Genericity, Generosity, and Tori

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June 22, 2006

Contents

Themes

Connected groups of finite Morley rank (in general)

- Generic covering and conjugacy theorems
- Definable hulls of p-tori

Essential Notions

- Morley rank $(\mathrm{rk}\,(X))$
- Generic set: $\mathrm{rk}\,(X) = \mathrm{rk}\,(G)$
- Connected group

  $[G : H] < \infty \implies G = H.$

  $X, Y \subseteq G$ generic $\implies X \cap Y$ generic
- $d(X)$: definable subgroup generated by X.

Types

- *p*-torus: divisible abelian *p*-group
- Types:

  Degenerate: No infinite 2-subgroup
  Even: Nondegenerate, no nontrivial 2-torus (“characteristic two type”)
- *p*-unipotent: definable, connected, bounded exponent, nilpotent *p*-group

Structure

- Without 2-tori

$$1 \leq O_2(G) \leq G$$

$O_2(G)$: maximal unipotent 2-subgroup

$$\bar{G} = G/O_2(G)$$

$$\bar{G} = U_2(\bar{G}) * \hat{O}(\bar{G})$$

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Theorem (G_2)

The generic element of G belongs to $C^\circ(T)$ for a unique maximal 2-torus T.

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A simple group of even type is algebraic.

Theorem (D)

A connected degenerate type group contains no elements of order two.

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Theorem (D)

A connected degenerate type group contains no elements of order two.

Methods: Black box group theory, genericity arguments—soft

Good Tori

Theorem (E)

A simple group of even type is algebraic.

1st wave: No bad fields, no degenerate type simple sections.
2nd wave: No degenerate type simple sections.
3rd wave: General case (*tori*)

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Definition

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Rigidity properties:

R-I $N^\circ(T) = C^\circ(T)$

R-II Any uniformly definable family of subgroups of T is finite.

Ref: Altinel-Cherlin, *Limoncello* (J. Alg. **291** (2005), 371–413)

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But $M^\circ = C^\circ(U)$ in view of

(a) the absence of p-tori;
(b) Wagner's theorem: the multiplicative group of a field of finite Morley rank in positive characteristic is a good torus;

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forcing finitely many involutions in U.

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Properties of $H = C^\circ(T)$:

- Almost self-normalizing (Rigidity-I)
- Generically disjoint from its conjugates: $H \setminus (\bigcup H^{[G \setminus N(H)]})$ generic in H.

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Lemma (Genericity Lemma)

If a definable subgroup H of G is almost self-normalizing and generically disjoint from its conjugates then:

- *$\bigcup H^G$ is generic in G;*
- *For $X \subseteq H$, we have $\bigcup X^G$ generic in G if and only if $\bigcup X^H$ is generic in H.*

Definition

X is generous in G if the union of its conjugates is generic in G.

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Poizat's Problem

Problem

Let G be a connected group of finite Morley rank which satisfies the condition

$$x^n = 1$$

generically. Then G satisfies the condition

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identically.

Theorem

G as above. If $x^n = 1$ *generically on G, and n is a power of 2, then* $x^n = 1$ *identically on G.*

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More generally:

Theorem

G as above. If $x^n = 1$ generically on G, and $n = 2^k n_O$ with n_O odd, then $G = U * G_1$ with U a 2-group of bounded exponent and G/U a group satisfying $x^{n_O} = 1$ generically.

Generic equations

Analysis:

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  - $x^n = 1$ generically in G
  - $x^n = 1$ generically in H
  - $x^n = 1$ generically in Ta some $a \in H$
  - $x^n = 1$ generically in T
  - $T = 1$
- $G = U * G_1$ with U a 2-group of bounded exponent and G/U containing no involutions.
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Carter Subgroups

Definition

A Carter subgroup of G is a connected definable nilpotent subgroup which is almost self-normalizing.

Theorem (Frécon-Jaligot)

They exist.

Theorem (Frécon)

If the group G involves no bad groups and no bad fields, and T_0 is a maximal divisible torsion subgroup, then $C^\circ(T_0)$ is a Carter subgroup.

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See (or hear) Frécon . . .

Selected Examples

- Degenerate type groups
- Limoncello
- Toricity

Degenerate type groups

Sylow 2-subgroup finite, nontrivial.
Minimal example, simple (without loss).
Any 2-element will lie *outside* any proper definable connected subgroup of our ambient group G.
Useful simplification:

Lemma (EA)

The Sylow 2-subgroup of G is elementary abelian.

Genericity argument

Afterward, other techniques are brought to bear.

Lemma EA

Elementary abelian Sylow 2-subgroup = no elements of order 4.

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- Covariant: $H_{t^g} = H_t^g$
- Almost selfnormalizing: $N^\circ(H_t) = H_t$.

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Here $t \neq 1$, and H_t a proper connected definable subgroup for $t \neq 1$.
Definition: $H_t = N^\circ(\ldots N^\circ(C^\circ(t))\ldots)$. One takes connected normalizers until it stabilizes.
This is only interesting for t a 2-element, in which case $t \notin H_t$ (by minimality).

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Claim

For any 2-element $t \neq 1$, the coset tH_t is generous.

For $a \in tH_t$ and t a 2-element, $[d(a) : d^\circ(a)] = o(t)$. So the

Lemma EA

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Proof.

A variation on the standard genericity argument:

- $N^\circ(tH_t) = H_t$
- The conjugates of tH_t are pairwise disjoint

□

The initial configuration

Even type.
A "uniqueness" case, weak embedding, $M \leq G$ "big".

$$M \cap M^g \text{ contains a nontrivial unipotent 2-subgroup}$$
$$\text{iff}$$
$$g \in M$$

Aim: $G = SL_2$ (char. 2) and M a Borel subgroup

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$A \leq M$ elementary abelian, $M/C^\circ(A)$ $2^\perp$.
In fact $A = \Omega_1(O_2{}^\circ(M))$.

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Case division

Subcase 2: SL_2 sits as a proper subgroup of G.
Technically, we want to shift the line of division to:
Subcase 2^*: There are distinct conjugates A_1, A_2 of G with
$H = C^\circ(A_1, A_2) > 1$.
Then $L = \langle A_1, A_2 \rangle \leq C^\circ(H) < G$ and this gives us $L \simeq SL_2 < G$.

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$L \simeq SL_2 < G$.

Case 2^*, The main line

L contains 1-dimensional algebraic tori T—good tori (Wagner)
We learned in earlier "waves" of analysis that we want to look at the set $\mathcal{T}$ of conjugates of T lying in M, and eventually prove *they are all conjugate* under the action of M. This part of the analysis originally depended on M being *solvable*.

Conjugacy of tori

$\mathcal{T}$: some good tori contained in M.
Objective: $\mathcal{T}$ consists of a single conjugacy class under the action of $\mathcal{M}$.

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Lemma

Maximal good tori in $\mathcal{M}$ are generous in M, and are conjugate.

Lemma

Let $\mathcal{F}$ be a uniformly definable family of good tori, invariant under conjugation in M. Then $\mathcal{F}$ breaks up into finitely many M-conjugacy classes.

Proof.

T_0 a maximal good torus of M.
$\mathcal{F}_0$ the set of conjugates of tori in $\mathcal{F}$ that lie in T_0.
$\mathcal{F}_0$ is a uniformly definable family of subgroups of T_0, hence finite. □

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A little history: The published version of Limoncello runs this way—but the results it quotes are based on arguments found in early drafts of Limoncello.

Groups without unipotent p-subgroups

"$p^\perp$-type" (mainly, $p = 2$).

Theorem

Let G be a group of finite Morley rank of $p^\perp$ type. Then every p-element is p-toral (belongs to a p-torus).

Corollary

Let G be a connected group of finite Morley rank of $p^\perp$ type, and T a maximal p-torus. Then every p-element a of $C(T)$ belongs to T.

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Proof.

a belongs to a maximal torus T_0.
T, T_0 are maximal p-tori of $C(a)$, hence conjugate in $C(a)$.
Forcing $a \in T$. □

Toricity via generosity

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Proof.

Again, Ha turns out to be generically disjoint from its conjugates (in a suitable sense). □

Permutation Groups

$$(G, X)$$

Definably primitive: no nontrivial G-invariant definable equivalence relation.

(MPOSA)

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Proposition

(G, X) definably primitive. Then the degree of multiple transitivity of G is bounded by a function of rk (X).

(Special case of the theorem, but sufficient.)

Bounds on t

Lemma

T abelian divisible and definable, T_∞ its maximal torsion free definable subgroup of T. Then $rk(T/T_\infty) \le rk(X)$.

(In other words, the stabilizer in T of a point of X which is generic over the torsion subgroup is torsion free.)

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If G is not algebraic we are in $2^\perp$ type and we consider the definable hull T of a maximal 2-torus (not in G, but in a suitably chosen stabilizer of a small set of points).
The generic multiple transitivity gives us an action of Sym_n .

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The generic multiple transitivity gives us an action of Sym_n .

- If the action is nontrivial then T/T_∞ blows up and we get a contradiction.
- If the action is trivial then we get a 2-element outside T centralizing T and we contradict the corollary to “toricity”.

Challenges

- Algebraicity of simple K^*-groups of odd type
- Absolute bounds on Prüfer rank of groups of odd type
- Generosity of (some) Carter subgroups
- Construction of bad groups
- Construction of bad field towers.
- Sharp bounds on definably primitive groups
- Explicit classifications of generically 2-transitive actions of simple algebraic groups in the fMr category
- Representation theory of algebraic groups in the fMr category.