CLASSIFYING HOMOGENEOUS STRUCTURES III

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Abstract. On metrically homogeneous graphs.

1. The Problem

Definition 1.1. A connected graph is *metrically homogeneous* iff when considered as a metric space in the graph metric, it is homogeneous.

Example 1. An n-cycle; a regular tree.

Remark 1.2. *A homogeneous metric space is derived from a metrically homogeneous graph iff the following hold.*

- *The metric is $\mathbb{Z}$-valued.*
- *If the distance δ occurs, then a geodesic path of length δ occurs.*

Remark 1.3. *If Γ is a bipartite graph which is homogeneous as a metric space with bipartition, then each part of Γ is also a metrically homogeneous graph with edge relation $d(x, y) = 2$.*

Lemma 1.4 (Macpherson)**.** *Let $T_{r,s}$ be an s-regular tree of r-cliques. Then $T_{r,s}$ is metrically homogeneous.*

Proof. Let $T(r, s)$ be an (r, s)-regular bipartitioned tree. Then $T(r, s)$ is homogeneous as a metric space with bipartition. To see this, we have to check that the convex closure of a finite set can be computed from the metric structure. For example, if v_1, v_2, v_3 are points at distances d_1, d_2, d_3 from a common center v, then the perimeter of the triangle (v_1, v_2, v_3) is $2(d_1+d_2+d_3)$, so the metric information gives us d_1, d_2, d_3 and we can locate the center.

The induced metrically homogeneous graphs on the parts are $T_{r,s}$ and $T_{s,r}$. □

Remark 1.5.

1. If G is a connected graph of diameter at most 2, then G is metrically homogeneous iff G is homogeneous.

2. If G is metrically homogeneous and $v \in G$, then the connected components of the induced graph $\Delta_i(v)$ are metrically homogeneous. In particular $\Delta_1(v)$ is a homogeneous graph, hence finite, imprimitive, or (up to complementation) a Henson graph.

Problem ([Cam98, Che11]). *Classify the metrically homogeneous graphs.*

2. A catalog

The finite metrically homogeneous graphs were classified by Cameron (1977) as follows.

- The finite homogeneous graphs;
- The cycles;
- In diameter 3: the antipodal double of an independent set, a 5-cycle, or $E(K_{3,3})$.

Conjecture 1. *The metrically homogeneous graphs are the following.*

- *Connected homogeneous graphs* ($\delta \leq 2$).
- *Finite antipodal graphs of diameter 3*
- *Tree-like graphs* $T_{r,s}$ *(*Γ_1 *is finite or imprimitive, or* $r = 2$*,* $s = \infty$*)*
- *Komjáth-Mekler-Pach/Henson graphs* $\Gamma^\delta)_{K,C;\mathcal{H})}$*—to be explained*
- *One further variation* $\mathcal{A}^\delta_{ap,n}$

3. The Henson construction

A clique (or simplex) is a set of vertices at mutual distance 1. A $(1, \delta)$ space is a set of vertices at distances 1 or δ.

If $\delta \geq 3$ is the diameter, and $\mathcal{H}$ is a set of $(1, \delta)$-spaces, we denote by $\mathcal{A}^\delta_\mathcal{H}$ the class of $\mathcal{H}$-free metric spaces of diameter δ.

Lemma 3.1. $\mathcal{A}^\delta_\mathcal{H}$ *is an amalgamation class.*

More precisely, the range of possible values r for $d(u, v)$ over the base A is $d^- \leq r \leq d^+$ with

$$d^- = \max(i - j \mid i = d(u, x), j = d(x, v))$$
$$d^+ = \min(i + j \mid i = d(u, x), j = d(x, v))$$

Here $d^- < \delta$ and $d^+ > 1$ so the values $1, \delta$ may be avoided.

4. The Komjáth-Mekler-Pach construction

[KMP88]: There are countable universal $\mathcal{C}$-free graphs where $\mathcal{C}$ consists of

- All odd cycles with length below some bound $2K + 1$; or
- All cycles of girth at length C

We will deal with the perimeters of triangles rather than the lengths of cycles. We define the following classes of triangles depending on four numerical parameters K_1, K_2, C_0, C_1.

- Even perimeter at least C_0;
- Odd perimeter at least C_1;
- Odd perimeter below $2K_1 + 1$;
- Odd perimeter above $2(K_2 + i)$ where i is an edge length.

Definition 4.1.

With $K = (K_1, K_2)$ and $C = (C_0, C_1)$, $\mathcal{A}^\delta_{K,C}$ is the class of finite integral metric spaces of diameter at most δ, omitting the triangles of type (K, C).

When $\mathcal{A}^\delta_{K,C}$ is an amalgamation class, then $\Gamma^\delta_{K,C}$ is the associated metrically homogeneous graph. We call this KMP-type.

Proposition 4.2. *If a non-exceptional metrically homogeneous graph corresponds to an amalgamation class given by triangles, then it is a KMP-type graph.*

We may also pass to $\mathcal{A}^\delta_{K,C;\mathcal{H}}$ by combining KMP and Henson constraints, under mild conditions on $\mathcal{H}$.

But we have not yet addressed the following.

Question 1. *What conditions on δ, K, C correspond to having an amalgamation class?*

Some a priori considerations: the property of amalgamation can be stratified by the size of the amalgamation diagram. Let A_k denote the amalgamation property up to size k.

Lemma 4.3. *For each k, there is a set of linear inequalities and congruences on the parameters δ, K, C which corresponds to the property A_k.*

Proof. Note that the class of forbidden triangles associated with δ, K, C is a uniformly definable family of relations in Presburger arithmetic, and hence A_k is a definable property in Presburger arithmetic. Apply quantifier elimination. $\square$

Corollary 4.4. *The following are equivalent.*

- *The amalgamation property for $\mathcal{A}^\delta_{K,C}$ is equivalent to A_k for some fixed k;*
- *The amalgamation property for $\mathcal{A}^\delta_{K,C}$ is equivalent to a finite combination of linear inequalities and congruences on the parameters.*

We now exhibit such a set of conditions.

Amalgamation

$\delta \geq 2$; $1 \leq K_1 \leq K_2 \leq \delta$ or $K_1 = \infty$, $K_2 = 0$;
C_0 even, C_1 odd; $2\delta + 1 \leq C_0, C_1 \leq 3\delta + 2$

and

(I) $K_1 = \infty$ and $K_2 = 0$, $C_1 = 2\delta + 1$; if $\delta = 2$ then $C' = 8$;

or

(II) $K_1 < \infty$ and $C \leq 2\delta + K_1$, and
- $\delta \geq 3$;
- $C = 2K_1 + 2K_2 + 1$;
- $K_1 + K_2 \geq \delta$;
- $K_1 + 2K_2 \leq 2\delta - 1$

(IIA) $C' = C + 1$ or

(IIB) $C' > C + 1$, $K_1 = K_2$, and $3K_2 = 2\delta - 1$;

or

(III) $K_1 < \infty$ and $C > 2\delta + K_1$, and
- If $\delta = 2$ then $K_2 = 2$;
- $K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$;
- If $C' > C + 1$ then $C \geq 2\delta + K_2$.

Method of proof: in one direction, give an explicit amalgamation procedure. In the other direction, give many explicit amalgamation arguments, involving diagrams of order 4 or 5.

As a corollary: amalgamation is equivalent to A_5.

5. Antipodal variations

Definition 5.1. A graph of finite diameter δ is *antipodal* if for every vertex v there is a unique vertex v' with $d(v, v') = \delta$.

In some contexts, one requires only that the relation $d(x, y) \in \{0, \delta\}$ should be an equivalence relation.

In the context of metrically homogeneous graphs, the antipodal graphs are the ones with no triangle of perimeter greater than 2δ. They satisfy the useful *anitpodal* *law*

$$d(u, v') = \delta - d(u, v)$$

These graphs are in our catalog as KMP-type, but the Henson variations are unusual. We let $\mathcal{A}^{\delta}_{ap,n}$ denote the downward closure of the class of antipodal graphs with no n-clique. In intrinsic terms, the conditions are the stated bound on perimiters together with the omission of all pseudo-cliques (A, B) on n-points; here A, B are cliques and the distance between points of A and B is exactly $\delta - 1$. By the antipodal law, if we wish to omit an n-clique then we must omit these pseudo-cliques as well.

6. Evidence for the Conjecture

We collect a number of prior results with more recent work, some in collaboration with Amato and Macpherson, as follows.

Theorem 1. *Any metrically homogeneous graph* Γ *not in the catalog satisfies the following two conditions.*

- Γ_1 *is primitive;*
- *Any two vertices at distance* 2 *have infinitely many common neighbors*
- *The diameter is at least* 4*.*

We refer to the first two conditions as *generic type*. This can be written in a more explicit way, in terms of the following breakdown.

- Γ_1 is a Henson graph or random graph; or
- Γ_1 is an independent set and each vertex of Γ_2 has infinitely many neighbors in Γ_1.

There is no detailed plan of attack for the conjecture, but the natural approach is to proceed by induction on diameter when δ is finite, and to get the classification in infinite diameter either directly from the finite case, or from its proof.

In more detail one wants the following.

- For δ finite, show inductively:
  - The constraints on triangles correspond to the anticipated amalgamation class;
  - Given specified constraints on triangles, any configuration omitting the Henson constraints embeds;
- For δ infinite, show that the classification follows from the classification for δ finite (this can be attacked directly).

For the diameter 3 classification [AChMc13] we followed this plan.

Hubička's language: The parity metric
$d_2(u, v) = (d_o(u, v), d_e(u, v))$ gives the shortest walk of odd (resp. even) length.

This also has amalgamation with bounds d^-, d^+ similar to the metric case. But it does not seem that a bound on diameter necessarily bounds the sizes of these numbers.

Problem (Hubička)**.** *What are the diameter 2 graphs which are homogeneous for the parity metric?*

Note: $d_e(u,u) = 0$ but $d_o(u,u)$ may be a nontrivial function. We may wish to require this to be constant.

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