Ultraproducts and Complex Reflection Groups

Gregory Cherlin

Newton Institute

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- Coxeter groups and Dynkin Diagrams
- Complex Reflection Groups
- Groups of Finite Morley rank
- Recognition of Coxeter Groups
  (via Ultraproducts)

I

Coxeter groups and Dynkin diagrams

Kaleidoscopes

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  Crystallographic condition: preserves a lattice

An example

The symmetric group Sym_n on $\mathbb{R}^{n-1}$ viewed as $e^{\perp}$ with $e \in \mathbb{R}^n$, $e = (1, \ldots, 1)$.

Reflections: transpositions (ij).
Associated vectors $\pm(e_j - e_i)$.
Root system $e_j - e_i$.

Fundamental roots $e_{i+1} - e_i$ (elementary transpositions) correspond to a set of generating reflections.

Angles: $2\pi/3$ or $\pi/2$, mostly the latter.
(Orders of products: 3 or 2.)

Coxeter-Dynkin diagram: $\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\cdots-\!\!-\!\bullet$

Classification

Via root systems

Positive and negative roots
Fundamental roots: $r = \sum \lambda_i r_i$ with constant sign (and integer entries, in the crystallographic case).
Obtuse or right angles, mostly the latter.
Dynkin diagram:

- Vertices = Fundamental roots
- Edges = obtuse angles
- Labelled if the angles are not $2\pi/3$
- Oriented if the lengths differ (by Dynkin)

Diagram chasing

What are these things?

Singularities, representations of quivers,
simple complex Lie algebras / simple algebraic groups
Reference: M. Hazewinkel, W. Hesselink, D. Siersma, and F. D. Veldkamp, *The ubiquity of Coxeter-Dynkin diagrams (an introduction to the* $A - D - E$ *problem)* Nieuw Arch. Wisk. 25:257–307, 1977

Combinatorial data given in the language of group theory.
Maximal notions of symmetry
(Galois, crystallography/chemistry).

Reference: Hermann Weyl, **Symmetry**

Princeton University Press, Princeton, N. J., 1952, 168 pp.

Weyl groups

Simple (or reductive) algebraic groups.
T maximal torus, B a Borel subgroup. $W = N(T)/T$
($T = C(T)$): automorphisms of T.

Theorem
W is a crystallographic Coxeter group, and $G = BWB$.

Example: $G = \mathrm{GL}_n$, $W = S_n = \mathbf{A_{n-1}}$, namely permutation matrices, and B consists of upper or lower triangular matrices.
$G = LWU = UWU$ (since L is conjugate to U under W).
$W =$ failure of limited Gaussian elimination.

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Root groups: Minimal T-invariant groups with nontrivial action.
Structure: F_+, with T acting via $\chi : T \to F^\times$
Characters: $\chi(t) = t^u$ ($t = (t_1, \ldots, t_n)$, $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$)
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Calculation: $E_{ij}(a)^{\delta(t_1,\ldots,t_n)} = E_{ij}([t_i^{-1} t_j] a)$;
$\chi \sim (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0) \sim e_j - e_i$ in $\mathbb{Z}^n$.

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This gives also the crystallographic condition.

Curtis-Tits Theorem

Structure of simple algebraic groups (Chevalley)

- Generated by root subgroups
- *Non-opposite:* relations live in the root system $\langle r, s \rangle$
- *Opposite:* generate a *root* SL_2 (or PSL_2) $L_{\pm r}$

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Theorem (GLS-III.81') $H \sim \Sigma$ of rank at least three. X_r root groups. $H_J = \langle X_{\pm r} : r \in J\rangle$. Then H is isogenous to the free amalgam of all H_J with $|J| \leq 2$.

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The Dynkin diagram gives the isomorphism types of the Lie rank two subgroups H_J: ($A_1 \times A_1$, A_2, B_2, or G_2).

II

Complex Reflection Groups

Complex Reflections

Real reflections: semisimple, with eigenvalues $(1, \ldots, 1, -1)$.
Complex reflections: semisimple, with eigenvalues $(1, \ldots, 1, \zeta)$.

Fully classified (Shephard and Todd, Arjeh Cohen).

What are they?

Theorem
Let G be a finite group of linear transformations on a space of dimension n. Then G is a complex (or real!) reflection group if and only if the algebra of G-invariant polynomials is generated by n algebraically independent homogeneous polynomials. In this case, the product of their degrees is the order of G.

E.g., Sym_n on $\mathbb{R}^n$, symmetric polynomials, $n! = \ldots$.

They appear also in singularity theory, but with one exception are not, as yet, "ubiquitous".

"*The quaternionic versions of the Coxeter and Shephard-Todd groups* are still to be defined and classified." (Arnold)

Recognizing Real Reflection Groups

Theorem W finite, $I \subseteq W$ subset, n fixed.

1. The set I generates W, consists of involutions, and is closed under conjugation in W;
2. The graph Δ_I with vertices I and edges (i, j) for noncommuting pairs $i, j \in I$ is connected;
3. For all sufficiently large prime numbers ℓ, W has a faithful representation V_ℓ over the finite field $\mathbb{F}_\ell$ in which the elements of I operate as "complex" reflections.

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1. $I \subseteq I(W)$, $I \lhd W$, $\langle I \rangle = W$.
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3. *Reflection representation* V_ℓ over all $\mathbb{F}_\ell$, without fixed vectors. (*Ubiquitous*)

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Then one of the following occurs.

(a) W is dihedral in dimension $n = 2$, or cyclic of order two;

(b) W is an irreducible crystallographic Coxeter group: A_n, B_n, C_n, D_n ($n \geq 3$), F_n ($n = 4$), or E_n ($n = 6, 7$,or 8);

(c) Or: W is the **binary octahedral group (#12)**, in dimension two.

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If W also has some irreducible reflection representation of dimension at least $\mathbf{3}$ (over any field), then W is an irreducible crystallographic Coxeter group.

(Proof later)

III

Groups of finite Morley rank

Rank=Dimension

Algebraic Groups: dimension.

δ : subsets $\rightarrow \mathbb{N}$ (dimension of Zariski closure).

Well-behaved on definable sets.

Morley rank: abstract notion of dimension (in general ordinal valued)

Algebraicity Conjecture: A simple group of finite Morley rank is algebraic.

Conjectured by Zilber, who now conjectures the opposite, and has good chances to be right at least once, if not more.

Recap

Objects

Groups with a dimension

Main question

Are the infinite simple ones algebraic?

Important actors

Involutions

Source of inspiration

Finite group theory

Classification of **F**inite **S**imple **G**groups

Some results

Theorem [ABCJ et al.] A simple group of finite Morley rank of infinite 2-rank is algebraic.

Theorem [B^3 CJ et al.] A *minimal nonalgebraic* simple group of finite Morley rank has Prüfer 2-rank at most two.

Generic Recognition

Theorem [Berkman-Borovik]
Let G be a simple K^*-group of finite Morley rank, and p a prime. Let T_0 be a maximal p-torus in G, of Prüfer rank at least 3. Assume:

(A) G is generated by the subgroups

$$C^\circ_G(x) \text{ for } x \in T_0 \text{ of order } p$$

(B) For every element x of order p in T_0 we have:

(B$_S$) the group $C^\circ_G(x)$ contains no nontrivial p-unipotent subgroup;

(B$_R$) $C_G{}^\circ(x) = F^*(C_G{}^\circ(x))$.

Then G is a Chevalley group over an algebraically closed field of characteristic other than p.

Proof

We use root SL_2-subgroups.
Σ: $L \leq G$ definable, T_0-invariant, of type SL_2.
T_L: $C_L(T_0)$. $T = \langle T_L : L \in \Sigma \rangle$
r_L; $W_0 = \langle r_L : L \in \Sigma \rangle$.

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- Σ generates G.
- Pairs from Σ generate Lie rank 2 groups.

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- Pairs from Σ generate Lie rank 2 groups.
- W is a crystallographic Coxeter group
  - I is closed under conjugation in W_0
  - Δ_I is connected
  - W_0 on $(T_0)_\ell$ is a faithful and irreducible on all nontrivial $T_0[\ell]$ ($\ell > 2$)

IV

Recognition of Coxeter groups

Recognizing Real Reflection Groups

Theorem W finite, $I \subseteq W$ subset, n fixed.

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3. *Reflection representations* V_ℓ over $\mathbb{F}_\ell$.

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(c) W is the binary octahedral group in dimension two.

Complex reflection groups

Suppose the group is a *complex reflection group* which also has faithful representations over most F_ℓ.

The center acts by scalars:

$z \in Z(W)$, $i \in I$:
z acts on $[i, V]$, this gives an eigenvalue, and z is a scalar.
$|Z(W)|$ divides most $\ell - 1$, hence divides 2.

After dropping the crystallographic Coxeter groups (and ignoring a family of imprimitive groups to be treated separately) we are down to

$$\#4, \#12, \#23, \#24, \#30, \#33$$

The remainder

#	Dim.	Name	$\|W\|$	$\|Z(W)\|$	$\|r\|$
4	**2**		$2^3 * 3$	2	[3]
12			$2^4 * 3$	2	[2]
23	**3**	H_3	$2^3 * 3 * 5$	2	[2]
24			$2^4 * 3 * 7$	2	[2]
30	**4**	H_4	$2^6 * 3^2 * 5^2$	2	[2]
33	**5**		$2^7 * 3^4 * 5$	2	[2]

#23,#24, #30, #33: order must divide $|\mathrm{GL}_n(\ell)|$.
E.g. *#33:* $\ell \equiv 2 \mod 3^4$, [Dirichlet]
$|\mathrm{GL}_5| \equiv 2^{10}(2^5-1)(2^4-1)(2^3-1)(2^2-1)(2-1)$
$2^4 - 1$ and $2^2 - 1$ give a factor of 3^2 only

Why complex?

Ultraproducts

Given:

- Structures M_x indexed by a set X,
- A total finitely additive 2-valued probability measure μ on X

we define an "average" structure M_μ as $\lim\limits_{\mu} M_i$

Idea: $M_\mu = \{\lim_\mu(a_x) : a_x \in M_x\}$;

$R(\lim_\mu a_x, \lim_\mu b_x)$ iff $P_\mu(R(a_x, b_x)) = 1$.

…where $\lim_\mu(a_x)$ is the sequence itself, modulo probable equality … (which was already specified!)

(Example: asymptotic cones, à la van den Dries-Wilkie)

Łoś' theorem

Theorem For any *first order* property ϕ, ϕ holds in the μ-limit (ultraproduct) iff its probability is one.

Corollary

If something is true of a sequence of models, it is true of their ultraproduct.

Application

Theorem
If G has irreducible reflection representations with respect to the finite connected set I, over fields of arbitrary large characteristic, then it has one also in characteristic zero.

Take an ultraproduct of the representations. *First, specify:*

1. A Language
2. A Theory

Language: Field F, Vector space V, operators $M_g : V \to V$.

Theory:
(Field, F-vector space, linear operators)
multiplication in G, I acting by reflections
irreducibility

Classification of groups of even type

1. Treat rank one (uniqueness theorems)
2. Treat rank two (amalgam method)
3. Prove reductivity and semisimplicity for elements of odd prime order (conditions B_R, B_S)
4. Apply Generic Recognition
   (a) Find root SL_2-subgroups
   (b) Build a torus and Weyl group optimistically
   (c) Show the Weyl group is a crystallographic Coxeter group, using the action on a torus
      i. It is is a complex reflection group by ultraproducts
      ii. It is a crystallographic Coxeter group by Shephard-Todd
   (d) Apply Curtis-Tits