A Census of Homogeneous finite dimensional Permutation Structures (After Sam Braunfeld)

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MFO

- Overview
  - Cameron's Homogeneous Permutations
  - Structure of the Lattice of $\emptyset$-definable equivalence relations
  - The current Census (after Braunfeld): problems and conjectures
  - 2-constraint and 3-constraint
- Details
  - Genericity criterion
  - Representation Theorem: generalized ultrametric spaces
  - The role of distributivity
  - 2-constrained classes

1 Overview

2 Details

Homogeneous Permutations

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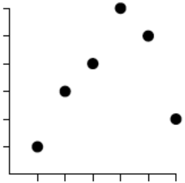
2-dimensional diagrams:



A plot of the permutation 134652

(Waton)

Classification of homogeneous permutations

1. Trivial: $|A| = 1$
2. Nontrivial primitive: $<_2 = <_1^{\pm}$ or $<_1, <_2$ independent (generic)
3. Imprimitive: $(\mathbb{Q}^2; <_1, E_1)$ lexicographic realized as $(\mathbb{Q}^2, <_1, <_2)$ in one of two ways.

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Problem I. The n-dimensional case.
Remark. All homogeneous ordered graphs have an obvious source; to what extent does adding an order to a language lead to new examples?
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Problem I is the base case!
Problem II. When does a countable universal permutation exist for a family determined by finitely many constraints? (More relevant to the study of permutation pattern classes, but we leave it aside.)

Higher dimensions: first census

1. Trivial
2. Nontrivial primitive: apart from restrictions $<_j = <_i^{\pm}$, no other variations known.
3. Imprimitive: Lexicographic $\mathbb{Q}^k$, up to $k = 2^{n-1}$ (with the corresponding chain of equivalence relations definable).

WHAT ELSE?

Higher dimensions: second census

Sam Braunfeld's examples:

Theorem

Any finite distributive lattice can occur as the lattice of all $\emptyset$-definable equivalence relations in a finite dimensional permutation structure.

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Problem

Normal subgroup structure of the automorphism groups; there is a metric element, as we shall see.

Example

$(\mathbb{Q} \times \mathbb{Q}, E_1, E_2)$ (product, boolean lattice with two atoms). Extends to $(\mathbb{Q}^2, E_1, E_2, <_1^*, <_2^*)$ by generically ordering the quotient Qq^2/E_i. This allows a change of language to $(\mathbb{Q}^2, <_1, <_1', <_2, <_2')$.

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This is a difficult example to understand abstractly, and does not give a good model for the proof of the representation theorem (as far as I know).

New Census

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Classify the 3-constrained examples explicitly!

Remark. The same problem arose in the case of metrically homogeneous graphs. In that case the solution is a family of examples which is uniformly definable in Presburger arithmetic.
There is no obvious parallel to look for in the present case.

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Genericity Criterion

Theorem (Cameron)

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Conjecture

This holds for all homogeneous finite dimensional permutation structures.

What is known?

Genericity Criterion

Proposition

Suppose k, n satisfy the following condition.

$$\frac{k!}{(k-\ell)!} > n \cdot 2^{\ell} \qquad \ell = \lfloor k/2 \rfloor$$

Then any homogeneous n-dimensional permutation structure which realizes all $(k-1)$-types is generic.

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3	2
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A less numerical version of the argument pushes $k-1$ down to 3 when $n = 3$, confirming the conjecture in this case.

Genericity Criterion: Proof

Proof.

We show that any structure of order k is the unique amalgam of two substructures of order $k-1$.
The numerical condition allows us to choose ℓ pairs of indices (i, j) such that for any one of the n orders, one of these pairs is non-adjacent with respect to that order.

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Then we can add $\ell-1$ points so that every pair becomes non-adjacent with respect to every order, and view the extended structure on $k+\ell-1$ points as the unique amalgam resulting from factors of order

$$k+(\ell-1)-\ell = k-1$$

(remove one point from each of the ℓ pairs). □

Realization of lattices

Let Λ be a finite distributive lattices.
(1) A Λ-metric space is Λ-valued with triangle inequality

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(Corresponds to: $E_\lambda(x, y) \iff d(x, y) \leq \lambda$.)

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Lemma

If $\mathbb{0}$ is meet irreducible, then the universal homogeneous Λ-metric space has an expansion by linear orders to a homogeneous structure in which all meet irreducible equivalence relations are convex with respect to at least one such.

Realization, continued

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If $\mathbb{O}$ is not meet irreducible, replace Λ by $\Lambda' = [\mathbb{O}', \Lambda]$ and then factor out $E_{\mathbb{O}}$
The last step is admittedly not very plausible: $E_{\mathbb{O}}$ is not convex and it is hard to see what structure is inherited by the quotient, or why it should be homogeneous ....

Is distributivity necessary?

Lemma

If Γ is a non-trivial homogeneous n-dimensional permutations structure, then any proper inclusion $F < E$ in the lattice of $\emptyset$-definable equivalence relations has infinite index.

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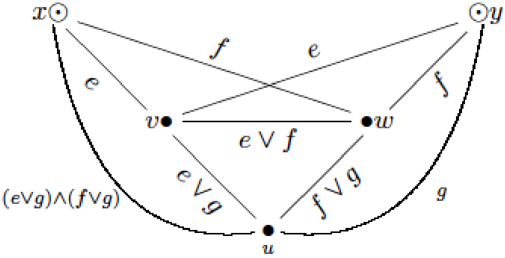
This does not prove the necessity of distributivity: maybe the reduct to the language of equivalence relations is not homogeneous!

But it makes it very plausible ....

Proof of distributivity



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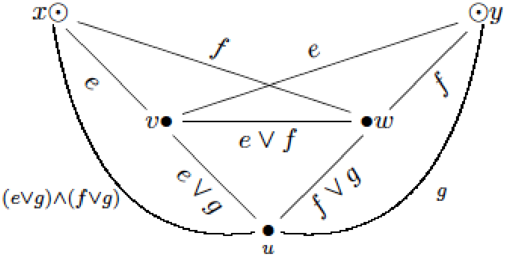

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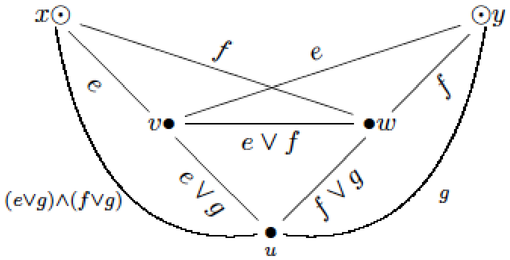

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How do we get the factors? An analog of Neumann's Lemma

2-Constrained Classes

Proposition

If Γ is 2-constrained then it is of standard primitive type: that is, we impose a set of conditions $<_j = \pm <_{j'}^{\pm}$ and nothing else.

(If 2-constraints determine the 3-constraints then similarly.)

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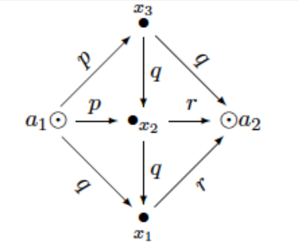
Proof.



This says from p, q, r we get the 2-type *majority*(p, q, r). □

Reference

Sam Braunfeld, *Homogeneous n-dimensional permutation structures,* preprint, December 2015, 23 pp.