Urysohn, Fraïssé, and Henson in the Third Millennium

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

Logic and Mathematics 2011
Sep. 4, Urbana-Champaign

Henson, 1971–1973

1971. *A family of countable homogeneous graphs*, PJM 38
1972. *Countable homogeneous relational structures and $\aleph_0$-categorical theories*, JSL 37.
1973. *Edge partition properties of graphs*, CJM 25

Around Homogeneity

- Uniqueness and Universality
- Free amalgamation, antichains, and Turing chaos
- Partition properties
- Automorphism groups

I. Universality and Uniqueness

Definition

A structure is homogeneous if its automorphisms induce all isomorphisms between f. g. substructures M and M_1.

I. Universality and Uniqueness

Definition

A structure is homogeneous if its automorphisms induce all isomorphisms between f. g. substructures M and M_1.

Homogeneity $\implies$ Uniqueness (Hausdorff for $(\mathbb{Q}, <)$)
Homogeneity $\implies$ Universality (Urysohn, $\mathbb{U}_{\mathbb{Q}}$ and $\mathbb{U}$)

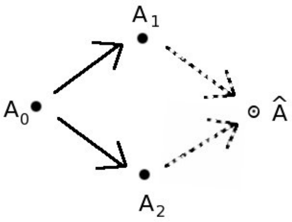
> *. . . a quite powerful condition of homogeneity: the latter being, that it is possible to map the whole space onto itself (isometrically) so as to carry an arbitrary finite set M into an equally arbitrary set M_1, congruent to the set M.*
>
> *(Urysohn to Hausdorff, cited by Hušek, 2008)*

I. Universality and Uniqueness

Definition

A structure is homogeneous if its automorphisms induce all isomorphisms between f. g. substructures M and M_1.

Homogeneity $\implies$ Uniqueness (Hausdorff for $(\mathbb{Q}, <)$)
Homogeneity $\implies$ Universality (Urysohn, $\mathbb{U}_{\mathbb{Q}}$ and $\mathbb{U}$)
Amalgamation $\implies$ Existence (Fraïssé)



I. Universality and Uniqueness

Definition

A structure is homogeneous if its automorphisms induce all isomorphisms between f. g. substructures M and M_1.

Homogeneity $\implies$ Uniqueness (Hausdorff for $(\mathbb{Q}, <)$)
Homogeneity $\implies$ Universality (Urysohn, $\mathbb{U}_{\mathbb{Q}}$ and $\mathbb{U}$)
Amalgamation $\implies$ Existence (Fraïssé)

Applications (Henson)

- A universal homogeneous graph Γ_∞ exists and is unique:

$$\Gamma_\infty \simeq \text{Rado's Graph} \simeq \text{Random Graph} \simeq \Gamma_\infty^c$$

- The universal homogeneous K_n-free graph Γ_n exists and is unique.

Further Applications

Komjáth/Mekler/Pach 1988

- Universal P_n-free graphs
- C_{2n+1}-homomorphically free graphs
- $C_{\geq N}$-free graphs

Further Applications

Komjáth/Mekler/Pach 1988

- Universal P_n-free graphs
- C_{2n+1}-homomorphically free graphs
- $C_{\geq N}$-free graphs

Cherlin/Shelah/Shi

For $\mathcal{C}$ a finite set of connected finite graphs the following are equivalent:

1. The theory $T^*_{\mathcal{C}}$ of e. c. $\mathcal{C}$-free graphs is $\aleph_0$-categorical.
2. The corresponding algebraic closure operator $\mathrm{acl}_{\mathcal{C}}$ is locally finite.

Further Applications

Komjáth/Mekler/Pach 1988

- Universal P_n-free graphs
- C_{2n+1}-homomorphically free graphs
- $C_{\geq N}$-free graphs

Cherlin/Shelah/Shi

For $\mathcal{C}$ a finite set of connected finite graphs the following are equivalent:

1. The theory $T^*_{\mathcal{C}}$ of e. c. $\mathcal{C}$-free graphs is $\aleph_0$-categorical.
2. The corresponding algebraic closure operator $\mathrm{acl}_{\mathcal{C}}$ is locally finite.

Proof.

The set of types over a finite alg. closed set is finite □

(The model is homogeneous for a finite relational language.)

Further Applications

Komjáth/Mekler/Pach 1988

- Universal $\mathrm{P_n}$-free graphs
- $\mathrm{C_{2n+1}}$-homomorphically free graphs
- $\mathrm{C_{\geq N}}$-free graphs

Cherlin/Shelah/Shi

For $\mathcal{C}$ a finite set of connected finite graphs the following are equivalent:

1. The theory $\mathrm{T}^*_{\mathcal{C}}$ of e. c. $\mathcal{C}$-free graphs is $\aleph_0$-categorical.
2. The corresponding algebraic closure operator $\mathrm{acl}_{\mathcal{C}}$ is locally finite.

Example For any $\mathcal{C}$, there is a universal $\mathcal{C}$-homomorphically free graph. — acl is degenerate in this case.

Decision Problems

Problem 1.

(A) Is $T^*_{\mathcal{C}}$ $\aleph_0$-categorical?

(B) Is there a universal $\mathcal{C}$-free graph?

Decision Problems

Problem 1.

(A) Is $T^*_{\mathcal{C}}$ $\aleph_0$-categorical?

(B) Is there a universal $\mathcal{C}$-free graph?

Komjáth/Füredi

If C is 2-connected then the following are equivalent.

1. There is a universal C-free graph.
2. T^*_C is $\aleph_0$-categorical.
3. C is complete.

Decision Problems

Problem 1.

(A) Is $T^*_{\mathcal{C}}$ $\aleph_0$-categorical?

(B) Is there a universal $\mathcal{C}$-free graph?

Komjáth/Füredi

If C is 2-connected then the following are equivalent.

1. There is a universal C-free graph.
2. T^*_{C} is $\aleph_0$-categorical.
3. C is complete.

Problem 2. (Solidity Conjecture) Show that if there is a universal C-free graph then its blocks are complete.

Ongoing . . .

Undecidability

Theorem (Here there be Tygers)

The universality and $\aleph_0$-categoricity problems are undecidable for finite sets of forbidden induced subgraphs (universal theories).

Proof.

Wang tilings $\rightarrow$ black/white tilings $\rightarrow$
binary relations on $(\mathbb{Z}, \mathrm{s})$ $\rightarrow$ graphs on $\mathbb{Z}$ and:

Many tilings $\implies$ no universal graph;

No tilings $\implies$ local finiteness $\implies$ universal graph

(and recursive inseparability) □

II. Antichains and Turing Chaos

- Free amalgamation
- Indecomposability
- Antichains

II. Antichains and Turing Chaos

- Free amalgamation
- Indecomposability
- Antichains

Examples

- WGB-graphs
- Directed graphs (Henson 1972)
- Nilpotent rings and groups of class 2 [CSW 1993]

Antichains and WQO

- WQO

Antichains and WQO

- WQO

Tournaments: Henson 1972
Permutation Patterns: Tarjan 1972

- Minimal antichains

Nash-Williams 1963, Gustedt 1993/1999
Subgraphs: Cycles and anchored paths (Ding 1992)

III. Classification

It is an interesting and apparently open question if there are any homogeneous graphs G *(with* $c(G) = \aleph_0$*) which have* G *and* $\overline{G}$ *connected, other than* U, G_p *and* $\overline{G}_p$ *(*$p \geq 3$*).* [H1971]

III. Classification

It is an interesting and apparently open question if there are any homogeneous graphs G *(with* $c(G) = \aleph_0$*) which have* G *and* $\overline{G}$ *connected, other than* U*,* G_p *and* $\overline{G}_p$ *(*$p \geq 3$*).* [H1971]

Classifications

Graphs Lachlan/Woodrow 1980

Tournaments Lachlan 1984

Directed Graphs Cherlin 1998

III. Classification

It is an interesting and apparently open question if there are any homogeneous graphs G *(with* $c(G) = \aleph_0$*) which have* G *and* $\overline{G}$ *connected, other than* U*,* G_p *and* $\overline{G}_p$ *(*$p \geq 3$*).* [H1971]

Classifications

Graphs Lachlan/Woodrow 1980

Tournaments Lachlan 1984

Directed Graphs Cherlin 1998

- Chaos?

The Ramsey Method

Homogeneous Tournaments: Catalog

Orders L_1, $\mathbb{Q}$

Strict Local Orders $S[L_1]$, $S(\mathbb{Q})$

Generic T^∞

The Ramsey Method

Homogeneous Tournaments: Catalog

Orders L_1, $\mathbb{Q}$

Strict Local Orders $S[L_1]$, $S(\mathbb{Q})$

Generic T^∞

The critical tournament $C^+ = [L_1, \vec{C}_3]$ (or its dual).

Theorem (Lachlan)

$C^+ \implies T$ *for all finite* T.

$C^+ \implies T$

Definition

$$\mathcal{A}^\ell \quad \{A \in \mathcal{A} : \forall B = A \cup L \,(B \in \mathcal{A})\}$$
$$\mathcal{A}' \quad \{A \in \mathcal{A} : \forall B = L[A] \cup \{v\} \,(B \in \mathcal{A})\}$$

$C^+ \implies T$

Definition

$$\mathcal{A}^\ell \quad \{A \in \mathcal{A} : \forall B = A \cup L\,(B \in \mathcal{A})\}$$
$$\mathcal{A}' \quad \{A \in \mathcal{A} : \forall B = L[A] \cup \{v\}\,(B \in \mathcal{A})\}$$

Lemma

1. $\mathcal{A}^\ell$ *has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

$C^+ \implies T$

Definition

$$\mathcal{A}^\ell \quad \{A \in \mathcal{A} : \forall B = A \cup L\,(B \in \mathcal{A})\}$$
$$\mathcal{A}' \quad \{A \in \mathcal{A} : \forall B = L[A] \cup \{v\}\,(B \in \mathcal{A})\}$$

Lemma

1. *$\mathcal{A}^\ell$ has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

Proof of the Theorem; If all tournaments of order $n-1$ are in $\mathcal{A}$ then taking "$\mathcal{A}$" to be $\mathcal{A}^\ell$, all tournaments of order n are in $\mathcal{A}$. □

Proof of the Lemma

Lemma

1. $\mathcal{A}^\ell$ *has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

Proof.

Proof of the Lemma

Lemma

1. *$\mathcal{A}^\ell$ has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

Proof.

1. Formal

Proof of the Lemma

Lemma

1. *$\mathcal{A}^\ell$ has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

Proof.

1. Formal
2. Interesting, but not as interesting as

Proof of the Lemma

Lemma

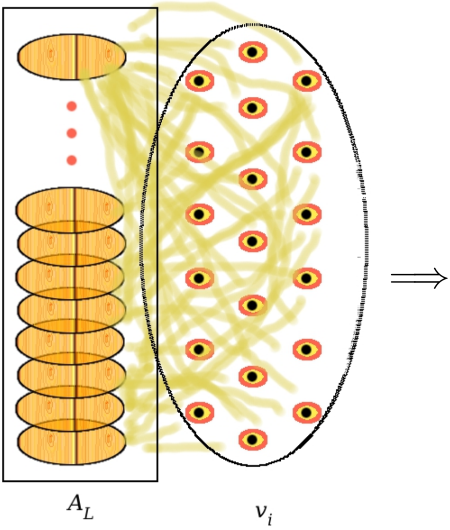
1. $\mathcal{A}^\ell$ *has amalgamation*
2. $C^+ \in \mathcal{A} \implies C^+ \in \mathcal{A}'$
3. $\mathcal{A}' \subseteq \mathcal{A}^\ell$

Proof.

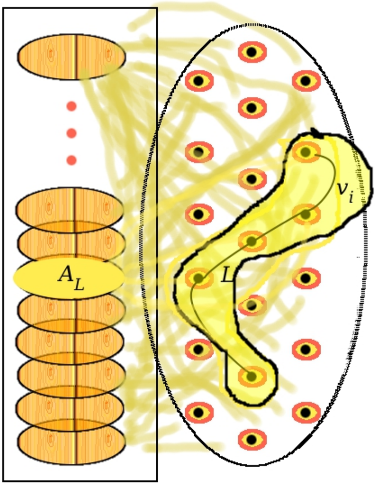
1. Formal
2. Interesting, but not as interesting as
3. Ramsey's Theorem: details to follow.

□

The Ramsey Argument



The Ramsey Argument



Homogeneous Directed Graphs

Theorem

The homogeneous directed graphs consist of the following.

1. *Henson digraphs $\Gamma_{\mathcal{T}}$ with $\mathcal{T}$ an antichain of tournaments;*
2. *Countably many others (explicitly listed).*

Homogeneous Directed Graphs

Theorem

The homogeneous directed graphs consist of the following.

1. *Henson digraphs $\Gamma_{\mathcal{T}}$ with $\mathcal{T}$ an antichain of tournaments;*
2. *Countably many others (explicitly listed).*

The general method.

1. Make catalog.
2. Reduce completeness proof to a finite number of base cases.
3. Verify by explicit amalgamation [else, revise catalog]

Homogeneous Directed Graphs

Theorem

The homogeneous directed graphs consist of the following.

1. *Henson digraphs $\Gamma_{\mathcal{T}}$ with $\mathcal{T}$ an antichain of tournaments;*
2. *Countably many others (explicitly listed).*

The general method.

1. Make catalog.
2. Reduce completeness proof to a finite number of base cases.
3. Verify by explicit amalgamation [else, revise catalog]

Is there Chaos?

Conjecture: Yes, but not seen yet.

Lachlan Chaos

Problem 3. Decision Problem: For $\mathcal{A}_0$, $\mathcal{B}_0$ finite:

$$\bigwedge \mathcal{A}_0 \implies \bigvee \mathcal{B}_0?$$

i.e. $$|\{\mathcal{A} : \mathcal{A}_0 \subseteq \mathcal{A}, \mathcal{A} \cap \mathcal{B}_0 = \emptyset\}| = 0?$$

Lachlan Chaos

Problem 3. Decision Problem: For $\mathcal{A}_0$, $\mathcal{B}_0$ finite:

$$\bigwedge \mathcal{A}_0 \implies \bigvee \mathcal{B}_0?$$

i.e. $$|\{\mathcal{A} : \mathcal{A}_0 \subseteq \mathcal{A}, \mathcal{A} \cap \mathcal{B}_0 = \emptyset\}| = 0?$$

Variations: $$|\{\mathcal{A} : \mathcal{A}_0 \subseteq \mathcal{A}, \mathcal{A} \cap \mathcal{B}_0 = \emptyset\}| =??$$

Example

- Lachlan's Decision problem is decidable in the class of homogeneous directed graphs.

Lachlan Chaos

Problem 3. Decision Problem: For $\mathcal{A}_0$, $\mathcal{B}_0$ finite:

$$\bigwedge \mathcal{A}_0 \implies \bigvee \mathcal{B}_0?$$

i.e. $$|\{\mathcal{A} : \mathcal{A}_0 \subseteq \mathcal{A}, \mathcal{A} \cap \mathcal{B}_0 = \emptyset\}| = 0?$$

Variations: $$|\{\mathcal{A} : \mathcal{A}_0 \subseteq \mathcal{A}, \mathcal{A} \cap \mathcal{B}_0 = \emptyset\}| = ??$$

Example

- Lachlan's Decision problem is decidable in the class of homogeneous directed graphs.
- The decision problem for "$\exists 2^{\aleph_0}$" is equivalent to the following.

Problem 3'. For $\mathcal{T}$ a finite set of tournaments, is there an infinite antichain of finite $\mathcal{T}$-free tournaments?

The *WQO* decision problem

Problem 4. Decision Problem. $(\mathcal{Q}, \leq)$ a class of finite structures with the substructure relation (or induced substructure). $C \subseteq \mathcal{Q}$ finite. $I_C =$

$$\{q \in \mathcal{Q} : \neg\exists c \in C\,(c \leq q)\}$$

Is I_C WQO?

The *WQO* decision problem

Problem 4. Decision Problem. $(\mathcal{Q}, \leq)$ a class of finite structures with the substructure relation (or induced substructure). $C \subseteq \mathcal{Q}$ finite. $I_C =$

$$\{q \in \mathcal{Q} : \neg\exists c \in C\,(c \leq q)\}$$

Is I_C WQO?

Harvey Friedman: There is a recursive locally finite partial order $(\mathcal{Q}, \leq)$ for which the *WQO* decision problem is a complete Π^1_1 set.

On the other hand ...

Finiteness Theorem

Theorem (Cherlin/Latka 2000)

For any k there is a finite set of infinite antichains Λ_k such that for any set $\mathcal{T}$ of finite tournaments with $|\mathcal{T}| \leq k$ the following are equivalent.

1. $\mathcal{Q}_\mathcal{T}$ *is wqo;*
2. $\exists I \in \Lambda_k \; I \subseteq_* \mathcal{Q}_\mathcal{T}$.

- Tournaments $|\Lambda_1| = 2$: Latka
- Permutation Patterns Vatter, Waton, Brignall, . . .
- Induced graphs Gustedt

Motivation: implications for computational problems.

NB. Gustedt develops a general theory of minimal antichains.

Finiteness Theorem

Theorem (Cherlin/Latka 2000)

For any k there is a finite set of infinite antichains Λ_k such that for any set $\mathcal{T}$ of finite tournaments with $|\mathcal{T}| \leq k$ the following are equivalent.

1. $\mathcal{Q}_{\mathcal{T}}$ *is wqo;*
2. $\exists I \in \Lambda_k \; I \subseteq_* \mathcal{Q}_{\mathcal{T}}$.

- Tournaments $|\Lambda_1| = 2$: Latka
- Permutation Patterns Vatter, Waton, Brignall, . . .
- Induced graphs Gustedt

Motivation: implications for computational problems.

NB. Gustedt develops a general theory of minimal antichains.

Problem 5. Encode permutations by tournaments, preserving (known) minimal antichains.

Classification Problems

Problem 6. Classify homogeneous structures of the following types:

Partitioned Graphs

Metrically Homogeneous Graphs [Cameron 1998]

k-Dimensional Permutations [Cameron 2002]

Graphs as Metric Spaces

Definition

A graph is metrically homogeneous if it is homogeneous as a metric space.

Example r-regular tree.
(Note: the metric on n vertices determines the metric on their convex closure!)

Metrically Homogeneous Graphs

Example Macpherson's Graphs $T_{r,s}$.

Construction

$\Gamma = (A, B)$ bipartite and homogeneous as a metric space with bipartition.

Then $\frac{1}{2}A$ is a metrically homogeneous graph.

Application $\Gamma = T(r, s)$ a semi-regular (r, s)-branching tree. Macpherson's graph $T_{r,s}$ is $\frac{1}{2}A$.

Theorem (Macpherson)

Distance transitive infinite locally finite graphs are of the form $T_{r,s}$ with $2 \leq r, s < \infty$.

CATALOG

Known Metrically Homogeneous Graphs

1. $\delta \leq 2$ [Lachlan/Woodrow], deg ≤ 2 [C_n], or finite [Cameron]
2. $T_{r,s}$ $(2 \leq r, s \leq \infty)$
3. $\Gamma^{\delta}_{\Delta,\mathcal{S}}$ with $\mathcal{A}^{\delta}_{\Delta}$ 3-constrained
4. $\Gamma^{\delta}_{a,n}$ with $\delta \geq 4$ if $n < \infty$.

CATALOG

Known Metrically Homogeneous Graphs

1. $\delta \leq 2$ [Lachlan/Woodrow], deg ≤ 2 [C_n], or finite [Cameron]
2. $T_{r,s}$ $(2 \leq r, s \leq \infty)$
3. $\Gamma^{\delta}_{\Delta,\mathcal{S}}$ with $\mathcal{A}^{\delta}_{\Delta}$ 3-constrained
4. $\Gamma^{\delta}_{a,n}$ with $\delta \geq 4$ if $n < \infty$.

$\mathcal{S}$: Henson constraints $(1, \delta)$-spaces (an independent set of cliques).

Amalgamation:

- $d^{+}(b_1, b_2) = \min_a(d_1(b_1, a) + d_2(b_2, a))$
- $d'(b_1, b_2) = \min(d^{+}(b_1, b_2), \delta - 1)$.

Metric Triangle Constraints

Initial Conjecture $\mathcal{A}^\delta_\Delta = \mathcal{A}^\delta_{K,C}$ (KMP-type). I.e.

$$\begin{aligned} 2K + 1 &\le P \qquad (P \text{ odd}) \\ P &\le C \end{aligned}$$

Metric Triangle Constraints

Initial Conjecture $\mathcal{A}^\delta_\Delta = \mathcal{A}^\delta_{K,C}$ (KMP-type). I.e.

$$2K + 1 \leq P \qquad (P \text{ odd})$$
$$P \leq C$$

Theorem (But actually:)

$\mathcal{A}^\delta_\Delta = \mathcal{A}^\delta_{K,C}$ with $K = (K_1, K_2)$ and $C = (C_1, C_2)$, subject to "Presburger" conditions.

$$2K_1 + 1 \leq P < 2K_2 + 2d(a,b) \qquad (P \text{ odd})$$
$$P \leq C_i \text{ where } P \equiv i \mod 2$$

Metric Triangle Constraints

Initial Conjecture $\mathcal{A}^\delta_\Delta = \mathcal{A}^\delta_{K,C}$ (KMP-type). I.e.

$$2K + 1 \leq P \qquad (P \text{ odd})$$
$$P \leq C$$

Theorem (But actually:)

$\mathcal{A}^\delta_\Delta = \mathcal{A}^\delta_{K,C}$ with $K = (K_1, K_2)$ and $C = (C_1, C_2)$, subject to "Presburger" conditions.

$$2K_1 + 1 \leq P < 2K_2 + 2d(a,b) \qquad (P \text{ odd})$$
$$P \leq C_i \text{ where } P \equiv i \mod 2$$

Lemma

$\mathcal{A}^\delta_{K,C}$ has amalgamation iff it has 5-amalgamation.

Classification Conjecture

With a few notable exceptions

1. If $\delta < \infty$ then $\mathcal{A}^\delta \cap \Delta$ has amalgamation.
2. If $\mathcal{A}^\delta \cap \Delta = \Delta^\delta_{K,C}$ then $\mathcal{A}^\delta = \mathcal{A}^\delta_{K,C;\mathcal{S}}$ or $\mathcal{A}^\delta_{a,n}$.
3. If $\delta = \infty$ and Γ is not bipartite, then $\Gamma = \lim \Gamma_i$ with Γ_i metrically homogeneous of diameter $2i$, for large i.

Classification Conjecture

With a few notable exceptions

1. If $\delta < \infty$ then $\mathcal{A}^\delta \cap \Delta$ has amalgamation.
2. If $\mathcal{A}^\delta \cap \Delta = \Delta^\delta_{K,C}$ then $\mathcal{A}^\delta = \mathcal{A}^\delta_{K,C;\mathcal{S}}$ or $\mathcal{A}^\delta_{a,n}$.
3. If $\delta = \infty$ and Γ is not bipartite, then $\Gamma = \lim \Gamma_i$ with Γ_i metrically homogeneous of diameter $2i$, for large i.

Problem. Show that $(1,2) \implies (3)$.

IV. Indivisibility

COROLLARY 4.2. Let $p \geq 3$ *and suppose that* $|G_p| = A_1 \bigcup \cdots \bigcup A_n$*. Then for some* $j = 1, \ldots, n$ *the graph* $G_p|A_j$ *admits every finite graph which does not admit* K_p*.*

. . . We raise the question of whether or not the conclusion of Corollary 4.2 can be strengthened to read: "$G_p|A_j$ *admits* G_p*, for some* $j = 1, ..., n$*."?*

IV. Indivisibility

COROLLARY 4.2. Let $p \geq 3$ and suppose that $|G_p| = A_1 \bigcup \cdots \bigcup A_n$. Then for some $j = 1, \ldots, n$ the graph $G_p|A_j$ admits every finite graph which does not admit K_p.

... We raise the question of whether or not the conclusion of Corollary 4.2 can be strengthened to read: "$G_p|A_j$ admits G_p, for some $j = 1, ..., n$."?

Yes

G_3 Komjáth/Rödl 1986

G_n El-Zahar/Sauer 1989

$\mathbb{U}^\delta$ van Thé/Sauer 2009

Free Amalgamation EZ/S 2003, 2005

Ref: L. van Thé, AMS Memoir 968, 2010

Indivisibility and Free Amalgamation

Theorem (El-Zahar, Sauer)

Let L be a finite binary language, Γ a countable homogeneous L-structure, and assume $\mathcal{A}_L$ has free amalgamation. Then Γ is indivisible iff any two orbits of Γ are comparable up to a finite partition.

Orbits are orbits for the stabilizer of an arbitrary finite subset.

The relation $O \leq O'$ which we call comparability up to a finite partition is defined by: *there is a finite partition of O whose pieces embed into O'*.

V. Structural Ramsey Theory

Ref: Again, L. van Thé, AMS Memoir 968, 2010
This is the finitary variant.

Hungarian Notation $N \to (n)^m_k$

Generalized $C \to (B)^A_k$.
Restrict B, or add a linear order to the language.

Nešetril-Rödl Amalgamation Method

V. Structural Ramsey Theory

Ref: Again, L. van Thé, AMS Memoir 968, 2010
This is the finitary variant.

Hungarian Notation $N \to (n)^m_k$

Generalized $C \to (B)^A_k$.
Restrict B, or add a linear order to the language.

Nešetril-Rödl Amalgamation Method

Familiar homogeneous structures acquire the Ramsey property when equipped with a suitable ordering, or a bit more.

Expanding Homogeneous Structures

Problem 7. Is every finitely determined relational homogeneous structure in a finite language a reduct of a class with the Ramsey property?
(Asked often by Bodirsky.)

Expanding Homogeneous Structures

Problem 7. Is every finitely determined relational homogeneous structure in a finite language a reduct of a class with the Ramsey property?
(Asked often by Bodirsky.)

Problem 7'. Give a model theoretic description of the way to add a linear order which covers the known cases.

Expanding Homogeneous Structures

Problem 7. Is every finitely determined relational homogeneous structure in a finite language a reduct of a class with the Ramsey property?
(Asked often by Bodirsky.)

Problem 7'. Give a model theoretic description of the way to add a linear order which covers the known cases.

Examples
Random graph: freely
Urysohn space: freely
Partial order: compatibly
Boolean algebra: lexicographically

VI. Automorphism groups

Henson 1971

• There is an $\alpha \in Aut(\Gamma_n)$ with a single orbit iff $n \leq 3$ or $n = \infty$.

• $(\Gamma, Aut(\Gamma)) \hookrightarrow (\Gamma_\infty, Aut(\Gamma_\infty))$ (with unique extensions)
Corollary. $S_\infty \hookrightarrow Aut(\Gamma_\infty)$
Jaligot-Bilge, extensions, in progress.
Hasson-Kojman-Onshuus: Symmetric indivisibility

Problem 8. Given a finite partition of Γ_n and a K_n-free graph Γ, is there an embedding of Γ into one piece so that automorphisms of Γ extend uniquely to Γ_n?

Normal Subgroups

S_∞/Bdd is simple

Theorem (Tent-Ziegler)

If C is an unbounded conjugacy class in $Aut(\mathbb{U})$ then $(C \cup C^{-1})^8 = \mathbb{U}$.

Corollary $Aut(\mathbb{U})/\mathrm{Bdd}$ is simple
Note: Work with $\mathbb{U}_S$ with S countable and additively closed, and vary S.

Tent (ongoing): this is a formal consequence of canonical amalgamation $B_1 \otimes_A B_2$ and a formalism reminiscent of stability theory (in fact, includes the case of strongly minimal sets as treated by Lascar).

Topological Dynamics

Kechris/Pestov/Todorcevic

Extreme Amenability: Fixed points of compact actions.
Equivalent to the finitary Ramsey property.

Topological Dynamics

Kechris/Pestov/Todorcevic

Extreme Amenability: Fixed points of compact actions. Equivalent to the finitary Ramsey property.

In the absence of linear order, characterize the universal minimal compact flow.

Topological Dynamics

Kechris/Pestov/Todorcevic

Extreme Amenability: Fixed points of compact actions. Equivalent to the finitary Ramsey property.

In the absence of linear order, characterize the universal minimal compact flow.

Definitely Millennium III . . . motivates Bodirsky's favorite question.

Problems

Problem 1. Universal $\mathcal{C}$-free graphs and $\aleph_0$-categoricity.
Problem 2. Solidity Conjecture (Complete blocks)
Problem 3. [Lachlan] For $\mathcal{A}_0, \mathcal{B}_0$ finite: $\bigwedge \mathcal{A}_0 \implies \bigvee \mathcal{B}_0$?
Problem 4. $(\mathcal{Q}, \leq)$ finite L-structures with the substructure or induced substructure relation. $C \subseteq \mathcal{Q}$ finite.
$I_C = \{q \in \mathcal{Q} : \neg\exists c \in C\,(c \leq q)\}$. Is I_C WQO?
Problem 5. Encode permutations by tournaments, preserving (known) minimal antichains.
Problem 6. Classify homogeneous partitioned graphs, metrically homogeneous graphs, and primitive homogeneous k-dim. permutations.
Problem 7. (A) Is every f.d. relational homogeneous structure in a finite language a reduct of a Ramsey class?
(B) Give a model theoretic description of the right linear order.
Problem 8. Indivisibility with unique extendability.