A central point of the BN-program, revisited

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Central Extensions

Proposition (ABC08, §II.3)

Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a simple Chevalley group. Then G is a Chevalley group over the same field. In particular, $Z(G)$ is finite.

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Main ingredient Steinberg's description of the universal central extension of the simply connected Chevalley group of given type, over a field k

$$\pi : \hat{G}(\Phi, k) \to G(\Phi, k)$$

Another ingredient

> The kernel of the covering map π is known in K-theory as $K_2(k)$, where k is the field over which the Chevalley group is defined
>
> ABC08, §II.3, p. 139

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So let me try to set the record straight.

Selected Passages from Correspondence with Friends 1 no. 4 (2013) pp. 13–28

> Reread my letter five or six times, precisely because everything in it is haphazard, not in strict logical order; and yet the fault is yours.
>
> N. Gogol, Selected Passages from Correspondence with Friends

MATHEMATICS DISCOVERED, INVENTED, AND INHERITED
ALEXANDRE BOROVIK

. . . I re[a]d the mathematical biography of Richard Brauer written by Walter Feit [**29**] with great interest. I was struck by the following paragraph (pp. 11–12).

> *A few years ago, as I was preparing a lecture on the history of group theory, I came across a paper of Burnside in which he . . .*

l.c.—The universal extension of $G(\Phi, k)$

Ingredients Irreducible root system Φ, field k.

Theorem (Steinberg)

If $|k| > 4, \neq 9$ then the universal central extension of $G(\Phi, k)$ is given by generators $x_\alpha(t)$ ($\alpha \in \Phi$, $t \in k$) and relations

(A) *Additivity of x_α;*

(B) *Commutator formula for $[x_\alpha(s), x_\beta(t)]$ when $\alpha + \beta \neq 0$*

except in rank 1 . . . ; and in either case one gets G as the quotient by additional relations

(C) *h_α is multiplicative (h_α defined in the customary way from $x_{\pm\alpha}$).*

The kernel of $\pi : \hat{G} \to G$ is contained in the image of h_α for any long root α; the extension is determined by the corresponding extension of a torus in $\mathrm{SL}_2(k)$; the Steinberg symbol is the corresponding 2-cocycle as a function on k^*.

Bilinear Steinberg symbols

The kernel of the homomorphism $St(\Lambda) \to GL(\Lambda)$ will be called $K_2\Lambda$

Milnor, Def. 5.1

Theorem (Steinberg, Matsumoto)

Suppose G is not of symplectic type (and, in particular, not of rank 1). Then

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And what about the remaining cases?

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(Feit) *…a paper of Burnside in which he characterized the groups* $\mathrm{SL}_2(2^a)$ *for* $a > 1$ *as the only simple groups of even order in which the order of every element is either* 2 *or odd [***23***]. In this paper Burnside used some of the basic properties of involutions in a way quite similar to the way that Brauer used them fifty years later. However Burnside did not realize the importance of this approach […] I don't know of any mathematical paper by Burnside or anyone else that refers to this paper.*

⋮

Sasha But let me return my story back to the early 1980s and my first attempts to study groups of finite Morley rank. …

$l.c.$—Central extensions, symplectic case

Theorem (Moore in rank 1; Matsumoto)

In the symplectic case, including rank 1, the Steinberg symbol is characterized by the following relations,

(S1) *(Cocycle condition)* $\{x,y\}\{xy,z\} = \{x,yz\}\{y,z\}$

(S2) $\{1,1\} = 1$, $\{x,y\} = \{x^{-1},y^{-1}\}$

(S3) $\{x,y\} = \{x,(1-x)y\}$ *(for* $x \neq 1$*)*

Furthermore, the function $\{x,y\}^{\natural} := \{x,y^2\}$ *is bi-multiplicative.*

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Corollary

If the field k is quadratically closed then the universal central extension of a simply connected Chevalley group is given by the universal bi-multiplicative Steinberg symbol.

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Cf. Sah and Wagoner 2nd HLGMD (1977), Prop. 1.11

Using (Milnor) K-theory

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If G^* is a perfect central extension of $G(\Phi, k)$ then G^* interprets the structure $(k, Z(G^*); +, \cdot, \{\ ,\ \})$

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Example (Freitag/Minchenko, 2016)

A non-trivial bi-multiplicative $\aleph_0$-stable Steinberg symbol. (k, D_1, D_2) differentially closed

$$\{x, y\} = D_1(\ln x)D_2(\ln y) - D_1(\ln y)D_2(\ln x)$$

with values in $(k, +)$.

Ali Nesin, Çatalca Istanbul, 10/15/1995

I found a teaching job in a school called Istanbul Schoool of International Sciences (ISIS). The school gives degrees of two English schools: the London School of Economics and Portsmounth University. ISIS is a branch of these schools and we have to obey the British rules.

Bilkent (Ankara) did not want me. I do not know their official reason yet, but I know the real reason: when I was there for a year, I caused some trouble.

p. 1

My children go to a French school. This is the best news I can give you.

I go to a research institute once a week. This is the only time I can do research.

The construction of the buildings of the orphanage is still going on. I have to worry about everything, from the heating to the ceiling, from the windows to the door handles.

I also have to worry about the children, from their toothbrush to their school books.

p. 2

$l.c.$—-Steinberg symbols of finite rank

Fact (Newelski-Wagner)

Let k be a field of finite Morley rank and X a definable subgroup of $k^\times$ which contains an infinite subfield F of K, not assumed definable. Then $X = k^\times$.

Fact

Let k be the algebraic closure of a prime field ($\mathbb{F}_p$ or $\mathbb{Q}$). Then $K_2(k) = 1$ (all Steinberg symbols are trivial).

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Proof of triviality, finite rank case.

$$a^\perp = \{b \mid \{a,b\} = 1\} = \{b \mid \{b,a\} = 1\}$$

$(\{a,b\} = \{b^{-1}, a\})$

For $a \in k_0^{alg}$, $a^\perp$ contains $(k_0^{alg})^\times$, so $a^\perp = k^\times$ (N-W).

So for any a, $a^\perp$ contains $(k_0^{alg})^\times$, and $a^\perp = k^\times$. □

K-theoretic ingredients

$K_2(\mathbb{F}_p^{alg})$ is trivial: is trivial (Steinberg)
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(1) For k algebraically closed (more generally, for $k^\times$ divisible and containing all roots of unity), $K_2(k)$ is torsion free and divisible.
(2) For k a number field, $K_2(k)$ is a torsion group.

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Proof of (1).

$K_2(k) = (k^\times \otimes k^\times)/\langle\text{Steinberg relation}\rangle$; $k^\times \otimes k^\times$ is torsion-free.
Claim: the subgroup generated by Steinberg relations $a \otimes (1-a)$ is also divisible.

$$1 - a = \prod_i (1 - \alpha_i)$$

$$a \otimes (1-a) = \sum \alpha_i^n \otimes (1 - \alpha_i) = n \left(\sum \dots \right) \qquad \square$$

my first attempts to study groups of finite Morley rank. …I realised that I knew *nothing* about $SO_3(\mathbb{R})$ …
…I think now that my further study of groups of finite Morley rank would be impossible without this amusing re-discovery of the geometry of involutions in $SO_3(\mathbb{R})$.

…I soon discovered that there were no way for me to publish my results in any form more serious than some obscure preprints [**9,10**]. In these sad years not only the theme was unfashionable—the name of Zilber was unfashionable in Novosibirsk, too. The situation changed only when …I started to develop collaboration with colleagues in the international mathematical community [**15**][1].

[1] But even these preprints would not appear without an expression of interest and support from Otto Kegel …

The o-minimal case

The o-minimal theory of central extensions is much richer, and tied up with fundamental issues (Pillay, Peterzil, Hrushovski, Berarducci, Edmundo, Conversano, Gismatullin).

Fact ([BPP10, Proposition 2.2])

If H is a definably connected group definable in an o-minimal expansion of the real field, and G is a central extension of H with finite kernel, then G is definable in the same structure iff G carries a group topology which makes this a topological covering of Lie groups.

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Problem

Is every finite central extension of a Lie group equivalent to a topological covering?

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Fact ([CoPi-2, 2015])

Let G be definable in an o-minimal structure. Then the quotient $G^{\circ\circ}/G^{\circ\circ\circ}$ is abelian, and is naturally isomorphic to the quotient of a certain connected compact commutative Lie group by a dense finitely generated subgroup.

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Problem

Is there a group G with $G^{\circ\circ}/G^{\circ\circ\circ}$ non-commutative?

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Ali Nesin, Çatalca Istanbul, 10/15/1995

…In particular, I used some ideas from my proof of Burnside's Theorem in my work with Ali Nesin on CIT groups of finite Morley rank [**13,14**].

I have to worry about a pirate university that my father founded.

I have to worry about a company that my father founded to publish a daily newspaper.

Yesterday, two ladies came to chat with me. They have troubled children – asocial and hyper-active I believe. They seeked my advise on how to educate them. I spent two hours with the two ladies.

I always worked a lot. But I had at most one or two things to do. Now, I have hundreds of things to do. They are all in different domains.

Now mathematics: I sent the doubly transitive and the Suzuki paper to Lyons. The n-gon paper is quite nice in my opinion. I am sure that you will appreciate it more if you looked through the calculations, or if you tried to prove them yourself. They are not always easy. It is a wonder that everything works, because so many equalities

p. 4

Selected Passages from Correspondence with Friends 1 no. 4 (2013) pp. 13–28(footnote)

> *…support from Otto Kegel who visited Novosibirsk in May 1983 …(On receiving the news of his visit, algebraists from all over Siberia rushed to Novosibirsk to give a talk on their latest research in front of Otto Kegel, who, slightly stunned by the unexpected burden but attentive and supportive, was patiently listening to everyone for three or four days in a row. But this Bulgakovesque episode deserves to be told in a separate tale.)*

Meanwhile the dogs were lustily barking in all possible tones: one of them, with his head thrown back, indulged in such conscientious ululations as if he were receiving some prodigious pay for his labors; another hammered it out cursorily like your village sexton; in between rang out, similar to the bell of a mailcoach, the persistent treble of what was probably a young whelp; and all this was capped by a basso voice belonging presumably to some old fellow endowed with a tough canine disposition, for his voice was as hoarse as that of a basso profundo in a church choir, when the chorus is in full swing with the tenors straining on tiptoe in their eagerness to produce a high note and all the rest, too, throwing their heads back and striving upwards—while he alone with his bristly chin thrust into his neckerchief, turns his knees out, sinks down almost to the ground and issues thence that note of his which makes the window-panes quake and rattle.

L'envoi (from the oral tradition)

I do not aim to minimize the number of false things I say, but to maximize the number of true things I say.

Saharon Shelah

Ya tutarsa !

Nasrettin Hoca

Mathematics is the most dangerous thing you can do.

Sasha Borovik, addressing Rutgers math majors

How much and how many things happened since then. What a life when I think of it!

Ali Nesin

All's Well that Ends Well

Stephen Leacock

…Gertrude and Ronald were wed. Their happiness was complete. Need we say more? …

All's Well that Ends Well

Stephen Leacock

... Gertrude and Ronald were wed. Their happiness was complete. Need we say more? ... Yes, only this. The Earl was killed in the hunting-field a few days after. The Countess was struck by lightning. The two children fell down a well. Thus the happiness of Gertrude and Ronald was complete.