Universal Graphs (with Forbidden Subgraphs)

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The players

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$\mathcal{G}_C$: the set of countable C-free graphs

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Problem

Is there a countable universal C-free graph?

(What does this say about C?)

Four Problems

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2. *Is there a countable universal C-free graph with oligomorphic automorphism group?*
3. *If so, how does one make the graph homogeneous?*
4. *And is the universal minimal flow metrizable (what is the structural Ramsey theorem)?*

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Note: We can give a criterion for #2 which implies that in #3 we can take a finite relational language.

The Model Theoretic Problem

$\mathcal{C}$ a finite set of finite relational structures
$T_{\mathcal{C}}$ the theory of $\mathcal{C}$-free structures
$T^*_{\mathcal{C}}$ the theory of existentially closed $\mathcal{C}$-free structures

Problem

Can one compute the model theoretic properties of $T^*_{\mathcal{C}}$ from the data $\mathcal{C}$?

- Countable universal $\mathcal{C}$-free with oligomorphic automorphism group: $T^*_{\mathcal{C}}$ is $\aleph_0$-categorical;
- Countable universal $\mathcal{C}$-free: $T^*_{\mathcal{C}}$ is small;
- stable, ω-stable, simple, ...

Playing with blocks

Block: Maximal 2-connected.
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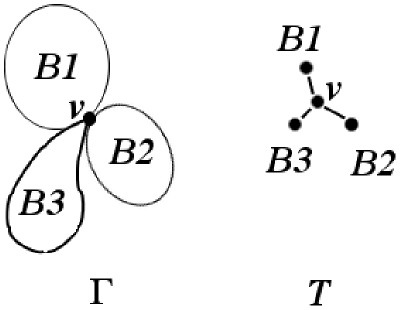
Example: starlike with three blocks.



The case of 1 block

Fact (Füredi-Komjáth)

Let C be 2-connected. Then the following are equivalent.

- *There is a countable universal C-free graph;*
- *There is a countable universal C-free graph with oligomorphic automorphism group;*
- *C is complete.*

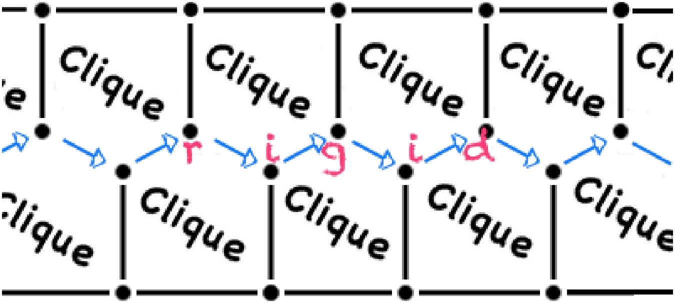
The case of 2 blocks

Fact (FK,Kom,ChT)

Let C have two blocks, of orders $m \leq n$. Then the following are equivalent.

- *There is a countable universal C-free graph;*
- *There is a countable universal C-free graph with oligomorphic automorphism group;*
- *The blocks are complete, with $m \leq 5$ and $(m, n) \neq (5, 5)$.*

Why not $(5,5)$?



$(K_5 \wedge K_5)$ - *free*

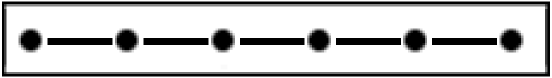
(Rigidity)

The case of trees

Fact (KMP,ChT,ChSh)

Let C be a tree. Then the following are equivalent.

- *There is a countable universal C-free graph with oligomorphic automorphism group;*
- *C is a path.*

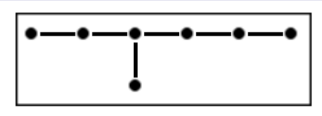


The case of trees

Fact (KMP,ChT,ChSh)

Let C be a tree. Then the following are equivalent.

- *There is a countable universal C-free graph;*
- *C is a near-path.*



Reasonable Conclusions

If we want universal C-free graphs, then apparently

- the blocks should be complete;
- the block structure should be path-like;
- there is not much difference between the oligomorphic case and the general case;
- and where they differ, the oligomorphic case is cleaner.

Reasonable Conjectures

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If there is a countable universal C-free graph, then

1. *Solidity The blocks of C are complete;*
2. *Pathlike After removal of some external paths (whiskers), the tree of blocks becomes a path.*

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Theorem (Cherlin/Shelah)

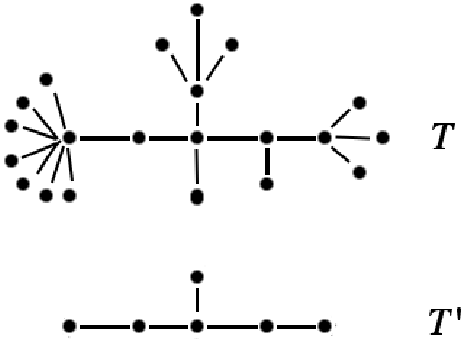
If there is a countable universal C-free graph and C is a block-path, then the blocks of C are complete. Therefore the second conjecture implies the first.

The Three Exclusion Methods

- Corner Pruning;
- Symmetric Pruning;
- The Hypergraph Template (Rigidity)

Pruning Trees

The case of trees: $T' = T \setminus$ leaves



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Let T be a finite tree for which there is a countable universal T-free graph. Then there is a countable universal T'-free graph. The same reduction applies if we require an oligormphic automorphism group.

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This suffices to classify the trees T allowing a countable universal T-free graph: one considers only trees T which prune to a near-path.

Pruning Corners

Segment: Connected union of blocks
Corner: A segment formed by a cut vertex and one component of its complement.

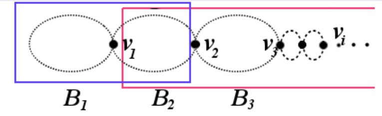
Lemma

Let C be a finite graph for which there is a countable universal C-free graph. Let (v, S) be a corner of C and C' the result of pruning the corner. Then there is a countable universal C'-free graph. The same reduction applies if we require an oligomorphic automorphism group.

Application

Lemma

Let C be a block path with ℓ blocks. Suppose that there is a countable universal C-free graph, but that some block is not complete; and let ℓ be minimal. Then (up to a reversal of the numbering) we have the following.

- *B_i is complete for $1 < i < \ell$ and also for $i = \ell$ unless $B_1 \simeq B_\ell$;*
- *$\ell \geq 3$ (Füredi/Komjáth);*
- *L_2^- embeds into R_1^+:* 

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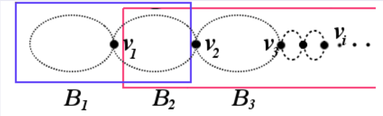


(Prune the terminal segment R_2.)

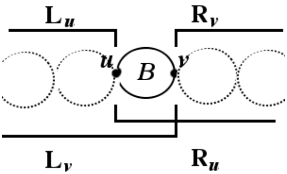
Symmetric Local Pruning

Lemma

Let C be a block path, B a block of C containing two cut vertices u, v*, and let*

$$L_u, R_u, L_v, R_v$$

be the corners rooted at u, v *respectively, to the left and right according to some orientation.*
If $L_v \setminus \{v\}$ *embeds into* $R_u \setminus \{u\}$*, then* (v, R_v) *is detachable.*



C-algebraic closure

Definition

A is C-algebraically closed in Γ if the free amalgam

$$\Gamma^\infty/A$$

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Theorem (ChShSh)

The following are equivalent.

- *There is a countable universal C-free graph with oligomorphic automorphism group;*
- *Every finite subset of a C-free graph is contained in a finite C-algebraically closed subset.*

Minimal Bases

Setting: $X \subseteq A \subseteq B \subseteq \Gamma$ *with* Γ *e.c. and* C*-free.*

Definition

B is free over A if $\bigoplus_{i=1}^{n}(B \setminus A)$ embeds over A into Γ, all n.
A is a base for B over X if A is minimal so that $X \subseteq A$ and B is free over A.

Example

$C = (K_3 \wedge K_3)$, $B = T$ a single triangle, $X = \{x\}$, $x \in T$.
Then T cannot be free over x.
If there is an $a \in T$ so that (a, x) is contained in infinitely many triangles, then T is free over (a, x).
Otherwise, T is a base for T over x.

Algebraic Closure

Lemma

Let $A \subseteq \Gamma$ with Γ C-free and algebraically closed, and A finite. Then the following are equivalent.

- *A is algebraically closed;*
- *For any $A \subseteq B \subseteq \Gamma$ with B embeddable in C, B is free over A.*

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Application to $(K_3 \wedge K_3)$-free:
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Application to $(K_3 \wedge K_3)$-free:
Any set X is contained in an algebraically closed set Y with $|Y| \leq 4|X|$ (in the worst case, Y is a union of K_4's).
For the general 2-bouquet (m, n) with $m \leq 5$: apply the Δ-system lemma to the copies of K_n involved; come down eventually to $n = 5$ and the heart is empty, which leads back to the case $(5, 5)$.

Candidates

Conjecture (At the moment)

If C contains no trivial blocks, and there is a countable universal C-free graph, then C is a block path with complete blocks and of one of the following types $(n_1, n_2, \ldots, n_\ell)$.

- $3^{\ell-1}\, n$, $3^{\ell-2}\, n\, 3$, $3^{\ell-2}\, 44$; *or*
- *One of the following forms:*

ℓ	Types
5	$(n_1, 3, 3, 3, n_5)$
"	$(3, n_2, 3, 3, n_5)$; $(n_1, 3, 3, 4, 4)$
4	$(n_1, 3, 3, n_4)$ $(n_4 \geq n_1 + 2)$
"	$(3, n, 3, n)$; $(4, 4, 4, n)$ $(n > 4)$;
"	$(3, 4, 4, n)$; $(4, 4, 3, n)$; $(3, 4, 3, n)$;
3	$(n_1, 3, n_3)$, $(n_1, 4, n_3)$
2	$(4, n)$; or $(5, n)$ with $n \geq 6$

Final Remarks

— In the *mixed case* there should be the hairy ball graphs: K_n plus one path at each vertex (existence of the universal object not proved!)

— If we forbid induced graphs then the existence of a countable universal graph is undecidable (Wang's dominos).

— Forbidding substructures reduces to forbidding graphs equipped with a partition of the vertices into two classes (Ch-Shi).

— We do not know what happens with permutation patterns, which is a very interesting case. Our theory does not apply there in its present form.