam Ab_1b_2 which should omit some specified class of forbidden structures $\mathcal{C}$. We are looking for an amalgam in a stable homogeneous structure and it is assumed that the preliminary analysis of k, μ, and so on, has been carried out in advance as described above.*

2 The standard amalgamation procedure *for amalgamation problems $(A; b_1, b_2)$ under the specified conditions is the following.*

1.1 Find $E_1, E_2 \subseteq A$ with $|E_i| \leq k$ and $\mathrm{rk}\,(b_i/E_i)$ minimized. (For $|E_i|$ of this size, $\mathrm{rk}\,(b_i/E_i)$ has been given a definite meaning.) Set $A_1 = E_1 \cup E_2$.

1.2 For $X \subseteq A$ containing A_1, let $\mathcal{A}(X)$ be the set of amalgams of b_1A_1, b_2A_1, and X over A_1 which omit the specified forbidden structures and satisfy:

$(*)_X$ *For $Y \subseteq X$ with $|Y| \leq k$, $\mathrm{rk}\,(b_1b_2/Y) \geq \mathrm{rk}\,(b_1b_2/A)$*

These amalgams are not required to be compatible with b_iX.

1.3 Check whether $|\mathcal{A}(X)| \leq \mu$ for all $X \subseteq A$ with $A_1 \subseteq X$ and $|X - A_1| \leq \rho\binom{\mu}{2}$. If not the procedure fails (and halts) at this stage.

1.4 Check whether for all $X \subseteq Y \subseteq A$ with $A_1 \subseteq X$, $|X - A_1| \leq 2k + \rho \cdot \binom{\mu}{2}$, and $|Y - X| \leq 2\rho$, each element of $\mathcal{A}(X)$ extends to an element of $\mathcal{A}(Y)$. If not, fail and halt.

1.5 At this point if the procedure has not failed then $\mathcal{A}(A) \leq \mu$. Run through the possibilities in $\mathcal{A}(A)$; if one extends Ab_1 and Ab_2, the procedure succeeds.

Lemma 8.1.4. *Let $\mathcal{C}$ be a finite set of constraints (forbidden structures) for the finite relational language $\mathcal{L}$ of arity ρ, all of size at most N. Let k, μ be the invariants associated to a hypothetical stable homogeneous $\mathcal{L}$-structure $\mathcal{M}$ with constraints $\mathcal{C}$, that is the rank and a bound on the sizes of finite quotients of $\mathcal{M}^2$ by equivalence relations definable from $2k$ parameters, computed according to the canonical quantifier elimination procedure from $\mathcal{C}$.*

1 *If there is in fact a stable homogeneous $\mathcal{L}$-structure with finite substructures exactly those omitting $\mathcal{C}$, then the standard amalgamation procedure will succeed for any appropriate data $(A; b_1, b_2)$.*

2 *If the standard amalgamation procedure fails for (A, Ab_1, Ab_2) then there is $A' \subseteq A$ of order at most $2k + \rho \cdot \binom{\mu}{2} + \mu \cdot \max(\rho, N)$ for which it fails.*

Proof. The first point has essentially been dealt with in the previous lemmas, modulo the basic properties of independence. For the second, a failure at stage 1.3 or 1.4 produces a corresponding subset of size at most $2k+\rho\cdot\binom{\mu}{2}+2\rho$ over which the procedure fails. If the procedure continues successfully to the final step, then $|\mathcal{A}(X)| \leq \mu$ for any X containing A_1. Fix a subset A' of A containing A_1 such that any two possible amalgams differ on $A'b_1b_2$, and $|\mathcal{A}(A')|$ is as large as possible. We may take $|A'| \leq 2k+\rho\binom{\mu}{2}$. For Y containing A' with $|Y-A'| \leq \rho$ each element of $\mathcal{A}(A')$ extends uniquely to $\mathcal{A}(Y)$. With step 1.4 this gives a unique extension satisfying the definition of $\mathcal{A}(A)$ apart from the omission of $\mathcal{C}$. Those which omit the forbidden substructures are incompatible with Ab_1 or Ab_2. Thus μ sets of size N or ρ suffice to eliminate all potential solutions to the standard amalgamation procedure, over A'.

Proposition 8.1.5. *Problem B is decidable; hence Problem A is decidable.*

Proof. Compute the putative rank k and the invariant μ. Attempt the standard amalgamation procedure for all $(A; b_1, b_2)$ with $|A|$ satisfying the bound of the previous lemma. If this fails then the desired structure does not exist. If it succeeds, then there is at least a homogeneous structure $\mathcal{M}$ corresponding to the specified constraints. Furthermore the quantifier elimination procedure used is correct for $\mathcal{M}$, so in particular its rank has been correctly computed and it is stable.

8.2 EFFECTIVITY

We continue in the spirit of quasifinite axiomatizability and Ziegler's Conjecture, with attention to issues of effectivity. Recall the notion of a skeletal type and skeletal language L_{sk} from §4.2. From the results in §4.5 we may derive:

Lemma 8.2.1. *With the language L and skeletal language L_{sk} fixed, there is a finite set $\mathbf{X}_0(L, L_{\mathrm{sk}})$ of pseudo-characteristic sentences such that:*

1 If $\mathcal{M}$ is a Lie coordinatized L-structure with full skeleton $\mathcal{M}_{\mathrm{sk}}$, then some pseudo-characteristic sentence χ is true in $\mathcal{M}$.

2 With $\mathcal{M}$, χ as in (1), every proper model of χ is isomorphic to an envelope of $\mathcal{M}$.

3 $\mathbf{X}_0$ is recursive as a function of L and L_{sk}.

The prefix *pseudo* is called for as no claim is made that *all* of these formulas actually have models. This is the price to be paid, initially, for requiring effectivity.

Proof. This is proved in Proposition 4.4.3 with a potentially infinite set $\mathbf{X}_0$. The finiteness (without regard to effectivity) is in Proposition 4.5.1, by compactness. Paying attention to the effective (and explicit) axiomatizability of the class of structures with the given full skeleton, the effectivity follows from the same argument (via an unlimited search until a proof of a suitable disjunction is found).

Evidently this is not satisfactory, and we wish to prune off the bogus characteristic sentences, preferably carrying along some side information about dimensions as well, as in the following definition.

Definition 8.2.2. *Assume L and L_{sk} are given.*

1 A skeletal specification *Δ for L_{sk} consists of a skeletal type augmented by dimension specifications for each of the geometries of the forms: "$= n$"; "$\geq n$"; or "$= \infty$", where n stands for a specified finite number (≥ 0 is acceptable, of course). The specification is* complete *if "$\geq n$" does not occur.*

2 If Δ is a skeletal specification then $\mathbf{X}_1(L, L_{\mathrm{sk}}, \Delta)$ is the set of sentences from $\mathbf{X}_0(L, L_{\mathrm{sk}})$ that have a model $\mathcal{M}$ with full skeleton satisfying the specification Δ.

3 If Δ is a skeletal specification, then Δ^∞ denotes its most general completion*: each specification $\geq n$ is replaced by the specification $= \infty$.*

By definition, Lemma 8.2.1 holds in a sharper form for $\mathbf{X}_1(L, L_{\mathrm{sk}}, \Delta)$. We claim further:

Proposition 8.2.3. $\mathbf{X}_1$ *is effectively computable as a function of* L, L_{sk}, *and* Δ.

This requires substantial argument. We will use induction on the height of the Lie coordinatization. The remainder of this section is devoted to that argument. In particular L, L_{sk}, and Δ are given. However we first make some reductions.

First reduction

We replace Δ *by* Δ^∞ *(so that the characteristic sentences become complete, modulo the underlying theory).*

To justify this reduction, note that for any Δ, $\mathbf{X}_0 \setminus \mathbf{X}_1$ is in any case recursively enumerable since it consists of sentences which are inconsistent with the base theory. The problem is to enumerate $\mathbf{X}_1$ effectively. However each formula ϕ in $\mathbf{X}_1(L, L_{\mathrm{sk}}, \Delta)$ is derivable from another in $\mathbf{X}(L, L_{\mathrm{sk}}, \Delta')$ with Δ' complete (working always modulo a background theory). It suffices to handle all the Δ' (uniformly), and as $\Delta' = {\Delta'}^\infty$ the first reduction is accomplished.

Second reduction

We assume that $\mathcal{M}$ *is non-multidimensional and has no "naked" vector spaces.*

The point is that these are conservative extensions; if a characteristic sentence holds in some $\mathcal{M}$, then that structure can be expanded to a non-multidimensional one in which, furthermore, every vector space comes equipped with an isomorphism to its definable dual. Compare §5.3. If we can recognize the characteristic sentences in this context, then we can find one that implies the original one (and find the derivation as well). This reduction changes the skeletal type, in an effective way.

Note that if we happen to be interested only in the stable category, at this point the proof leaves that category in any case.

To take advantage of the nonmultidimensionality it is convenient to relax the notion of skeleton, allowing the bottom level to consist of finitely many orthogonal Lie geometries sitting side by side. At higher levels we may restrict ourselves to finite covers and affine covers, with the dual affine part present and covering a self-dual linear geometry lying at the bottom.

As the first level presents no problems, we have only to deal with the addition of subsequent levels, in other words with finite or affine covers. The problem is the following. If $\mathcal{M}$ is the given (hypothetical) structure, and $\mathcal{M}^-$ is the structure obtained from $\mathcal{M}$ by stripping off the top level, then assuming that we can effectively determine what the possibilities for $\mathcal{M}^-$ are, we must determine what the possibilities for

$\mathcal{M}$ are. Actually the emphasis at the outset is on pseudo-characteristic sentences, which while possibly contradictory have at least the virtue of actually existing, rather than the more nebulous $\mathcal{M}$ and $\mathcal{M}^-$ which may not in fact exist. Still the criterion that a pseudo-characteristic sentence χ be acceptable (relative to a given specification Δ) is that there should be an associated χ^- already known to be acceptable, and hence associated with a structure $\mathcal{M}^-$, such that χ^- "says" (or rather implies) that $\mathcal{M}^-$ has a covering of the appropriate type, with the property χ. So we may concern ourselves here with a reduction of the properties of a hypothetical $\mathcal{M}$ to those of a real $\mathcal{M}^-$.

The case of a finite cover

We have $\mathcal{M}^-$, or equivalently a characteristic sentence χ^- for it (which is complete when supplemented by the appropriate background theory including the relevant Δ^- extracted from Δ). We have also a characteristic sentence χ putatively describing a finite cover $\mathcal{M}$ of $\mathcal{M}^-$. Here the details of the construction of these sentences, in the proof of quasifinite axiomatizability, become important. The point is that χ gives a highly overdetermined recipe for the explicit determination of all structure on $\mathcal{M}$, proceeding inductively along an Ahlbrandt-Ziegler enumeration; if one begins with the structure $\mathcal{M}$, one of course writes down the facts in $\mathcal{M}$, but to capture all possible χ is a matter of writing down all conceivable recipes, most of which presumably have internal contradictions. The problem is to detect these contradictions effectively by confronting χ with $\mathcal{M}^-$.

Let K be a bound for the various numbers occurring in the proof of Proposition 4.4.3, say $K = 2k + \max(k^*, k^{**}) + 1$. Let d be the Löwenheim-Skolem number associated with K in $\mathcal{M}^-$, i.e.: any K elements of $\mathcal{M}^-$ lie in a d-dimensional envelope in $\mathcal{M}^-$ (effectively computable, by Lemma 5.2.7). Test χ by testing the satisfiability of χ in a finite cover of such a d-dimensional envelope (by a search through all possibilities). Here we should emphasize that χ is of the specific form given in the proof of Proposition 4.4.3, so that if true in some $\mathcal{M}$ it would pass to this particular envelope.

Conversely, if χ passes this test, we claim that the construction of $\mathcal{M}$ according to χ succeeds. Running over an Ahlbrandt-Ziegler enumeration of $\mathcal{M}^-$, at each stage we have covered certain elements of $\mathcal{M}^-$ by appropriate finite sets with additional structure, and have the task of covering one more element a of $\mathcal{M}^-$ by a finite set, and specifying its atomic type over everything so far.

Look for a formula $\theta(x, \mathbf{y})$ where x refers to the elements of the fiber being added, and $\mathbf{y}$ (of length at most k) refers to k previously constructed elements, with the following properties:

1 χ implies that such an x exists (more on this momentarily);

2 the multiplicity of x over everything so far is minimized, according to θ.

Let us consider (1) more carefully. We require previously constructed elements $\mathbf{z}$ and a valid atomic formula $\rho(\mathbf{y}, \mathbf{z})$ so that:

$$\chi \quad \Longrightarrow \quad \forall \mathbf{y}, \mathbf{z}[\rho(\mathbf{y}, \mathbf{z}) \Longrightarrow \exists x \theta(x, \mathbf{y})]$$

We then hope:

3 For all $\mathbf{y}'$, there are $\mathbf{z}'$ so that χ together with the atomic type of $\mathbf{y}, \mathbf{y}', \mathbf{z}'$ will imply the atomic type of $x, \mathbf{y}'$.

4 After adding x as specified, the universal part of χ holds.

If any of these hopes are disappointed then the failure is witnessed by at most K elements and hence is also visible in the envelope with dimensions d.

One of the simplifying features in this case is that "everything is algebraic". In the case of affine covers, the behavior of algebraic closure in the hypothetical cover is one of the sticking points. For this the affine dual is helpful.

The case of affine covers

We first shift the notation slightly. We may suppose that the dual-affine part of the cover is absorbed into $\mathcal{M}^-$, since it is a finite cover of a linear geometry in $\mathcal{M}^-$ – just apply the previous case.

The following remark may be useful as motivation. Since the dual affine part is present in $\mathcal{M}^-$, $\mathcal{M}$ is rigid over $\mathcal{M}^-$; that is, the extension is canonical, but not definable. Questions of multiplicity do not arise, and the question of existence of $\mathcal{M}$ is transformed into a different question: does the *canonical* $\mathcal{M}$ have the posited property χ? It will suffice to *express* this in $\mathcal{M}^-$.

We fix the following notation: $V = V^*$ is the linear geometry in $\mathcal{M}^-$; A^* is an affine cover (with components A_t^*, each a finite cover of V^*); A is the affine cover, in $\mathcal{M}$ but not in $\mathcal{M}^-$, with components A_t dual to A_t^*.

The elements $a \in A_t$ will be identified with hyperplanes in A_t^* which project bijectively onto V^*. From this point of view the problem is one of elimination of a second-order quantifier (for such hyperplanes) from the language of $\mathcal{M}^-$.

Lemma 8.2.4. *Let $\mathcal{M}_0$ be the reduct of $\mathcal{M}$ including all structure on $\mathcal{M}^-$ (which we take to include the affine duals A^*) as well as the geometrical structure on A: affine space structure of A_t over V, and duality with A_t^*. Then this is the full structure on $\mathcal{M}$ (all 0-definable relations remain 0-definable).*

Proof. It suffices to show that if two tuples a, b have the same types in the reduct then they have the same types. Take an envelope E containing them and view the affine elements in a, b as predicates (for hyperplanes). These predicates are conjugate under the automorphism group of E^- (the top layer is stripped off) by assumption, and any such automorphism extends to one of E. Thus a, b have the same type in the full language.

Lemma 8.2.5. *Let $\mathcal{M}^-$ be a countable (or hyperfinite) Lie coordinatizable structure with distinguished sorts T, V, V^*, A^* with the usual properties, e.g. A^* is a T-parametrized family of affine dual covers of V^* (or more generally V_t^*), possibly with additional parameters fixed. Then there is a cover by an affine sort $A = \bigcup_T A_t$ compatible with the affine duals A_t^*, in the geometric language of the previous lemma, and its theory is uniquely determined.*

Proof. For the existence, we may assume $\mathcal{M}^-$ is non-multidimensional (as we have been, in any case) and does not have quadratic geometries (it suffices to adjoin some parameters). The issue of orientability falls away and $\mathcal{M}$ can be thought of as nonstandard-finite. In this case existence follows from the finite case: adjoin all internal linear sections for the maps $A_t^* \longrightarrow V_t^*$ in a nonstandard universe, and this is locally Lie, hence Lie.

For uniqueness of the theory, fix a formula, and shrink a given affine expansion to a finite envelope large enough to test the truth of the formula; at the finite level the expansion is completely canonical, so the answer is determined.

Lemma 8.2.6. *In the context of the previous lemma, the theory of the affine expansion $\mathcal{M}$ can be computed from the theory of $\mathcal{M}^-$.*

Proof. Follow the line of the previous argument. One needs to determine the theory of a finite envelope $\mathcal{M}_d$. This is the canonical expansion of a finite envelope $\mathcal{M}_d^-$. Its theory can be determined by inspection.

8.3 DIMENSION QUANTIFIERS

In this section we consider enhancements of first order logic expressing numerical properties of geometries in large finite (or nonstandard-finite) structures. That some such expansion is necessary to carry through the analysis of Lie coordinatization in a definable and effective way is made clear by the following example given in [**HBa**].

Let V be a finite dimensional vector space over a finite field, and let m, n be distinct nonnegative integers. Let $V^3_{m,n}$ be a free cover of the cartesian cube V^3 by finite sets of sizes m or n; the triple (v_1, v_2, v_3) will be covered by a set of size m if $v_3 = v_1 + v_2$, and by a set of size n otherwise. Let $\mathcal{M}(m, n)$ be the reduct of $V^3_{m,n}$ in which the vector space structure of V is forgotten. We can view this as having sorts V and V^3 in addition to the covering M, with the covering map $\pi : M \longrightarrow V^3$ and the projections from V^3 to V. The collection $\mathcal{M}(m, n)$ should be thought of as a uniform family of examples, but the recovery of the vector space structure from the covering is nonuniform with respect to first order logic. In the usual approach to effectivity one sorts out all the structures under consideration into finitely many classes, each axiomatizable in first order logic. We propose to follow much the same route here, after augmenting the logic to allow us to decode numerical information of the type used here: note that it is not necessary to know the value of m and n, but only which is larger (or actually, with a little more care, that they are different). This will be done using a *dimension comparison quantifier* to be introduced shortly.

The specific quantifier introduced in [**HBa**] in its "most general form" is actually too general, as we will now indicate. The simplest way to add the desired numerical quantifier would be with a *less than* quantifier $<$. Applied to two formulas ϕ, ϕ' involving the variable x, and possibly other free variables, the formula $< x(\phi; \phi')$ would represent the formula: the cardinality of the set defined by ϕ is less than the cardinality of the set defined by ϕ'; as usual, variables other than x which are free in ϕ or ϕ' remain free in the quantified expression. The problem with this is that it encodes undecidable problems, namely any diophantine problem over $\mathbb{Z}$, into the basic properties of structures with a bounded number of 4-types (in fact, directly into a multi-sorted theory of pure equality). A polynomial equation $p(\mathbf{x}) = 0$ may be encoded as an equation $p_1(\mathbf{x}) = p_2(\mathbf{x})$ with nonnegative coefficients, and after interpreting multiplication as cartesian product and sum as disjoint union, the solvability of such an equation is equivalent to the existence of a model $\mathcal{M}$ of the theory of equality with a number of sorts equal to the number of variables $\mathbf{x}$, satisfying one additional sentence involving the cardinality quantifier (which expresses the stated equality). We require a less expressive logic, for which we can determine effectively whether a Lie coordinatizable structure with a specified number of 4-types exists, having any specified property expressible in the logic.

Strictly speaking, we will make use of three enhancements of first order logic: a finite set of *fully embedded geometry* quantifiers G_t, a *dimension comparison* quantifer $D<$, and the standard quantifier $\exists^\infty$ – there are infinitely many. The second has a natural model only in finite structures, where the third encounters a frosty reception, so we will have to pay some attention to *weak* (i.e., non-canonical) interpretations of the logic as well. We will need completeness and compactness theorems for various combinations of these notions, in a limited context (essentially the context of Lie coordinatizable structures). Our specification of intended interpretations below will be less useful from a technical point of view than the axioms specified subsequently, determining the notion of a "weak" interpretation.

Definition 8.3.1 *1 A* type *t (of geometry) is one of the following: (i) set; (ii) linear; (iii) orthogonal$^-$; (iv) orthogonal$^+$; (v) symplectic; (vi) unitary. For each type t, the quantifier G_t has the syntax of an ordinary quantifier: if ϕ is a formula, then $G_t x\phi$ is also a formula, with x bound by G_t. The intended interpretation in a model $\mathcal{M}$ is that the subset of $\mathcal{M}$ defined by $\phi(x)$ is a fully embedded geometry of type t. The distinction between the two types of orthogonal geometry has a clear meaning only in the finite case, but will be carried along formally in all cases (in other words, the Witt defect is included in the type). As usual, variables other than x which are free in ϕ remain free in $G_t x\phi$, and have the effect of auxiliary parameters.*

2 The lesser dimension *quantifier $D<$ acts on pairs of formulas ϕ, ϕ' to produce a new formula $Dx(\phi < \phi')$. The intended meaning in a structure $\mathcal{M}$ is that:*

i ϕ and ϕ' define fully embedded canonical projective geometries J, J' of the same type; and

ii $\dim J < \dim J'$.

Evidently (i) is already expressible using the G_t.

3 The quantifier $\exists^\infty$ is the usual quantifier "there exist infinitely many". It may also have nonstandard interpretations in finite models, essentially of the form "there exist a lot".

4 The logics $\mathcal{L}^G$, $\mathcal{L}^D$, $\mathcal{L}^{D\infty}$ are obtained syntactically by augmenting first order logic by, respectively: all the G_t; all the G_t, and $D<$; all the G_t, $D<$, and $\exists^\infty$. In each case the logic is taken to be closed under iterated applications of all the operations.

Context. Our basic context will consist of a fixed finite language together with a specified bound k on the number of 4-types; the latter is formalized by a theory which we denote $B4(k)$; more exactly $B4(L,k)$ where L is the logic in use. (The richer the language, the more powerful this theory becomes.) In finite models with at most k 4-types, the language $\mathcal{L}^D$ has a canonical interpretation. We write $C_4(L,k)$ for the class of finite L-structures with at most k 4-types.

Proposition 8.3.2 (Effective coordinatizability). *There is a computable function $b(L,k)$ such that with the language L and the bound k fixed, every $M \in C_4(L,k)$ has a Lie coordinatization via formulas in $\mathcal{L}^D$ of total length at most $b = b(L,k)$.*

Proof. Both the boundedness and the effectivity are at issue.

For the boundedness, we use a modified compactness argument. Suppose toward a contradiction that $\mathcal{M}_n \in C_4(L,k)$ has minimal coordinatization of total length at least n, for each n. Without loss of generality these all involve the same skeleton (but the actual definitions of the geometries vary erratically). Consider the first order structure $\mathcal{M}_n^*$ obtained by adjoining predicates to $\mathcal{M}_n$ for all formulas in $\mathcal{L}^D$, as well as predicates giving the appropriate coordinatization. (Note that as $\mathcal{M}_n$ is finite, this does not affect definability in the individual structures, but does change the collection of uniformly definable relations as n varies.) Pass to an ultraproduct $\mathcal{M}_\infty^*$. This is weakly Lie coordinatized. Let $\mathcal{M}_\infty$ be the reduct of $\mathcal{M}_\infty^*$ to $\mathcal{L}^D$ (or rather the first-order language used to encode $\mathcal{L}^D$ in the $\mathcal{M}_n$). By the *theorem on reducts* this is also Lie coordinatizable, definably. One would like to say that this "property" is inherited by the $\mathcal{M}_n$. By the proof of quasifinite axiomatizability, there is a sentence which characterizes the *envelopes* in $\mathcal{M}_\infty$, for models whose dimensions are true (constant over geometries parametrized by realizations of the same type). Use of $\mathcal{L}^D$-definable predicates ensures that the $\mathcal{M}_n$ have true dimensions in this sense, and hence are envelopes. In particular they are Lie coordinatizable uniformly, contradicting their choice.

Now we turn to the effectivity of $b(k)$. There is a set of formulas in the language $\mathcal{L}^D$ which is adequate for the Lie coordinatization of any structure in our class. We wish to argue that this is a first order property and is a consequence of an explicitly known theory, and then to conclude via the completeness theorem.

As a base theory one may take a first order theory in which all $\mathcal{L}^D$ formulas occur as atomic predicates, and their definitions – to the extent that they have definitions – are included as axioms. To a very large extent the $\mathcal{L}^D$ formulas do have first order definitions, since it is possible to say in a first order way what the dimension is when it is finite. Thus we may include in the axioms: if a given dimension is finite (i.e., specified explicitly) then it is *formally* less than another if and only if it is, in fact, less than that other. These axioms leave open what happens when the dimensions are infinite. (It is a good idea to require in general that "less than" be transitive, but this is not yet relevant.)

Now for $b \geq b(k)$, there is a finite disjunction of potential Lie coordinatizations, and a corresponding collection of characteristic sentences (in the sense of the previous section) for which in fact one of the coordinatizations works within every structure of our class, and one of the corresponding characteristic sentences is valid. This is a first order sentence. Furthermore, whenever the appropriate characteristic sentence is valid, the corresponding Lie coordinatization is in fact a valid Lie coordinatization. This is the delicate point: to verify that a potential Lie coordinatization is in fact valid, it is necessary to have complete control over definability; for example, one must know that if no vector space structure is specified on a set, then it has no definable vector space structure. The characteristic sentences give this kind of control.

Accordingly, one can search for a provable first order sentence of the desired form, and when it is found then one has found an effective bound on $b(k)$ (we are not concerned here with the minimum value of $b(k)$).

Now we will develop a completeness theorem for $\mathcal{L}^D$ and use it to produce more explicit results on effectivity.

Definition 8.3.3. *$TF4_k$ is the following axiom system, whose models are called* weak models *for $\mathcal{L}^D$.*

1. *Background axioms as in the preceding proof: predicates correspond to all formulas of $\mathcal{L}^D$ and the axioms force "formal less than" to mean "less than" when at least one of the numbers is finite.*
2. *There are at most k pairwise contradictory formulas in 4 variables.*
3. *For the quantifiers G_t, assert that when they hold then the corresponding geometry is embedded and stably embedded.*
4. *Some group of formulas of total length less than $b(k)$ (from the preceding lemma) forms a Lie coordinatization. Use the quantifiers G_t here.*
5. *Transitivity of the relation $\dim(J) < \dim(J')$. (Supplementing (1) above.)*
6. *If the definable set D is* not *a canonical Lie geometry, then some formula of length at most $b'(k)$ shows that it is not. Here $b'(k)$ is also effective; failure involves failure of primitivity, rank bigger than 1, or a richer Lie structure than the one specified is definable. In all cases there is a definable predicate that shows this. The bound $b'(k)$ can be found in the same way as $b(k)$.*

Proposition 8.3.4. *Let ϕ be a sentence in $\mathcal{L}^D$ which is consistent with the axioms given above. Then ϕ has a finite model with at most k 4-types.*

Proof. Begin with a weak model, which will be Lie coordinatized. Note that if it is finite then it already has all required properties as they are expressed by the theory in this case. Otherwise, shrink it (i.e., take an envelope), preserving the truth of ϕ by keeping infinite dimensions large. Note that the formal less than relation on the infinite dimensions determines a linear ordering of finite length and hence can be respected by the shrinking process. (Note that the position in this sequence of a given infinite dimension is part of the type of the associated parameter to begin with.)

Corollary 8.3.5. *$TF4_k$ is decidable, uniformly in k.*

Proposition 8.3.6. *Extend the logic by the quantifier $\exists^\infty$ to get $\mathcal{L}^{D,\infty}$. The theory remains decidable.*

Proof. One must extend the axiom system to get a suitable notion of weak model, then convert each weak model into one in which all sets whose size is formally not infinite become sets which are in fact finite. To avoid pathology (or paying more attention over the formalization) one may suppose all structures contain at least two elements.

The axioms are as follows. We use the term "finite" here for "definable and formally finite" rather than "of specified size".

1 $\exists^\infty$ implies the existence of arbitrarily many (the conclusion is a first order scheme).

2 If $\exists^\infty x \exists y \phi(x,y)$ then: $\exists y \exists^\infty x \phi(x,y)$ or $\exists^\infty y \exists x \phi(x,y)$. In other words, the image of an infinite set under a finite-to-one function is infinite.

3 A definable subset of a definable finite set is finite.

4 Given two embedded, stably embedded geometries, one of which is formally infinite, and the other having dimension at least as large, then the second geometry is also formally infinite. (This relates $\exists^\infty$ and the dimension quantifier.)

Note that (2) implies that a finite union of finite sets is finite.

The problem now is to take a formula ϕ which has a weak model and give it a model in which all sets asserted to be of finite size are in fact of finite size. We may assume that ϕ specifies a coordinatization, and using $(2,3)$ we may also assume that the only sets whose finitude or infinitude are asserted are subsets of canonical projective geometries (possibly degenerate), and in view of the nature of definability in such geometries, we reduce further to the finitude or infinitude of the geometry itself. So the problem is to shrink geometries which are asserted to be of finite size to ones which are finite, while leaving alone those asserted to be infinite, and preserving both the order relationships (for which (4) is clearly essential, and largely sufficient) and the other (essentially first order) properties asserted by ϕ. Note that axiom (1) is not required to "do" a great deal; but it guarantees that unmitigated sloth is an adequate treatment of the infinite case.

In order for all of this to make sense, one thing is necessary: the formally finite and the formally infinite canonical projective geometries should be orthogonal (otherwise there is no appropriate dimension function to begin with). This is guaranteed by (2,3).

8.4 RECAPITULATION AND FURTHER REMARKS

We return very briefly to the survey given in the Introduction. The theory of envelopes was summarized in Theorem 1 and in terms of finite structures in Theorem 6, the latter incorporating the numerical estimates of §5.2 and some effectivity. The families referred to in Theorem 6 are determined by a specific type of Lie coordinatization in the language $\mathcal{L}^D$ as well as a definite characteristic sentence. Evidently the truth of a sentence can be determined in polynomial time. Part (5) of Theorem 6 is dealt with in §5.2 as far as sizes go, and the construction is given by the characteristic sentence.

Theorem 2 gave six conditions equivalent to Lie coordinatizability. The first five conditions were dealt with by the end of §3.5; this is discussed at the beginning of that section. In particular, to get from Lie coordinatizability to smooth approximability one uses the theory of envelopes, notably §3.2. The converse direction was the subject of §3.5. For the validity of the last condition, use Lemma 5.2.7 and the estimate on the sizes of envelopes.

Theorem 3 is the theory of reducts, given in §7.5. Theorem 5 summarizes the effectivity results of §§8.1-8.3. Theorem 7 has been dealt with in §7.5.

We recall one problem mentioned in [**HBa**]: are envelopes "constructible" in time polynomial in the dimension function? As noted there, the underlying sets are in fact too large to be constructed in polynomial time, but the problem has a sensible interpretation: the underlying set can be treated as known, and one can ask whether the basic relations on it can be recognized in polynomial time (think for example of the basic case in which the envelope is simply a geometry of specified dimension). This problem has model theoretic content. The proof of quasifinite axiomatizability is based on a 1-way version of "back-and-forth" which may be called "carefully forth". We do not know how to give this proof in a "back-and-forth" format, and it seems that the polynomial time problem involves difficulties of the type which have been successfully eluded here.

8.4.1 The role of finite simple groups

In view of the special status of the classification of the finite simple groups it seems useful both to clarify the dependence of the present paper on that result, and to consider the possibilities for eliminating that dependence, and arguing in the opposite direction.

The work carried out here can be viewed as a chapter within model theory which is dependent in part on the classification of the finite simple groups for its motivation, but which in terms of its content is largely independent of that classification both logically and methodologically.

For example, Theorem 7 as we have stated it is independent from that classification. Similarly, the proof of Theorem 6 really involves Lie coordinatizable structures, and as such does not involve the classification of the finite simple groups, which is invoked at the end, via Theorem 2, to give the present statement of that result. As far as Theorem 2 is concerned, we combine the primitive case from [**KLM**], which may be taken here as a "black box", with independent model theoretic methods.

The proof of [**KLM**] is however strongly dependent on the classification of the finite simple groups. Theorem 7-Model Theory offers an array of model theoretic properties which can be taken as defining a certain portion of the theory provided by the classification of the finite simple groups. No such model theoretic version is known for the whole classification, and for that matter we are not aware of any other comparable portion of the classification that can be expressed in model theoretic terms. Initially one might try to assume Theorem 2-Characterizations (3) (i.e., 2 (6) with an arbitrary function), so that one has LC1 and LC2, and ask whether one can prove LC3-LC9 directly and non-inductively. The combinatorial flavor the properties (LC4-LC9) suggests that this may not be an unreasonable endeavor.

This issue was raised in [**HBa**] and remains both open and of considerable interest. It was noted there that the results on sizes of definable sets can be reversed to give a definition of rank and indpendence in purely combinatorial terms, that is in terms of asymptotic sizes of sets. In particular the properties (LC4) and (LC5) then become cleanly combinatorial. Property (LC4) becomes the statement that model-theoretically independent subsets of a single type over an algebraically closed base are statistically independent (giving unexpected support for the old term: "independence theorem"). We give a direct proof of this below. This proof is closely analogous to the proof of (LC4) from finite S1-rank given in [**HrS**]], but it emerged only on following up a suggestion of L. Babai regarding the similarity of the desired result with Szemeredi's regularity lemma, a similarity which will not be pursued here. The next challenge, accordingly, would be a direct proof of (LC5).

In the following, we work with the extension of first order logic by cardinality quantifiers, allowing us to assert that one definable set is smaller than another, and, via some definable encoding of disjoint unions, also allowing comparisons of the form $m|D| < n|D'|$. This could be recast more generally in a context where one has a definable probability measure on the definable sets. Indeed in general the relations between simplicity and the existence of such probability measures remains to be clarified.

Let $\mathcal{M}$ be a nonstandard member (e.g., an ultraproduct) of a family of finite structures, where cardinality quantifiers receive their canonical interpretations in finite structures, and the corresponding nonstandard interpretations in the ultraproduct. Call a definable set D *small* if $|D|/|M|$ is infinitesimal, where M is the underlying set of $\mathcal{M}$.

Lemma 8.4.1. *If D forks over $\emptyset$ then D is small.*

Proof. We may suppose that D divides over $\emptyset$, that is D has an an arbitrarily large indiscernible set $\{D_i\}$ of conjugates which is k-inconsistent for some fixed k. It follows by induction on k that D is small; more exactly (for the sake of the induction) that $|D_i|/|\bigcup D_i|$ goes to 0 as the size r of the set of conjugates increases. If $k = 1$ then these sets are empty, and for $k > 1$ we may consider for each i the $(k-1)$-inconsistent family $\{D_i \cap D_j\}$ for $j \neq i$. Then by induction $|D_i \cap D_j|/|D_i|$ goes to 0 as r increases, so the cardinality of a union of length n of conjugates D_i is of the order of $n|D_i|$, as long as $\binom{n}{2}|D_i \cap D_j|/|D_i|$ is negligible.

Lemma 8.4.2. *Suppose that $\mathcal{M}$ is a nonstandard member of a family of finite structures that realize boundedly many 4-types. Let p_1, p_2, p_3 be 1-types, and let p_{12}, p_{13}, p_{23} be 2-types projecting onto the corresponding 1-types appropriately. Then there is a formula $\phi(x, y)$ such that $\phi(a_1, a_2)$ holds if and only if $\{y : p_{13}(a_1, y) \& p_{23}(a_2, y)\}$ is small, and this formula is stable, and is even an equation in the sense of Srour* [**PS**].

Proof. The set $D = \{y : p_{13}(a_1, y) \& p_{23}(a_2, y)\}$ is definable from two parameters and can take on only a finite number of cardinalities in $\mathcal{M}$ (as this holds, with a bound, in the family of finite structures associated with $\mathcal{M}$). Hence ϕ can be defined. Now we must show that if (a_i, b_i) is an indiscernible sequence, and $\phi(a_i, b_j)$ holds for $i < j$, then $\phi(a_i, b_i)$ holds for all i. Let $D_i = \{y : p_{13}(a_i, y) \& p_{23}(b_i, y)\}$. Then by assumption $|D_i \cap D_j|/|M|$ is infinitesimal for $i \neq j$. As in the previous argument, if $|D_i|$ is not small relative to $|M|$ then $|D_i|$ is small relative to $\bigcup D_i$ and hence also relative to $|M|$, a contradiction.

Proposition 8.4.3. *With the hypotheses of the preceding lemma, suppose that there is no finite 0-definable equivalence relation splitting p_i ($i = 1$, 2, or 3), and that p_{ij} is not small relative to $\mathcal{M}^2$ for $i, j = 1,2; 1,3; 2,3$. Let P_{123} be the set of triples $(a_1, a_2, a_3) \in M^3$ such that $\mathcal{M} \models p_{ij}(a_i, a_j)$ for each pair $i, j = 1,2; 1,3; 2,3$. Then P_{123} is not small relative to M^3, and in particular is nonempty.*

Proof. We use similar notations P_i, P_{ij} for the loci of the given types.

Compute the number of triples (a_1, a_2, a_3) satisfying $p_{13}(a_1, a_3)$ and $p_{23}(a_2, a_3)$ by first choosing a_3 in $|P_3|$ ways, then choosing a_i for $i = 2, 3$

in $|P_{i3}|/|P_3|$ ways; this yields $|P_{13}||P_{23}|/|P_3|$, which is not small relative to $|M^3|$. It follows that for some a_1 satisfying p_1, the number of a_2 for which $\neg\phi(a_1, a_2)$ holds is not small relative to $|M|$, and hence the formula $\neg\phi(a_1, x)$ does not fork over $\emptyset$. Hence $\neg\phi(a_1, a_2)$ holds for some pair (a_1, a_2) which is ϕ-independent in the sense of local stability theory. Then by stability and our hypothesis on p_1, p_2, $\neg\phi(a_1, a_2)$ holds for all such independent pairs. Similarly, we can choose ϕ-independent (a_1, a_2) satisfying p_{12}. So all solutions to p_{12} satisfy $\neg\phi$, and the claim follows.

We have not touched on the other directions for further research which were already mentioned in [**HBa**]. As far as the diagonal theory envisaged there is concerned, the completion, or near-completion, of the foundations of geometric simplicity theory ought to be helpful in this connection.

References

[AZ1] G. Ahlbrandt and M. Ziegler, Quasifinitely axiomatizable totally categorical theories, *Annals of Pure and Applied Logic* **30** (1986), 63–82.

[AZ2] G. Ahlbrandt and M. Ziegler, What's so special about $(\mathbb{Z}/4\mathbb{Z})^\omega$?, *Archive Math. Logic* **31** (1991), 115–132.

[As] M. Aschbacher, *Finite group theory*, Cambridge studies in advanced mathematics **10** (1986), Cambridge University Press, Cambridge, England.

[BeLe] G. Bergman and H. Lenstra, Subgroups close to normal subgroups, *J. Algebra* **127** (1989), 80–97.

[Bu] S. Buechler, *Essential Stability Theory*, Perspectives in Mathematical Logic, Springer-Verlag, New York, 1996, xi+355 pp. ISBN 3-540-61011-1.

[CaP] P. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Society* **13** (1981), 1–22.

[CaO] P. Cameron, *Oligomorphic permutation groups*, London Math. Society Lecture Notes Series **152**. Cambridge University Press, Cambridge, 1990.

[CaK] P. Cameron and W. Kantor, 2-Transitive and antiflag transitive collineation groups of finite projective spaces, *J. Algebra* **60** (1979), 384–422.

[CaL] R. Carter, *Simple groups of Lie type*, Wiley Classics Library, London/New York (1989).

[Ch] G. Cherlin, Large finite structures with few types, in *Algebraic Model Theory*, eds. B. Hart, A. Lachlan, M. Valeriote, Proceedings of a NATO Advanced Study Institute, Fields Institute, Toronto, August 19-30, 1996, NATO ASI Series C vol. 496. Kluwer, Dordrecht, 1997.

[CHL] G. Cherlin, L. Harrington, and A. Lachlan, $\aleph_0$-categorical, $\aleph_0$-stable structures, *Annals of Pure and Applied Logic* **18** (1980), 227–270.

[CL] G. Cherlin and A. Lachlan, Stable finitely homogeneous structures, *Transactions American Math. Society* **296** (1986), 815–850.

[CPS1] E. Cline, B. Parshall, and L. Scott, Cohomology of finite groups of Lie type I, *Publ. Math. I.H.E.S.* no. 45, 169–191.

[CPS2] E. Cline, B. Parshall, and L. Scott, Cohomology of finite groups of Lie type II, *J. Algebra* **45** (1977), 182–198.

[CoAt] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups; Maximal Subgroups and Ordinary Characters for Simple Groups*, Clarendon Press, OUP, Oxford, 1985.

[EvSI] D. Evans, The small index property for infinite-dimensional classical groups, *J. Algebra* **136** (1991), 248–264.

[Ev] D. Evans, Computation of First Cohomology Groups of Finite Covers, *Journal of Algebra* **193**, (1997), 214–238

[EH] D. Evans and E. Hrushovski, On the automorphism groups of finite covers, *Annals of Pure and Applied Logic*, **62** (1993), 83–112.

[EIM] D. Evans, A. Ivanov, and H. D. Macpherson, Finite covers, pp. 1–72 in *Model Theory of Groups and Automorphism Groups*, Proceedings of a Summer School in Blaubeuren, 1995, ed. D. Evans, London Math. Society Lecture Notes Series 244, Cambridge University Press, 1997, xvi+212 pp.

[FJ] M. Fried and M. Jarden, *Field Arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band **11**, Springer-Verlag, New York, 1986.

[Ha] L. Harrington, Lachlan's homogeneous structures, lecture, ASL conference, Orsay (1985).

[Hi] D. G. Higman, Intersection matrices for finite permutation groups, *J. Algebra* **6** (1967), 22–42.

[HoPi] W. Hodges and A. Pillay, Cohomology of structures and some problems of Ahlbrandt and Ziegler, *J. London Mathematical Society* **50** (1994), 1–16.

[HrTC] E. Hrushovski, Totally categorical structures, *Transactions American Math. Society* **313** (1989), 131–159.

[HrUn] E. Hrushovski, Unidimensional theories: An introduction to geometric stability theory, in H.-D. Ebbinghaus et al. eds., *Logic Colloquium '87*, North Holland, Amsterdam, 73–103.

[HrS1] E. Hrushovski, Pseudofinite fields and related structures, §§4,6, preprint, 1992, 40 pp.

[HBa] E. Hrushovski, Finite structures with few types, in *Finite and Infinite Combinatorics in Sets and Logic*, eds. N. W. Sauer, R. E. Woodrow, and B. Sands, NATO ASI Series C vol. 411, Kluwer, Dordrecht, 1993.

[HP] E. Hrushovski and A. Pillay, Weakly normal groups, in *Logic Colloquium 85, Paris*, North Holland (1986).

[JP] W. Jones and B. Parshall, On the 1-cohomology of finite groups of Lie type, 313–318, in *Proceeding of the conference on finite groups, Utah*, F. Gross and W. Scott eds., New York, Academic Press 1976.

[KiTh] B. Kim, *Simple First Order Theories*, doctoral thesis, Notre Dame, 1996.

[Ki] B. Kim, Forking in simple unstable theories, *J. London Math. Society* **57** (1998), 257–267.

[KiP] B. Kim and A. Pillay, Simple theories, *Annals of Pure and Applied Logic* **88** (1997), 149–164.

[KL] J. Knight and A. Lachlan, Shrinking, stretching, and codes for homogeneous structures, in *Classification Theory*, ed. J. Baldwin, Lecture Notes in Mathematics **1292**, New York, Springer (1985).

[KLM] W. Kantor, M. Liebeck, and H. D. Macpherson, $\aleph_0$-categorical structures smoothly approximable by finite substructures, *Proceedings London Math. Society* **59** (1989), 439–463.

[LaPP] A. Lachlan, Two conjectures regarding the stability of ω-categorical theories, *Fundamenta Mathematicæ* **81** (1974), 133–145.

[La] A. Lachlan, On countable stable structures which are homogeneous for a finite relational language, *Israel J. Math.* **49** (1984), 69–153.

[Lu] E. Luks, Isomorphism of graphs of bounded valence can be tested in polynomial time, in **Proc. 21 IEEE Symposium on Foundations of Computer Science, Syracuse, 1980**, 42–49, IEEE, New York (1980).

[Mp1] H. D. Macpherson, Interpreting groups in ω-categorical structures, J. Symbolic Logic **56** (1991), 1317–1324.

[Mp2] H. D. Macpherson, Homogeneous and smoothly approximated structures, in **Algebraic Model Theory**, eds. B. Hart, A. Lachlan, M. Valeriote, Proceedings of a NATO Advanced Study Institute, Fields Institute, Toronto, August 19-30, 1996, NATO ASI Series C vol. 496. Kluwer, Dordrecht, 1997.

[PiGS] A. Pillay, *Geometrical Stability Theory*, Oxford Logic Guides **32**, Clarendon Press, Oxford, 1996, x+361 pp. ISBN 0-19-853437-X

[PiGr] A. Pillay, Definability and definable groups in simple theories, *J. Symbolic Logic* **63** (1998), 788–796.

[PS] A. Pillay and G. Srour, Closed sets and chain conditions in stable theories, *J. Symbolic Logic* **49** (1984), 1350–1362.

[Sch] G. Schlichting, Operationen mit periodischen Stabilisatoren, *Arch. Math. Basel* **34** (1980), 97–99.

[ShS] S. Shelah, Simple unstable theories, Annals of Mathematical Logic **19** (1980), 177–203.

[Sh] S. Shelah, *Classification Theory*, first edition 1978; revised edition 1990, xxxiv+705 pp.; Elsevier, Amsterdam, The Netherlands.

[Wa] F. Wagner, Almost invariant families, undated preprint, 4pp.

[Wi] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern, *Crelle's Journal* **176** (1937), 31–44. §1.

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