# Homogeneous Ordered Graphs

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# Abstract



**Theorem**

*All homogeneous ordered graphs are known.*

**Proof.**

[Cherlin1998, Chap. IV] — as modified in http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf. □

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# Homogeneity

**Definition (Urysohn, 1924, letter to Hausdorff)**

Any isomorphism between finite parts is induced by an automorphism.

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Homogeneous structures Γ $\iff$ Amalgamation Classes $\mathcal{A}$

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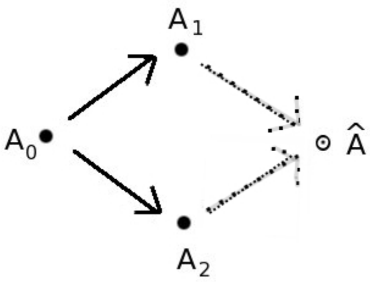

# Amalgamation and Homogeneity



**Remark (Fraïssé)**

*If $\Gamma$ is a homogeneous structure then the category $\mathrm{Sub}(\Gamma)$ of f.g. substructures has the amalgamation property and joint embedding.*

Proof.

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And conversely: The Fraïssé limit.

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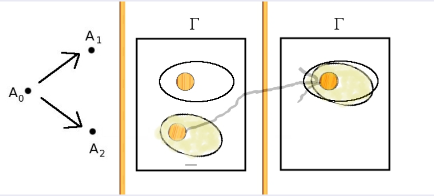


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# Examples



- The rational order $\mathbb{Q}$.
- The Random Graph $\Gamma_\infty$.
- The generic triangle-free graph $\Gamma_3$
- *The generically ordered version of any of the above*

The generically ordered order is the generic permutation.

Permutation: A structure with two orderings.

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# Structural Ramsey Theory





**Theorem (Ramsey)**

$$N \to (B)^A_k$$

*Given $A, B, k$ find $N$:*

*Coloring $\binom{[1,\ldots,N]}{A}$ makes some $B$-set $A$-monochromatic*

**Theorem Template (Structural Ramsey)**

$$\mathcal{N} \to (\mathcal{B})^{\mathcal{A}}_k$$

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Finite graphs, finite directed graphs, finite triangle-free graphs
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Finite orders, finite ordered graphs, finite ordered triangle-free graphs, finite metric spaces
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**Question (Bodirsky)**

*Does every finitely presented homogeneous structure in a relational language have a finite expansion with the Ramsey property?*

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$$\mathcal{L} \iff \mathbb{Q} \iff \mathrm{Aut}(\mathbb{Q}) \text{ (with topology)}$$

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KECHRIS/PESTOV/TODORČEVIĆ:

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# Linear Orders



**Remark**

*If $\mathrm{Aut}(\Gamma)$ is has fixed points on compact flows then $\Gamma$ has a definable linear order.*

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# From Homogeneity to Ramsey Theory?





Returning to

**Question (Bodirsky)**

*Given a homogeneous structure in a finite relational language, is there a homogeneous expansion with the same properties, and with a structural Ramsey theorem?*

What are some good test cases?

# Lessons of KPT



- Take ordered structures seriously.
- Take metric spaces seriously.

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## Remark on metrically homogeneous graphs

[Cherlin1998, Appendix]: 27 homogeneous structures with 4 nontrivial symmetric 2-types, not accounted for by general principles.

18 can be interpreted as metrically homogeneous,

3 are generic liftings of a metrically homogeneous graph of diameter 3 by generically splitting a type

6 remain unexplained.

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- Take metric spaces seriously.
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The present talk deals only with the first problem.

# Some Classifications



All the homogeneous structures of the following types (and others) have been classified.

- Partial Orders SCHMERL [1979]
- Graphs LACHLAN/WOODROW [1980]
- Tournaments LACHLAN [1984]
- *Directed Graphs* CHERLIN [1998]
- Homogeneous Permutations CAMERON [2003]
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But not all

# The Generic Local Order



Classification $\Longrightarrow$ exotic examples .

Lachlan's generic local order.

**Definition**

A *local order* is a tournament such that the in-neighbors and the out-neighbors of any vertex form a linear order.

**Theorem (Lachlan)**

*The infinite homogenous tournaments are*

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# Summing up



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Nguyen Van Thé's question:
Tthe classification of homogeneous graphs case is known,
can we add ordering?
And do we get exotic examples?

# Can we add ordering?—First Impressions



## Two Questions

- Is this hard?
- Is this easy?

# Looks Hard





**Question**

*Shouldn't we start with homogeneous ordered tournaments?*

- There are three infinite homogenous tournaments
- There are infinitely many homogenous graphs.
- Almost all can be *generically ordered* to give ordered versions

**Conjecture**

*The class of homogeneous ordered tournaments is simpler than the class of homogeneous ordered graphs.*

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**Proof.**

$$R \leftrightarrow S \triangle <$$

□

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**Remark**

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Disconcerting!

# So is this Hard?

Simplest guess:

- The homogeneous ordered tournaments are, mainly, the generically ordered homogeneous tournaments;
- The homogeneous ordered graphs are, mainly, the generically ordered homogeneous graphs.

Hopelessly false . . .

There are homogeneous ordered graphs which are not ordered homogeneous graphs!

What should the classification theorem say?

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# Why is this Easy?

Add order to previous proofs?

There are two proofs of the classification of the homogeneous graphs.

**Lachlan/Woodrow 1980** Introduced subtle inductive methods relating to amalgamation classes.

**Cherlin 1998, Chap. 4** A proof based on Lachlan's later classification of homogeneous tournaments.

The second proof unifies tournaments and graphs.

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# Cherlin 1998, Chap. 4

*“The proof given here is more complex than the one given [by Lachlan/Woodrow], but it generalizes . . . ”*

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## Objection

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# Objection Overruled



## Proposition

*A homogeneous ordered graph is*

- *A generically ordered homogeneous graph; or*
- *A generically ordered homogeneous tournament; or*
- *Something simpler (linear extension of partial order, equivalence relation with convex classes)*

# Strategy



Treat the generically ordered homogeneous tournaments as sporadic.

- Cameron treated linear expansions of $\mathbb{Q}$ (homogeneous permutations).
- The generically ordered random tournament is the generically ordered random graph!
- This leaves only the generically ordered local order $\mathbb{S}$ to be captured by other methods.

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# Graphs vs. Tournaments





**Corollary**

*The classification of homogeneous tournaments with trivial* acl *follows from the classification of homogeneous ordered graphs.*

**Proof.**

Generically order the tournament and view it as a homogenous ordered graph. □

# The three cases



Suppose that the graph contains an infinite independent set.

| | |
|---|---|
| Special | Omits some ordered form of the 3-cycle $C_3$. |
| *Target:* | Homogeneous permutations, Linear extensions of partial orders |
| Sporadic | Realizes both ordered forms of $C_3$ ($\vec{P}_3$, $\vec{P}_3^c$) and $\vec{I}_\infty$, but omits $\vec{I}_1 \perp \vec{P}_3$. |
| *Target:* | Generically ordered $\mathcal{S}$. |
| Generic | Realizes $\vec{I}_1 \perp \vec{P}_3$, $\vec{P}_3^c$, $\vec{I}_\infty$. |
| *Target:* | Generically ordered Henson graph |

http://www.math.rutgers.edu/~cherlin/Paper/HomOG3.pdf.

# Conclusion





**Theorem**

*All homogeneous ordered graphs were known before the classification was undertaken*

Essentially, since 1984.

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# Generalizations and related questions



**Question**

*Classify the homogeneous structures $(\Gamma, S, R)$ where $(\Gamma, R)$ is a graph and $(\Gamma, S)$ is a local order.*

Note: The generic local order has a particularly subtle expansion to a Ramsey class.

**Question**

*Classify the homogeneous partially ordered graphs.*

**Question (Cameron)**

*Classify the homogeneous metrically homogeneous graphs.*

**Question**

*Work out the structural Ramsey theory for suitably ordered versions of the known metrically homogeneous graphs.*