Universal Graphs with Forbidden Subgraphs

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Origins

[RADO64]: Universal Graphs

- Universal (countable) graphs exist
- Universal locally finite graphs do not exist (de Bruijn)

[KomPach91] (survey): WHEN do universal graphs exist?

Origins

[RADO64]: Universal Graphs
[KomPach91] (survey): WHEN do universal graphs exist?

[ERDŐS-RÉNYI63]: Automorphisms

- $\mathrm{Aut}(\Gamma) = 1$ for Γ random finite
- $\mathrm{Aut}(\Gamma)$ rich for Γ random infinite

> *Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.*

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[Rado64]: Universal Graphs
[KomPach91] (survey): WHEN do universal graphs exist?

[Erdős-Rényi63]: Automorphisms

> *Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.*

[KPT05] $\mathrm{Aut}\,\Gamma$ has fixed points $\iff$ Structural Ramsey
[Pes98] $\mathrm{Aut}(\mathbb{Q})$ has fixed points $\iff$ Ramsey

Universality

We follow Rado's line (or Komjáth/Pach's interpretation of it) ...

Homogeneity

Definition (Homogeneity)

$$A \simeq B \iff A \sim B \text{ (conjugate under } \mathrm{Aut}(\Gamma)\text{)}$$

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CONSEQUENCES

- Universality (modulo finite substructures)
- Uniqueness (modulo finite substructures)
- Oligomorphic (finitely many orbits on n-tuples)

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As observed by Urysohn in 1924 ...

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Urysohn 1924 (Letter)

> *"... [a] condition of homogeneity: the latter being, that it is possible to map the whole space onto itself ... so as to carry an arbitrary finite set M into an equally arbitrary set M_1, congruent to the set M."*

Ref: [Hušek08]

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$\mathbb{U}$: universal complete separable metric space
$\mathbb{U}_{\mathbb{Q}}$: universal rational-valued metric space

- $\mathbb{U}_{\mathbb{Q}}$ is a universal graph (edges: $d(u, v) = 1$)
  cf. Moss, Cameron ...

Limits of Homogeneity

Theorem (Lachlan/Woodrow 1980)

The homogeneous graphs are (up to complementation)

- C_5, $K_3 \otimes K_3$ *(9 vertices)*
- $m \cdot K_n$ $(m, n \leq \infty)$
- *Generic K_n-free [Henson71]*

However, a structural Ramsey theorem requires an order . . .

Classification Theorem with Order

Theorem (Cherlin 2013)

The homogeneous ordered graphs are

- *Generic linear extensions of homogeneous partial orders with edge relation "comparability" (cf. [Schmerl79])*
- *Generically ordered homogeneous graphs (cf. [LachWood80])*
- *Generically ordered homogeneous tournaments with edges "$a \to b \iff a < b$" (cf. [Lachlan84])*
- *Homogeneous permutations (cf. [Cameron03])*

1. Origins
2. Universality and Homogeneity
3. Universality without Homogeneity

A Decision Problem

Survey: [KomjathPach91]

Narrowing the focus:

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Problem

$\mathcal{C}$: finite set of finite, connected, forbidden subgraphs

Is there a universal $\mathcal{C}$-free graph?

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Variant

Forbid induced subgraphs

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Variant

Forbid induced subgraphs

- More general
- Undecidable via Wang's domino problem
- for the brave . . .

What so special about SUBGRAPHS?

- Sample Theorems
- Conjectures
- Underlying Theory [CheSheShi97]

Sample Theorems

Who,When	What	Which
KMP88	Forbid a long path	$\exists$
"	No short odd cycles	"
ChShe07	Tree	path or near-path
ChShi96	Set of cycles	short odd cycles
ChSheShi97	Hom-closed set	$\exists$
FürKom97	2-connected	complete
Kom99,ChTal07	2 blocks	(K_m, K_n): $\min(m, n) \leq 5$ not $(5, 5)$!

Conjectures (1 Constraint)

Conjectures on existence of universal C-free graphs

1 (Solidity) Blocks of C should be complete

2 (Block-Path) After pruning trees, C should become a block-path

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1 (Solidity) Blocks of C should be complete

2 (Block-Path) After pruning trees, C should become a block-path

Theorem (ChShe, in prep)

If the constraint C is a block path, and a universal C-free graph exists then C has complete blocks.

Corollary

$(2) \implies (1)$

Methods

- Pruning
- Algebraic Closure (+ Füredi-Komjáth method)

Algebraic Closure

Non-Definition — a is $\mathcal{C}$-algebraic over X if forbidding $\mathcal{C}$ bounds the number of vertices like a.

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Example

Let $\mathcal{C}$ contain a star (i.e., we bound the vertex degrees). Then

- Obviously algebraic means *neighbor*
- Algebraic means *in the connected component*

Algebraic Closure

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Example (cont.)

- Forbidding C_4 makes a common neighbor unique. *This can be iterated.*
- Forbidding C, 2-connected but not complete, with a, b non-adjacent, makes $\bar{a}$ unique over $C \setminus \{a, b\}$, where $\bar{a}$ results by setting $a = b$.

Oligomorphic Universality

Theorem

Let $\mathcal{C}$ be a finite set of finite connected forbidden subgraphs with all blocks complete. Then the following are equivalent.

- *There is a universal $\mathcal{C}$-free graph with oligomorphic automorphism group;*
- *The algebraic closure of a vertex is always finite.*

The halting problem for the relation *obviously algebraic in*

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Example

If $\mathcal{C}$ contains a star, decidable:

- Algebraic closure = connected component
- Oligomorphic iff some path forbidden

Pruning

The first method of pruning:

- For a tree, remove its leaves.
- Generally, remove a minimal block-leaf (or more generally, a “corner”)

Lemma

If C prunes to C', then a universal C-free graph will contain a universal C'-free graph. So we may argue inductively.

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— And probably . . .

A tentative Result

Theorem (CheShe, in progress)

Let C be a block-path with $\ell \geq 6$ blocks, all complete, of sizes $m_i = |B_i| \geq 3$ all i, and allowing a universal C-free graph. Then up to reversal the sequence (m_i) is one of: $(4, 4, 3^*), (3, m, 3^*), (m, 3^*)$

Is the end in sight? Not yet —

A Problem for Graph Theorists

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Is this a problem for graph theorists?

Think about acl_C ...

Menger's theorem?