Conjugacy of Good Tori

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A conjugacy theorem

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Divisible, abelian (“torus”)

with dense torsion (“decent”)

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in each definable subgroup ("good")

In affine algebraic groups, maximal tori are conjugate

Theorem *In groups with dimension, maximal* **good** *tori are conjugate.*

Existence

Theorem (Wagner)
A torus over a field with dimension, in positive characteristic, is good.

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Corollary (Borovik)
A connected solvable $\pi^\perp$*-group acting faithfully on a nilpotent* π*-group, where the whole thing is equipped with a notion of dimension, is a good torus.*

Groups with dimension

(I.e.: finite Morley rank)

Conjecture
A simple group with dimension is algebraic.

Finite groups: cardinality
Algebraic groups: dimension
(Link: Lang-Weil, $|X| \approx q^{\dim(X)}$.)

Free groups: a weak analog (Sela; Feighn, Bestvina)

Finiteness Theorem

Theorem (Finiteness) *Let G be a group with dimension, and $\mathcal{F}$ a uniform family of good tori. Then the groups in $\mathcal{F}$ belong to a finite number of conjugacy classes.*

Application

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But, suppose that:
$H = \langle A_1, A_2 \rangle \simeq \mathrm{SL}_2$ (conjugates of A), with $H < G$
(cf. [Jaligot, Thése]).

Goal: a contradiction.

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Definition *T a torus of H, M a point stabilizer.*
$\mathcal{T} = \{T^g : T^g \leq M\}$
Jaligot: if
$(*)$ *all proper simple definable sections of G are algebraic,*
then the tori in $\mathcal{T}$ are conjugate under the action of M.

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The minimality hypothesis $(*)$ is eliminated in two steps . . .
Cf.: [Altinel/Cherlin, *Limoncello*, J. Alg to appear]

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- $\mathcal{T}$ forms a single conjugacy class under the action of M.

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  . . . in between the two steps, we rework Jaligot using weaker information . . .

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Some details

Groups with dimension

(Groups of finite Morley rank)

1. $\text{rk}\,(X)$ (dimension); $\deg(X)$ (multiplicity)
2. $K < H$:
   $$[H : K] = \infty \implies \text{rk}\,(K) < \text{rk}\,(H)$$
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7. Saturation: a set of bounded cardinality is finite

Good tori: Generalities

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3. $T \leq H$ nilpotent $\implies T \leq Z(H)$.

Rigidity

R-I. $N^\circ(T) = C^\circ(T)$.
(Immediate)

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Note: By rigidity II the Finiteness Theorem follows from the Conjugacy Theorem.

Namely conjugate $\mathcal{F}$ into one maximal good torus.

III

The conjugacy theorem

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maximal good tori are conjugate.

Generic covering

Lemma (Generic covering)
G connected, $T \leq G$, $H = C^\circ(T)$. Then $\bigcup H^G$ is generic.

Conjugacy Theorem, proof:
Induction. T_1, T_2 max. G connected. Goal: $T_1 \sim T_2$.

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$H_i = C^\circ(T_i)$. We may suppose that $H_1 \cap H_2 \neq 1$:
$h \in H_1^\times \cap H_2^\times$

$T_1, T_2 \leq C(h)$:

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$T_1, T_2 \leq C(h)$: *(done)*

"*Generic covering $\implies$ conjugacy*"

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*The difficulty is in point #3. We require *non-genericity* under certain conditions. Point #2 will be useful . . .

Nongenericity

Glossary: $H = C^\circ(T)$, $\mathcal{F} = \{H \cap H^g : \dots\}$.
Recall that no group $X \in \mathcal{F}$ contains T.

Lemma *Let H be connected, $T \leq Z(H)$ a good torus, and $\mathcal{F}$ a uniform family of subgroups of H, where no member of $\mathcal{F}$ contains T.*
Then $\bigcup \mathcal{F}$ is not generic in H.

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Imagine for a moment that $H = T \times H_0$—how do

you distinguish small and large subsets of a product?

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Application: $K < H$ groups.

$X \subseteq H$ is generic

iff

$\{hK : X \cap hK$ generic in $hK\}$ is generic in H/K

—which implies that the set in question is nonempty.

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Proof We may suppose that $X \cap T = 1$ for $X \in \mathcal{F}$.
We will show that $V \cap \bigcup \mathcal{F}$ is finite for each coset $V = hT$, so by Fubini $\bigcup \mathcal{F}$ is not generic.

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$g, g' \in V \cap \bigcup \mathcal{F}$. $g \in X \in \mathcal{F}$, $X \cap T = 1$. So $d(g) \times T \leq H$.

$$d(g') \leq d(g) \times T, \quad d(g') \cap T = 1$$

So $\pi_1 : d(g') \to d(g)$ (injectively).

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By symmetry, we conclude $\pi_1 : d(g') \simeq d(g)$.

$$d(g') \sim h_{g'} : d(g) \to T$$

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…and the decent tori

Definition (Connected Frattini)
$\Phi(G) = \bigcap\{H : H \leq G$ *connected, maximal*$\}$.

Lemma *Let T be divisible abelian. Then the following two conditions are equivalent.*
1. $T = d(T_{\text{tor}})$
2. $T/\Phi(T)$ *is a good torus.*

($1 \implies 2$): T/M is a good torus, so $T/\Phi(T)$ is as well.

($2 \implies 1$): $T_t = d(T_{\text{tor}})$.
If $T_t < T$, then $T_t{}^\circ \leq M$ maximal connected. But $T/\Phi(T) \to T/M$, so T/M is good, contradiction.

The last word

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Theorem *Maximal decent tori are conjugate.*

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Theorem *Maximal decent tori are conjugate.*

In the analysis of $H = N^\circ(T)$ (nongenericity), we consider $H/\Phi(T)$ . . .

The rest is quite formal.

Summary

- Maximal good tori are conjugate.
- Algebraic tori are good, in the Zariski topology.
- Algebraic tori with dimension are good in positive characteristic.
- The conjugacy theorem is used in the classification of groups of finite Morley rank of even type (AC, building on ABCJ)—along with generic covering lemmas. In fact, it was abstracted out of that context.