GENERIX BEGINS: MIXED PUNCHLINES AND MILKSHAKES

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Contents

These notes have been prepared on the occasion of the conference at Lyon in memory of Éric Jaligot, June 25–26, 2014. They are rough notes, and will not be revised.

I will be discussing some early work of Éric Jaligot (and related work, both previously and subsequently). As this work was motivated for the most part by the algebraicity conjecture for groups of finite Morley rank, I will first place that in context. Then I will present the leading ideas of one of the main results in his thesis in some detail, and comment more briefly on the second main result of that thesis, which is probably more difficult, and certainly more elaborate, as well as broader in its implications. This work led ultimately to the complete classification of groups of even type and the elimination of mixed type.

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I will also discuss work that we did together (and in part with Burdges) on minimal simple groups, and some of the aftermath.

We first touch on some points of history.[1]

1. Morley, Marsh, Baldwin/Lachlan, Zilber; and Macintyre

Theorem 1 (Łoś Conjecture, Morley Categoricity Theorem)**.** *A countable theory T is categorical in one uncountable power if and only if it is categorical in all uncountable powers.*

Here Morley introduced his notion of Morley rank, an ordinal-valued measure of the size of a definable set, and the integer-valued Morley degree, which is the number of disjoint definable subsets of maximal rank,

Marsh, in his doctoral thesis, discussed sets of Morley rank and degree 1 under the name *strongly minimal sets.* Theories whose models are algebraic over a fixed strongly minimal set are said to be *almost strongly minimal.* These are the simplest kinds of uncountably categorical theories, and in studying the countable models of uncountably categorical theories, Marsh wrestled with the gap between the two notions. A little later Baldwin and Lachlan managed to overcome the gap, and then analyzed the situation further. The simple form of Zilber's result is the following.

Theorem 2 (Zilber)**.** *Let T be an uncountably categorical theory which is not almost strongly minimal. Then T interprets an infinite group of finite Morley rank, which may be taken to be abelian or simple.*

The issues that arise in the presence of group actions are illustrated in their most basic form by the example of a vector space and the corresponding affine space. Naming a point in the affine space allows us to identify it with the corresponding vector space: but it is necessary to name this point, and any point will do. This is reflected more concretely by the following: the group of automorphisms of affine space which induce the trivial action on the underlying vector space is the translation group, and infinite abelian group which is interpretable in the structure.

This interpretability of this relative automorphism groupis striking, if one compares it with the noninterpretability of the automorphism group of the vector space (or the noninterpretability of the full symmetric group in the theory of equality).

The abelian group $A = \mathbb{Z}/p\mathbb{Z}^{(\infty)}$ (an infiniite direct sum) is uncountably categorical. Here the set $A[p]$ of elements of order p is a vector space over $\mathbb{F}_p$ and the p-th roots of a given nontrivial element of $A[p]$ form a parametrized family of affine spaces over that vector space.

One may then proceed to investigate further abelian or simple groups of finite Morley rank. On the simple side, Zilber showed the following.

Theorem 3 (Zilber)**.** *Let G be a simple group of finite Morley rank. Then G is almost strongly mimimal.*

[1] A few of the necessary definitions not given in the text will be found in the Appendix.

Some deeper conjectures on the nature of strongly minimal sets suggest the following.

Conjecture 1 (Algebraicity)**.** *A simple group of finite Morley rank is isomorphic to the group of F-points of an algebraic group, for some algebraically closed field F.*

Another route toward this conjecture leads through Macintyre's theorem.

Theorem 4. *An $\aleph_0$-stable field is algebraically closed.*

That is the route I took, and accordingly I put the Algebraicity Conjecture in the $\aleph_0$-stable context.

Later we learned from Berline/Lascar that often the most satisfactory way to generalize the finite Morley rank theory to infinite rank is to pass directly to the more general superstable setting. This remains relevant to applications. For example, differentially algebraic groups (relative to finitely many commuting derivations) are $\aleph_0$-stable but the natural model theoretic context may be the theory of superstable groups (cf. work of Freitag).

And of course there have been further generalizations, sometimes with an eye toward very concrete applications (and frequently involving a structural analysis in the abelian case, of a more purely model theoretic nature).

2. The Borovik Program: CFSG and tameness

2.1. **The setting: tame or not.** Borovik proposed a program to explore the extent to which ideas used successfully in showing that most finite simple groups are algebraic (for short, CFSG) are applicable to the algebraicity conjecture for groups of finite Morley rank. One distinctive feature of the proposal was to allow free use of a *no bad fields* hypothesis. Here a bad field is a field of finite Morley rank in which there is a proper, infinite, definable subgroup of the multiplicative group. The naive version of the proposal would be to assume the nonexistence of bad fields, which was not very palatable then, and even less so now that their existence has been proved. However the relevant assumption was only that the groups under consideration did not interpret bad fields (and one could restrict this further, to the obvious sort of group theoretic interpretation, with however some effect on the force of the hypothesis).

The thinking here was that the project would be large enough even under this substantial restriction, and that if and when it reached some interesting stage of development, one could always reexamine the issues not previously addressed. In practice the process of reassessment took place very rapidly, with Jaligot's thesis the first notable step in that process. But throughout this 2-step approach has remained valuable.

Groups not involving bad fields (nor bad groups, which will not interest us here) are called *tame.* In addition, as one is dealing with various possible minimal counterexamples to the algebraicity conjecture in an inductive setting, the following terminology was taken over from the finite case.

Definition 2.1.

1. A K-group, or *group of known type*, is one whose definable infinite simple sections are algebraic.

2. A K^*-group is one whose proper definable sections are K-groups; and practically speaking we generally have in mind a simple group whose proper definable subgroups are K-groups, in other words a potential minimal counterexample to the algebraicity conjecture.

Eventually the program came to incorporate ideas of two kinds that were not part of the original CFSG.

- Some finite group theory (the amalgam method) that proposed an alternative approach to the classification problem, but appeared too late to play much of a role in the original proof;
- Other more geometric ideas which relate more naturally to the theory of algebraic groups.

And under the second heading, the topic also drifted to some extent from purely classification theoretic results to somewhat softer structural results. Here there is no clear line to be drawn, but there are discernible variations in taste. The work I'll be talking about was certainly part of a direct attack on the classification program but Éric was equally interested in a more conceptually motivated theory.

2.2. **The basic framework: even, odd, mixed, degenerate.** The starting point for the discussion is a consideration of the 2-Sylow subgroups of connected algebraic groups (or p-Sylow subgroups, but we soon need to specialize). This depends on the characteristic of the base field.

Characteristic	Type	Algebraic properties	Model theory
$= 2$	unipotent	bounded exponent, nilpotent, algebraic, connected	definable
$\neq 2$	semisimple (toral)	up to finite index, divisible, and dense in a maximal torus	not definable

Since the product of two algebraic groups is also a group of finite Morley rank, we have instances where the 2-Sylow structure is a mixture of unipotent and semisimple type.

This sets the stage for a model theoretic result.

Theorem 5 (Borovik-Poizat)**.** *The 2-Sylow subgroup of a group of finite Morley rank has a subgroup of finite index which is a central product (with finite intersection)*

$$U * T$$

where U is definable, connected, nilpotent, of bounded exponent, and T is abelian, divisible.

Accordingly we say that G is of even type if $S = U \neq 1$, and of odd type if $S = T \neq 1$, accepting the convention that 0 is odd. The remaining cases, $S = 1$ or both factors nontrivial, are the *degenerate* and *mixed* cases. The degenerate case remains recalcitrant, probably for a very good reason: one challenge is to understand whether there is an analog to $\mathrm{SO}(3, \mathbb{R})$ (without involutions, perhaps torsion-free) in this context.

The project was launched by Borovik's work on locally finite groups of odd type and Altınel's thesis on even type, all under the tame hypothesis. Jaligot's thesis took up the problem of removing tameness on the mixed and even sides, and later Burdges' thesis took up the corresponding problem on the odd side.

The project presupposed a substantial body of work on groups of finite Morley rank in general, much of it presented in the book of Borovik and Nesin. That investigation, which is by no means limited to the needs of the classification project, remains vigorously active, and indeed some byproducts of the classification project have fed back into the broader theory.

In particular, the study of minimal simple groups relies on a well developed theory of solvable groups to rely on, which was not fully in place, but advanced rapidly in the contemporary work of Frécon. Later Jaligot and I had occasion to request a "tailor-made" result from Frécon, which according to my recollection came more or less by return mail.

3. Mixed type

Now I turn to the first major result of Jaligot's thesis.

Theorem 6 (Jaligot)**.** *Let G be a simple K^*-group of finite Morley rank. Then G cannot have mixed type.*

This was proved under the tame hypothesis by ABC, and later proved outright (there is no simple group of finite Morley rank of mixed type), again by ABC. The latter proof requires one to first give a classification of the even type groups. This turns a portion of the K^* hypothesis into a theorem. However, one would not expect this to be enough. The key insight is in Altınel's habilitation, which suggests an approach to the even type and mixed type analyses which is inductive but only with respect to even type sections.

Both Jaligot's thesis and, later, Altınel's habilitation were exciting and stimulating developments which suggested great possibilities for the future: that is, that the technical obstacles that we were aware of standing in the way of very general results were potentially quite manageable.

3.1. Strong and weak embedding.

The general approach to the elimination of the mixed type case was inspired by ideas of finite group theory, notably the notion of *strong embedding,* which we now define.

Adapted to the finite rank context, strong embedding bifurcates into two notions, which we call respectively strong and weak embedding. They may be defined as follows.

Definition 3.1. If G is a group of finite Morley rank and M is a definable subgroup containing a Sylow 2-subgroup of G, then we say

- M is *strongly embedded* if M contains the normalizer of each nontrivial 2-subgroup of M;
- M is *weakly embedded* if M contains the normalizer of each nontrivial connected subgroup of M

Strong embedding represents an obstacle to using M to get more information about G, and weak embedding represents a first try at strong embedding. We will see this play out in a moment in the mixed type case.

The following points should be borne in mind (or at least, referred to as needed).

- if G has a strongly embedded subgroup, then all of its involutions are conjugate

- For weak embedding, it is enough to check the condition $N(P) \le M$ for two basic types of nontrivial connected 2-subgroup P of M:

$$P = U \text{ unipotent, } P = T \text{ divisible abelian}$$

- If M is weakly embedded but not strongly embedded in G then there are involutions $i \in M$ whose centralizer $C(i)$ is not contained in M; these involutions, which must be closely studied, are called *offending involutions*

3.2. **Toward weak embedding:** $\mathcal{U}(G)$**.** The following graph theoretical point of view is also inspired by the approach of finite group theorists.

Definition 3.2. Let G be a group of finite Morley rank. The graph $\mathcal{U}(G)$ has as its vertices the nontrivial 2-unipotent subgroups of G (i.e., the nontrivial definable, connected, nilpotent 2-subgroups of bounded exponent); the edges are pairs (U, V) which commute.

Here the graph $\mathcal{U}(G)$ is equipped with the G-action given by conjuation.

According to the algebraicity conjecture, for simple groups G $\mathcal{U}(G)$ should usually be connected, the exception being the case $G = \mathrm{SL}_2$ over an algebraically closed field of characteristic 2.

The graph $\mathcal{U}(G)$ gives rise to a very appropriate case division: if this graph is connected, we have the means to relate group theoretic information associated with different unipotent subgroups by moving along in stages within the graph. If the graph is disconnected, we have excellent chances to arrive at a weakly embedded subgroup, and from there to move to the strongly embedded case.

Lemma 3.3 (Model Lemma, under various hypotheses)**.** *Let G be a simple group of mixed type with $\mathcal{U}(G)$ disconnected, and let $\mathcal{U}_0(G)$ be a connected component of the graph, M the stabilizer of $\mathcal{U}_0(G)$. Then M is definable, contains a Sylow 2-subgroup of G, and $\mathcal{U}(M) = \mathcal{U}_0(G)$.*

The main point is that the condition $\mathcal{U}(M) = \mathcal{U}_0(G)$ gives us half of the criterion for weak embedding: namely, for $U \le M$ nontrivial and unipotent, the group $N(U)$ is the stabilizer of U, and since $U \in \mathcal{U}_0(G)$, this is contained in M.

We are just about ready to lay out the general strategy for the elimination of mixed type, under various hypotheses. But we need one more group theoretic notion.

3.3. **The subgroups** $B(G)$ **and** $D(G)$**.**

Definition 3.4. Let G be a group of finite Morley rank. Then

- $B(G)$ is the subgroup generated by the 2-unipotent subgroups of G (definable, connected, nilpotent, bounded exponent 2-subgroups).
- $D(G)$ is the definable subgroup generated by 2-tori, that is, the smallest definable subgroup containing all diviisible ableian 2-subgroups of G.

Now if G is a simple group of mixed type, then $G = B(G) = D(G)$, but this is not helpful. If G is a K-group one can see that $B(G), D(G)$ commute and that $B(G)$ is of even type, $D(G)$ of odd type, and so we need to show something similar in the general mixed type case to contradict the minimality of G. The natural setting here is that G should be a K^*-subgroup and we consider $B(H)$, $D(H)$ for various proper subgroups. We expect in particular that $D(G) = D(C(B(G))) = D(C(U))$ for each nontrivial unipotent subgroup U, which at a minimum inspires us to take a good look at $D(C(U))$.

In fact, the theory usually begins with a very formal study of the operators

$$B^\perp(X) = B(C(X)) \text{ and } D^\perp(X) = D(C(X))$$

but we will forego this here. But it is good to be aware that these are the main objects of study.

3.4. **The strategy.** Now we can lay out a "mixed type strategy" which can be carried out at various levels of generality. The aim is to prove that there can be no simple group G of mixed type, and it is best to imagine that G is a K^*-group for now.

Also, we should notice at the outset that the involutions in a group of mixed type are not all conjugate. This is elementary but must be checked. (One does not expect a "unipotent" involution (in a 2-unipotent subgroup) to be conjugate to a "semisimple" involution (in a 2-torus), but it can happen occasionally.) This means that if we reach a strongly embedded subgroup than rather than hitting a brick wall, we will be done.

One proceeds as follows.

- If $\mathcal{U}(G)$ is connected, show that $DC(U)$ is independent of the choice of U, hence normal, nontrivial, and proper in G, for a direct contradiction (this is life as it should be);
- If $\mathcal{U}(G)$ is disconnected, focus attention on the subgroup M stabilizing a connected component. We expect M to be strongly embedded, which would give an immediate contradiction. But we have to get there. So we still have to consider how one will get through the two intermediate cases.
  1. If M is not weakly embedded, then there is a nontrivial 2-torus T contained in M with $N(T)$ not contained in M.
  2. If M is weakly embedded but not strongly embedded then there is an *offending involution* $\alpha \in M$ whose centralizer escapes from M.

In both of the problematic configurations, there will be very sharp information under the K^* hypothesis, and in particular a canonical copy of the group $Q = \mathrm{PSL}_2$ over some algebraically closed field of characteristic 2 will appear, meeting M in a Borel subgroup, at which point a great deal of concrete information becomes available. We will need to push both configurations toward some contradiction. Here the group is beginning to look unlike an actually existing group so intuition becomes less useful. (It is well known that in the finite case a certain number of sporadic groups were proved nonexistent before being proved to exist.)

From this point on, details may vary somewhat according to context. Strategy is well enough, but there is always the question as to whether it can be carried through.

We learned how to complete this sketch by stages, learning something new at each stage.

ABC: the K^*-case, with tameness
Jaligot: the K^*-case

AC: the general case

So after [ABC] we had a sense of the general strategy as well as the kinds of specific configurations that arise along the way. Jaligot's method for treating these configurations is surprisingly direct, and relatively uniform. So we will focus on seeing what the two crucial configurations are, and how Jaligot drives them both toward the same punch line.

First, a word about the subsequent elimination of the K^*-hypothesis. One weakens this to the hypothesis that there is no section of mixed type (which is simply a minimality condition) together with the hypothesis that ever section of even type is algebraic, i.e. roughly half of the K^*-hypothesis, or a bit less. Then what one has to do is to carry out the even type classification somehow, also using only an inductive hypothesis on sections of even type. This latter does not seem very plausible, but Altınel suggested the main principles on which such an argument could plausibly be carried out, in his habilitation. So "the general case" is really, in the first instance, a somewhat generalized K^*-case, and then later through quite independent work is promoted to a completely general result.

In particular according to this plan Jaligot's thesis remains the model for what actually needs to be done in the mixed case to prove the result in general.

3.5. **The curious incident of the dog in the night.** Another powerful idea from finite group theory which serves very well to eliminate configurations which otherwise appear intractable is the *Thompson rank formula,* which is the rank-theoretic analog of a formula for group order in the finite case. This formula requires more than one conjugacy of involutions, but as we have seen that hypothesis will certainly be satisfied in the mixed type case. It is also preferable to have a finite number of conjugacy classes of involutions, and we will give it in that form. The starting point is the following.

Lemma 3.5. *Let i, j be non-conjugate involutions and suppose that the definable group $D = d(i, j)$ generated by i, j (i.e., the definable hull of the group generated by i, j) contains no nontrivial 2-torus. Then there is a unique involution k in the definably cyclic group generated by ij.*

Here the group D is a generalized dihedral group $C \cdot \mathbb{Z}/2\mathbb{Z}$ where C is definably cyclic, generated (as a definable group) by the element ij, and C is inverted by i (or j). It is reasonable that a definably cyclic group would contain at most one involution, but we do have to avoid nontrivial 2-tori to achieve this.

Suppose that Γ_1, Γ_2 are distinct conjugacy classes of involutions and that the lemma is applicable to pairs $(i, j) \in \Gamma_1 \times \Gamma_2$. One arrives at a definable map $\tau : \Gamma_1 \times \Gamma_2 \to I$, $\tau(i, j) = k$, with I the set of involutions in G, and if there are finitely many conjugacy classes of involutions then one restricts to a generically defined partial map

$$\tau' : \Gamma_1 \times \Gamma_2 \to \Gamma_3$$

connecting three conjugacy classes of involutions, and to the corresponding rank formula

$$\gamma_1 + \gamma_2 = \gamma_3 + f$$

with γ_i the rank of Γ_i and f the *fiber rank,* that is the rank of $f^{-1}(k)$. The ranks γ_i are conveniently expressed as $g - c_i$ with $g = \mathrm{rk}\,(G)$ and $c_i = \mathrm{rk}\,(C(t))$ for $t \in \Gamma_i$. When rewritten in this manner we arrive at the Thompson rank formula

$$g = c_1 + c_2 - c_3 + f$$

where c_1, c_2 are under our control in the sense that we may make a convenient choice of Γ_1, Γ_2, and c_3 is not under our control, and all possibilities require close consideration, though some may perhaps be quickly eliminated. Finally there is the fiber rank f, but since the relation $\tau(i,j) = k$ implies $i, j \in C(k)$, this again involves a consideration of the structure of the centralizer of k. So all the data occurring here may be considered to be local in the sense that they involve the structure of the centralizer of various involutions.

Perhaps surprisingly, in cases where this analysis can be carried through to a reasonably definite value for g, it tends to be violently inconsistent with what we already know. In other words, nonexistent groups are bad liars.

So now we have a full picture of the strategies and resources Jaligot had in hand when he took up the mixed type problem. Remarkably, the Thompson rank formula makes no appearance in his argument, because the analysis of involutions collapses at an early stage—for the same reason in both of the principle cases (failure of weak embedding, or weak embedding without strong embedding).

So we aim to describe in general terms how Jaligot's key configuration appears in each of two cases.

3.6. **Weak embedding.** Let G be a K^*-group of mixed type with a subgroup M containing a Sylow 2-subgroup of G, with M is halfway weakly embedded in the sense that $N(U) \le M$ for all nontrivial 2-unipotent subgroups of M, but not weakly embedded: so there is a nontrivial 2-torus T in M such that $N(T)$ escapes from M. Set

$$Q = B(N(T))$$

Now $N(T)$ is a K-group and Q is a subgroup of even type, while $Q \cap M$ is a weakly embedded subgroup of Q, since Q contains no 2-tori. But an algebraic group of even type with a weakly embedded subgroup is SL_2 in characteristic 2.

Now in the decomposition of a Sylow 2-subgroup of G (or M) as

$$U * T$$

we have $U \le Q$ and so the structure of U is very clear: it is a copy of the additive group of a field, and all involutions in U are conjugate under the action of the diagonal subgroup.

So we have a single well-defined conjugacy class I_u of involutions of unipotent type. We also consider some conjugacy class I_t of involutions of toral type. It turns out that our present configuration leads quite quickly to the following very reasonable sort of contradictoin.

$$I_u,\ I_t \text{ commute}$$

Since I_u, I_t are conjugacy classes of the simple group G, this is a blatant contradiction.

So suppose $i \in I_u$, $j \in I_t$ do not commute. We may suppose that $i \in U$.

Before describing the next stage of the analysis it is best to record a technical lemma which gives very great leverage throughout.

Fact 3.6. $C_G(i) = C_G(U)$

A few words on the proof. The fact that we have an abelian group (the diagonal subgroup of Q) acting transitively on U is essential. And we also need the computation $BC(i) = U$ which follows from the general theory of $B(\cdot)$ and $D(\cdot)$ in K-groups, and gives us the inclusion $C(i) \subseteq N(U)$.

Now the first few steps of the analysis are as follows, with an eye still on the Thompson rank formula.

(1) The definable group C generated by ij has no nontrivial 2-torus, so there is a unique involution k in C.
(2) $BC(k)$ is again a copy of SL_2 over a field of characteristic 2.
(3) $i \in BC(k)$, $jk \notin BC(k)$

For the first point: as C is inverted by i, if C contains a nontrival 2-torus T_0 and t_0 is an involution in T_0, then i centralizes t_0 and hence $U \leq C(t_0)$. But this puts U into $BC(t_0)$ and T_0 into $D(C(t_0))$, which commute: and then i commutes with T_0 for a contradiction.

For the second point: since $i \in C(k)$, also $U \leq C(k)$, so $U \leq BC(k)$. To eliminate the possibility $BC(k) = U$ we use the hypothesis that j does not centralize U. It follows easily that j does not normalize U, as otherwise it would centralize something, and then all of U. Now it is quite easy to deduce that the conjugates of U inside $BC(k)$ are pairwise disjoint and to deduce that $BC(k)$ is SL_2.

With the third point we approach the crucial configuration, so let us take our time here.

We have noticed $U \leq BC(k)$ and so $i \in BC(k)$. Now j is not conjugate to an element of U so certainly $j \notin BC(k)$.

The case of jk is more delicate. If $jk \in BC(k)$, then jk lies in a conjugate U_1 of U. But j centralizes jk and hence also U_1. But j lies in a 2-torus T, and as $T, U_1 \leq C(j)$ find that T and U_1 commute. So T commutes with jk and thus with k. At this point T commutes with $BC(k)$ and hence we i and we contradict our initial assumption on i, j.

But now one has only to realize that we have arrived at a contradiction (Jaligot's mixed type punchline).

Lemma 3.7. *Let G be a group of finite Morley rank with involutions i, j, k such that*

- *k is the unique involution in the definably cyclic group C generated by ij;*
- *$BC(k) \cong \mathrm{SL}_2$ in characteristic 2*
- *$i \in BC(k)$, $j \notin BC(k)$*

Then $jk \in BC(k)$.

The next time we encounter this lemma the statement will be a little more complicated: the role of k will be divided between two involutions k and k', and as a result some things which are obvious in the present formulation will need to be added explicitly.

Proof. The main point is

$$(ij)^2 \in BC(k)$$

This follows from $(ij)^2 = ii^j$.

Let $C_0 = C \cap BC(k)$ and $\overline{C} = C/C_0$. Then ij represents an involution in $\overline{C}$ and is therefore of the form $\overline{ij} = \overline{y}$ with y a nontrivial 2-element. If y is not an involution then some power of y^2 will be k, forcing k into $BC(k)$, which is nonsense. So $y = k$.

That is,

$$ijk \in BC(k)$$

and the claim follows. □

3.7. **Strong Embedding.** Let us see if we can find the more elaborate version of Jaligot's configuration in the configuration corresponding to failure of strong embedding. Here our starting point is a weakly embedded subgroup M of G and an *offending involution* $\alpha \in M$ with

$$C(\alpha) \not\leq M$$

One arrives quickly at a clear picture of the associated configuration.

- $C(\alpha)^\circ = Q \times H$;
- $Q = BC(\alpha) \cong \mathrm{SL}_2$; $Q \cap M$ is a Borel subgroup;
- H contains no involution

Here the main point is that if $C(\alpha)$ is to escape from M then $B(C(\alpha))$ must already escape from M.

Now there are three kinds of involutions which merit attention: the toral involutions (belong to some 2-torus), the offending involutions of M, and the corresponding involutions of unipotent type in $BC(\alpha) \cap M$. Let us call the latter involutions SL_2*-type*. We will use the interplay among these involutions to force all toral involutions into M, and thereby contradict the simplicity of G.

Let us suppose from the beginning that

$$j \text{ is a toral involution outside } M$$

Lemma 3.8. *If i is an involution of SL_2-type in M, then there is a unique involution k in the definably cyclic group C generated by ij, and it is an offending involution of M.*

Proof. There is less here than meets the eye. Let k be any involution commuting with i, j. Then since i is not an offending involution, and $j \notin M$, it follows that k is in M and is an offending involution. But then $k \notin C(k)^\circ$ so k is not toral.

In particular C has no nontrivial 2-torus and everything is clear. □

So now we consider the following involutions.

- i_0: some involution in M of SL_2-type;
- k_0: the corresponding involution in the definable hull of $\langle i_0 j\rangle$;
- i: some involution in $M \cap BC(k_0)$;
- k_1: the corresponding involution in the definable hull of $\langle ij] >$.

So now Jaligot's i, j, k have become i, j, k_0, k_1 with a "before" and "after" k. We have to throw away i_0 as we do not really know where it lies.

Let us now lay out the properties of this choice of i, j, k_0, k_1.

(1) k_1 is the unique involution in the definably cyclic group generated by ij.
(2) $i \in BC(k_0)$, $j \notin BC(k_0)$, but $j \in C(k_0)$;
(3) $k_1 \notin BC(k_0)$

Now Jaligot's argument should give us $jk_1 \in BC(k_0)$. Let us check.

We look at $\overline{C}$, the quotient by the intersection with $BC(k_0)$. Again $(\overline{ij})^2 = 1$, $\overline{ij} = \overline{y}$ with y a nontrivial 2-element, and as $k_1 \notin BC(k_0)$ we find $y = k_1$, $ijk_1 \in BC(k_0)$ and we conclude.

Now we look at the action of j on $BC(k_0)$, which must be conjugation by an involution in $BC(k_0)$. As j commutes with jk_1, j must centralize the corresponding 2-unipotent subgroup U_1 of $BC(k_0)$. But then a 2-torus containing j must commute with U_1, and hence with jk_1 and with k_1. But k_1 is an offending involution of M and centralizes no torus.

4. The weak embedding theorem

The next major result of Jaligot's thesis was the following.

Theorem 7. *Let G be a K^*-group of finite Morley rank and of even type, with a weakly embedded subgroup. Then $G \cong \mathrm{SL}_2$ over some algebraically closed field of characteristic 2.*

Again, this is a fundamental result for the analysis of even type groups in general. It can be used in many situations to argue that a minimal counterexample to some conjecture cannot exist, because the minimality often translates into a weakly embedded subgroup, and then this theorem says that the only reason the purported counterexample resists analysis is that it is too simple to be reduced to anything else. So for example if one has a large group G which one believes should in fact be a particular group G^*, then one may succeed in building from the local data an isomorphic copy of G^* within G, and there remains the question, why G^* is in fact G. But one may expect G^* to be a weakly embedded subgroup (if proper) and so this theorem applies.

This was preceded by Altınel's thesis in the tame case: but that in turn relied on remarkable work by Nesin and collaborators on permutation groups of finite Morley rank, by insanely computational methods in an honorable tradition (Tits, Timmesfeld). The general idea here is that when expressed explicitly in terms of generators and relations, the associative law in a group takes on a very implausible character, and if the group does not exist becomes outright contradictory. (If it does exist, but one is not careful, it may also become contradictory.)

This particular line is not one that Éric sought out, as far as I can see, but in this particular case he needed carry out these delicate computations in a distinctly looser setting than that encountered by Nesin. I had the opportunity to look at the configuration before the analysis was complete, and it looked quite desperate. One calculates in detail with elements of order 3. (Something that Deloro also takes pleasure in, in the very different context of representation theory.)

Again, Tuna and I came back later and reworked this in an even more general setting, taking Jaligot's thesis as a template for the calculations. The difference is that we had to worry about connected groups without involutions that might not be solvable, and how they intersect. To a large extent these groups sit on the side through the calculations and add some constant factor to the calculations, but when they are solvable it is very tempting to use what one knows about them, and their Fitting subgroups. So we had to strip down the calculation a bit more and get by. I won't say this gave us (or, anyway, me) any additional insight: it was more a matter of keeping a complicated edifice intact while knocking down a few walls.

At this point I'd like to move on to some work that Éric and I did together: more precisely, that Éric shanghaied me into when it was already well advanced.

5. Minimal simple groups

Éric had been thinking about minimal simple groups of odd type in Prüfer rank 1 and 2, with some unresolved configurations that remain challenging to this day. He dragged me into this, apparently with the hope that I would knock out one of them, but instead I asked the obvious question: if he could do that much in low Prüfer rank, what could he do in high Prüfer rank? So we took that up and eventually eliminated high Prüfer rank.

However, at that point, Éric was making unbridled use of the tameness hypothesis, which may seem out of character. The bound on Prüfer rank was a particularly shameless instance of that.

The results read as follows.

Theorem 8. *Let G be a tame minimal simple group of finite Morley rank and of odd type. Let S be a Sylow 2-subgroup of G, $A = \Omega_1(S\o)$, $T = C_G\o(S\o)$, $C = C_G\o(A)$, and $W = N_G(T)/T$, which is called the* Weyl *group. Then* $\mathrm{Pr}_2(G) \leq 2$ *and one has the following two possibilities:*

1. $\mathrm{Pr}_2(G) = 1$*:*
   a. *If C is not a Borel subgroup of G, then G is of the form $PSL_2(K)$ with K an algebraically closed field of characteristic different from 2.*
   b. *If C is a Borel subgroup of G and if $W \neq 1$, then $C = T$ is 2-divisible abelian, $|W| = 2$, W acts by inversion on T, and $N_G(T)$ splits as $T \rtimes \mathbb{Z}_2$. All involutions in G are conjugate.*
2. $\mathrm{Pr}_2(G) = 2$*:*
   Then $T = C = C_G(A)$ is nilpotent, $|W| = 3$, all involutions of G are conjugate, and G interprets an algebraically closed field of characteristic 3*. Furthermore:*
   a. *If C is not a Borel subgroup of G, then T is divisible abelian, and for each involution i in $S\o$, the subgroup $B_i = C_G\o(i)$ is a Borel subgroup of G of the form $O(B_i) \rtimes T$, where $O(B_i)$ is inverted by the two involutions in T different from i.*
   b. *Otherwise, T is a nilpotent Borel subgroup of G.*

And we added a few words about the possibilities in the non-tame case which had emerged by the time we had the results in final form, as follows.

> And without tameness? Burdges recently developed a new abstract notion of unipotence, leading to a robust signalizer functor theory without the tameness assumption [Bu02]. This allows one to prove a Trichotomy Theorem [Bo03]: a simple K^*-group of odd type is either a Chevalley group, or has small Sylow 2-subgroups, or has a proper "2-generated core". In the third case, the ambient group has recently been shown, without tameness, to be minimal simple in [**?**], provided it has large enough Sylow 2-subgroups. Thus, the problem of the limitation of the Prüfer 2-rank of a potential non-algebraic simple K^*-group of odd type reduces to the case of minimal simple groups without tameness. Assuming tameness, our result gives thus an absolute bound: 2. Unfortunately, tameness is used very intensively in our proof. On the one hand, it is used heavily to analyze the intersections of Borel subgroups. On the other hand, it is used in a critical arithmetical argument at the end of our proof that $\mathrm{Pr}_2(G) \leq 2$. Without tameness, such a bound remains a major open problem. To be continued, thus.

I will say a little about the tame version of this argument, namely the way tameness entered into the treatment of one of the main configurations.

The assumptions for now will be that the group T is a nilpotent Borel subgroup, and that it is strongly embedded in G. In particular the involutions of A are conjugate. The argument is based on consideration of the order of the Weyl group $W = N(T)/T$, which acts on S.

We find that apart from the fixed point 0, W acts regularly on its orbits, and in view of the strong minimality hypothesis also acts transitively on the involutions. So if we write n_p for the rank of the group of p-torsion in S, we find

$$|W| = 2^{n_2} - 1 \,|\, p^{n_p} - 1$$

But now the tameness assumption and some detailed analysis of various Borel subgroups shows that S is, as one might hope, a product of multiplicative groups of some field, and so apart from at most one prime p, the p-ranks n_p coincide with the Prüfer rank n.

So now, with the assistance of Dirichlet's theorem on primes in arithmetic progressions, we have the number theoretic condition

$$2^n - 1 \text{ divides } \ell^n - 1 \text{ for } \ell \text{ relatively prime to } 2^n - 1$$

This is close to a known number theoretic condition which is very relevant to finite group theory and was worked out completely by Feit (*large Zsigmondy primes*). After passing our condition through this filter we come down to a list of 7 possibilities, of which 5 survive inspection: $n = 1, 2, 4, 6, 12$.

We then have to eliminate the possibilities $n = 4, 6, 12$ and this can be done by further analysis of the very limited possibilities for the structure of W. Ron Solomon suggested a clever way to show that n must be prime. Another way is to show first that the centralizer of a 3-element in W must be a 3-group and then to show that the order of W does not allow this.

This ad hoc study of the Weyl group has matured over time into a more systematic enterprise which provides a powerful tool for the quick elimination of some configurations, some of which had previously been considered at length and eventually eliminated. This is also of interest in the degenerate case, where to speak of classification (i.e., complete elimination of this case) is currently out of the question, but something substantial can be said in the language of Weyl groups. A closely related issue is whether the Borel subgroups are self-normalizing, as would be the case in algebraic groups.

We mention some concrete results.

- In a minimal simple group of degenerate type, the Weyl group is cyclic of odd order. (Burdges/Deloro)
- In a connected group, if the Weyl group has odd order and p is the smallest prime divisor of the order, then there is unipotent p-torsion (Burdges/Cherlin)
- A non-nilpotent generous Borel subgroup of a minimal connected simple group is sef-normalizing (Altınel-Burdges-Frécon)

I single out this last result for reasons which will become clear shortly. It should be said however that the real thrust of the work by Altınel-Burdges-Frécon is a fundamental 4-part division of the class of degenerate type groups in which the question of the nontriviality of the Weyl group enters. This has been used by the same authors to develop a very satisfying Jordan decomposition in the context of connected minimal simple groups.

6. Strongly embedded Borels in the minimal simple nontame case

This work continued long distance, via what we then considered email, which did not include the ability to send attachments. So the relevant tex files would be part of the message body. As a result, the copy I have on file of Éric's message begins like this (but the emphasis is mine):

%Received: from mailhost.logique.jussieu.fr (turing.logique.jussieu.fr [134.157.19.1])
% by shiva.jussieu.fr (8.12.10/jtpda-5.4) with ESMTP id i1GHdrG3021595
% for <cherlin@math.rutgers.edu>; Mon, 16 Feb 2004 18:39:53 +0100 (CET)
%X-Ids: 166
%Received: from turing.logique.jussieu.fr (turing.logique.jussieu.fr [134.157.19.1])
% by mailhost.logique.jussieu.fr (Postfix) with ESMTP id 323A122EE82
% for <cherlin@math.rutgers.edu>; Mon, 16 Feb 2004 12:39:53 -0500 (EST)
%Date: Mon, 16 Feb 2004 18:39:53 +0100 (CET)
%From: Jaligot Éric <jaligot@logique.jussieu.fr>
%To: cherlin@math.rutgers.edu
%**Subject: Milkshake**
%Message-ID: <Pine.LNX.4.53.0402161831140.10315@turing.logique.jussieu.fr>
%MIME-Version: 1.0
%Content-Type: TEXT/PLAIN; charset=US-ASCII
%X-Miltered: at shiva.jussieu.fr with ID 40310069.000 by Joe's j-chkmail (http://j-chkmail.ensmp.fr)!
%X-Antivirus: scanned by sophie at shiva.jussieu.fr
%Status: RO
%X-Status:
%X-Keywords:
%
%**I made a slight milkshake with your new argument**
%and our paper to study the nontame case, with the
%standard Borel nilpotent.
%I think this gives the final bound on the Prufer
%rank, as apparently we don't need to understand
%Borels entirely.
%I did not look at the other case of Prufer rank 2
%of our paper yet, but I'm convinced that a close
%look at the cosets of C(i) in that case would work
%in more or less the same way.
%
%- Éric
%
%———————————————

So here is the milkshake.

> **Notations** $I = I(G)$. $i \in I$, fixed. $B = C(i)$ is a Borel subgroup (standard, nilpotent). $A = \Omega_1(S)$. $N(B)/B$ is nontrivial and of odd order, and most simply thought of as cyclic of prime order.
>
> **Furthermore:**
> 1. $g = \text{rk}\,(G)$, $c = \text{rk}\,(B)$, $c' = \text{rk}\,(C(\sigma))$ for $\sigma \in N(B) \setminus B$. This is constant, but in any case the generic value along any one coset would be sufficient.
> 2. $J = \{j \in I :$ There is $\sigma \in N(B)^\times$, j inverts $\sigma\}$.
>
> **Facts used:**
> $\text{rk}\,(I) = g - c$
> The strongly real elements of B are in A.
> For $\sigma \in N(B) \setminus B$, $C_B(\sigma) = 1$.
> Conjugates of B are disjoint.
> For $a \neq 1$ strongly real, $C(a) = C^\circ(a)$ is inverted by any involution inverting a.
> The elements of $N(B) \setminus B$ are strongly real (this is proved again along the way anyway).
>
> **Lemma 1** BI is generic in G.
>
> **Lemma 2** J is generic in I.
>
> **Lemma 3** $\text{rk}\,(I) = c + c'$
> Now fix $\sigma \in N(B) \setminus B$.
>
> **Lemma 4** $BC(\sigma)B$ is generic in G.
>
> **Lemma 5** $B[C(\sigma)^\times]B$ is disjoint from BI.
> In particular, we have two disjoint generic subsets of G, and a contradiction.

This became part of an argument by cases bounding the Prüfer rank of a minimal simple connected group. This is the schema.

- If the Prüfer rank is greater than 2 then there is a strongly embedded Borel subgroup B by a general line of argument with roots in finite group theory (*proper 2-generated core*).
- But if there is a strongly embedded Borel subgroup B then the Prüfer rank is at most 1, via two distinct lines of argument.
  (1) If the involutions of B are central the milkshake works;
  (2) If the involutions of B are not central then some more structural analysis based on the Burdges unipotence theory will work.

Now the second half of this analysis has been redone by Altınel, Burdges, Frécon using ideas mentioned above. So we will follow this line a little further.

7. Non-nilpotent strongly embedded Borel subgroups

Theorem 9 ([ABF13]). *Let G be a minimal connected simple group of finite Morley rank, and M a strongly embedded subgroup whose involutions do not centralize M°. Then the Prüfer rank of G is at most 1.*

I have said nothing about the proof given by Burdges, Jaligot, and myself. But it leaves one wondering if there is a more conceptual argument, and Altınel, Burdges, and Frécon brought this into the scope of their more general theory. The two proofs appear to be quite different, it is not entirely clear to me which of the two is more suitable for generalization. Certainly my impression is that the more conceptual methods must be the point of departure, with other methods kept in reserve. But since Deloro will be speaking about the continuation of his joint work with Jaligot, perhaps this will be clarified in his talk.

7.1. Preliminary remarks and notation.

Before entering into this more modern proof I should make a number of general remarks.

We begin with the state of affairs reached after the completion in [ABC08] of the vision in Altınel's habilitation [**?**]. This eliminates consideration of mixed type and reduces even type to inspection of the algebraic case, while degenerate case has Prüfer rank 0 by definition. So such questions now fall under the odd case, which has its own body of theory.

$$G \text{ has odd type}$$

Next, we give some general remarks of ABF on the structure of their proof. This presupposes the prior work [BuDe09] and [De08], and as it happens [De08] made use of [BCJ07]. So in order to disentangle that web of logical dependence they quote only [BuDe09] and redo what they need from [De08] in a self-contained way.

For the proof, one writes $B = M^\circ$ and one checks first that this is a generous Borel subgroup.

Furthermore, as we are in odd type it can be shown that the 2-elements of M belong to M°, in other words

$$M/M^\circ \text{ has odd order}$$

and thus our claim can be stated a little more clearly.

Proposition 7.1 (Theorem 9 above). *Let G be a minimal connected simple group of finite Morley rank and of odd type, and let B be a generous Borel subgroup with the following properties.*

- *$N(B)$ is strongly embedded in G;*
- *$N(B)/B$ has odd order;*
- *The Prüfer rank of B is at least 2.*

Then the involutions of B are in the center of B.

7.2. **The argument.** The idea now is to look at the action of $N(B)$ on the involutions of B by conjugation. By strong embedding, this action is transitive. The main result of [ABF13], to which we will return, is that $B = N(B)$. However we argue as follows.

- The connected Fitting subgroup $F(B)^\circ$ contains no 2-element.
- $B < N(B)$

As we are in odd type the first point is reasonably clear.

But $N(B)$ acts transitively on the involutions of $B/F(B)^\circ$, and this group is abelian, so if $B = N(B)$ then the Sylow 2-subgroup of $B/F(B)$ has a unique involution; but then the same applies to a Sylow 2-subgroup of B, contradicting the hypothesis on Prüfer rank.

It remains to see why $B = N(B)$, and we will find that there is still one point to be elucidated.

First, the punchline (where we add the admittedly superfluous hypothesis of odd type).

Proposition 7.2 ([ABF13, Thm. 3.12])**.** *Let G be a minimal connected simple group of finite Morley rank and odd type. Let B be a non-nilpotent generous Borel subgroup. Then*

$$B = N(B)$$

This ends the proof, apparently. However, as ABF take pains to explain, there are really two cases in the proof, which depend to a variable extent on the ideas of [De08], hence also (potentially) on [BCJ07]. When W has even order that paper is invoked directly. When W has odd order a self-contained argument is given. Since [De08] has some dependence on [BCJ07], to break the circle, one proves one more claim.

Claim. The Weyl group of G has odd order.

For this, one first takes a Carter subgroup Q of B, and one shows that $N(Q) \leq N(B)$, by strong embedding. This shows that Q is also a Carter subgroup of G, so $W = N(Q)/Q$. On the other hand, the solvable theory shows $N_B(Q) = Q$. So the Weyl group can be identified with a subgroup of $N(B)/B$ and therefore has odd order. (In fact, as $N(B) = BN(Q)$, W and $N(B)/B$ are even isomorphic in this setting.)

With this, we have completed an easy reduction of one of the two main results of [BCJ07] to Theorem 3.12 of [ABF13]. To go any further in our discussion would require taking up the theory of the Weyl group in earnest, which we will not do here.

8. Appendix: some definitions

We mention a few definitions that are not given in the text.

1. *Connected*

A group is *connected* if it has no proper definable subgroup of finite index. H° denotes the unique definable connected subgroup of H of finite index.

2. *Generic, generous*

A subset X of a group $G$of finite Morley rank is *generic* if $\mathrm{rk}\,(X) = \mathrm{rk}\,(G)$, and is *generous* if the union of its conjugates is generic. We make use of the fact that in a connected group, a generic set must have non-generic complement.

3. *Generation*

The definable group *generated* by a set X is the intersection of the definable groups containing X.

4. *Borel subgroup*

A *Borel subgroup* is a maximal connected solvable subgroup (these will be definable). In the context of minimal connected simple groups, these are the maximal proper definable connected subgroups.

5. *Fitting subgroup* $F(G)$

The maximal normal nilpotent subgroup. This is a definable subgroup. We are more interested in the largest connected normal nilpotent subgroup, which is $F(H)^\circ$. In a connected solvable group H the quotient $H/F(H)^\circ$ is divisible abelian.

6. *Weyl group*

The setting of groups of finite Morley rank allows for various notions of *Weyl group,* known to be equivalent in some important special cases. In [ABF13] the following definition is adopted.

$$W = N(T)/C(T)$$

where T is a *maximal decent torus*, that is, a maximal definable divisible abelian group which is generated as a definable group by its torsion subgroup.

But they also show that the definition is equivalent to other natural variants in the context of minimal connected simple groups of finite Morley rank, notably the following.

$$W = N(Q)/Q$$

with Q a Carter subgroup (connected, nilpotent, and of finite index in its normalizer). Under either definition the Weyl group is well defined as an abstract group as the relevant subgroups (T or Q) are conjugate under the stated conditions.

In the algebraic category, the definition is $W = N(T)/T$ with T a maximal torus.

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