Simple Groups of Finite Morley Rank

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Algebraicity Conjecture

An infinite simple group
of finite Morley rank
is
an algebraic group.

I. Ancient History

II. Borovik's program: The 4 types

III. Mixed and Even Type

IV. Odd type

Part I. **Ancient History**

Lindstrøm

Categoricity $\Rightarrow$ model-completeness for $\forall\exists$.

Morley, Baldwin, Zilber

Categoricity and (finite) Morley rank.

Macintyre

An infinite $\aleph_0$-stable field is algebraically closed.

Kegel & Wehrfritz

loc. finite $\mathcal{M}_c$-groups satisfying min-p (all p)

(centralizer)-connected

Baldwin-Saxl

Intersections of uniformly definable groups are uniformly definable.

Uncountable Categoricity: **Fine Structure**

Zilber If $\mathcal{M}$ is uncountably categorical and not almost strongly minimal then $\mathcal{M}$ interprets an infinite group G which is either:

(a) abelian; or

(b) simple.

Algebraicity Conjecture: A simple group of finite Morley rank is algebraic.

Two theorems of Zilber

I. If G is a simple group then the following are equivalent:
A. G is uncountably categorical
B. G has finite Morley rank.

II. If G is a solvable and centerless connected group then G has two sections K, T such that:

1. K carries the structure of the additive group of a field F;

2. T carries the structure of a multiplicative subgroup of F

3. T acts on K by conjugation, via multiplication

A broader view

Zilber Conjecture All structures of dimension 1 occur in nature.

With a little work (Weil, van den Dries, Hrushovski) this conjecture implies the algebraicity conjecture. However . . .

Hrushovski It's false. There are:

1. A 1-dimensional set on which two incompatible field structures coexist.

2. A 1-dimensional set of nonlinear type which does not involve any infinite group (much less field!).

(*On the other hand . . .*)

Part II. **Borovik's program: The Four Types**

Determine the possible 2-Sylow structures in a minimal counterexample.

Sporadics?

K^*: $\aleph_0$-stable, and every proper definable connected simple section is algebraic.

- 2-Sylow$^\circ$ subgroups
- The four *types*
- Bad fields

p-Sylow$^\circ$ structure in algebraic groups

Characteristic p:

unipotent – [bounded exponent, definable]

Model: Strictly upper triangular matrices.

Other characteristics:

semisimple – [divisible abelian]

Model: Diagonal matrices with entries suitable roots of unity.

2-Sylow$^\circ$ structure in groups of FMR

$S = U * T$:

2-Unipotent $\times$ 2-torus

with finite intersection

Types

T \ U	$\neq 1$	$= 1$
$\neq 1$	Mixed	Odd
$= 1$	Even	Degenerate

Bad fields

$(K;T)$ $T \leq K^{\times}$ proper, infinite.

Poizat

They "appear" to exist in characteristic 0.

Wagner

They appear not to exist in characteristic p, because:

a) The algebraic elements must form an elementary substructure.

b) There must be only finitely many p-Mersenne primes.

Consequences:

1. A simple group definable in a "pure" bad field of positive characteristic is algebraic.

2. The multiplicative group of a field of finite Morley rank and positive characteristic is a *good torus* in the sense that each subgroup is the definable closure of its torsion.

Part III. **Mixed Type and Even Type**

Theorem [ABC, Jaligot, Altinel]

G simple, FMR, with every definable section of *even type* algebraic

Then G is not of mixed type.

Quasi-Theorem
[. . . , punch-lines by Berkman and Tent]:

G simple, FMR, even type, with no degenerate simple sections

Then G is algebraic.

Reference:

`http://www.math.rutgers.edu/cherlin/Eventype`

Altinel's Jugendtraum

Can we treat even type *absolutely*?

Theorem [Weak Solvability]

G simple, FMR, even type, with a weakly embedded subgroup M.

Then $M/O_2(M)$ is of degenerate type.

Remark [Borovik]

X solvable connected $2^{\perp}$, faithful on U a connected abelian 2-group

Then X is a torus.

Proof:

$F(UX) = U$.

Remark [Altinel]

This X is a *good* torus.

Part IV. **Odd Type**

Borovik Trichotomy: Tame Case

(a) $\mathrm{pr}_2 \leq 2$; or

(b) proper 2-generated core; or

(c) Classical involution, and B-conjecture

Berkman: Case (c) is algebraic.

Borovik-Nesin (?): Case (b) is algebraic.

Issues:

Remove, or reduce, the dependence on tameness.

Handle the “small” cases of (a).

Elimination of Tameness

B-Conjecture: $O(C(i)) = 1$.

Tame

$\Rightarrow$

$O(C(i))$ is a *nilpotent signalizer functor*

$\Rightarrow$

B-conjecture

$$O(C(i)) \cap C(j) = O(C(j)) \cap C(i)$$

Idea: $U(C(i)) =$ *unipotent* part.

E.g. $U_p(C(i))$.

"U_0"

U_0

Root subgroups

A is abelian and indecomposable

$\bar{r}(A) = \mathrm{rk}\,(A/\mathrm{rad}\,A)$, maximal

$A/\mathrm{rad}\,A$ is torsion free

$$U_0(H) = \langle \text{root subgroups of } H \rangle$$

Properties of U_0

1. If H is solvable, then $U_0(H)$ is nilpotent.

2. $U_0(H)$ is a signalizer functor.

3. If all $U_p(H) = 1$, then H is a good torus.

Tame minimal simple groups of odd type

Theorem [CJ]

G tame minimal simple group FMR, of odd type, S Sylow 2-subgroup of G, $A = \omega_1(S^\circ)$, $T = C_G{}^\circ(S^\circ)$, $C = C_G{}^\circ(A)$, $W = N(T)/T$, (*Weyl group*). Then $\mathrm{pr}_2(G) \leq 2$ and one of:

1. $\mathrm{pr}_2(G) = 1$:

1a. C not a Borel: then G is $PSL_2(K)$

1b. C a Borel: If $W \neq 1$, then $C = T$ is 2-divisible abelian, $|W| = 2$, W acts by inversion on T, and $N_G(T)$ splits as $T \rtimes \mathbb{Z}_2$. All involutions in G are conjugate.

2. $\mathrm{pr}_2(G) = 2$:

$T = C = C_G(A)$ is nilpotent, $|W| = 3$, all involutions of G are conjugate, and G interprets an algebraically closed field of characteristic 3. Furthermore . . .