Twists and Twistability

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June 12, 2017, 4 P.M.

- Twisted isomorphism: examples
— The twisted automorphism group
- The random graph; descriptive set theory
— The m-random colored graph (Cameron/Tarzi)
— Metrically homogeneous graphs (Rebecca Coulson)
— Some topological dynamics
- Generalized ultrametric spaces (Sam Braunfeld)

Twisted isomorphism

Isomorphism *up to language* $\simeq^*$

Example

Graph and graph complement.

C_5: self-dual
Random Graph $\mathcal{R}$: self-dual
Generic triangle-free $\mathcal{H}_3$ (Henson), $\mathcal{H}_3^c$ is generic I_3-free.

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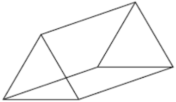
$C_6 \simeq^*$ Triangular prism



(bad example)

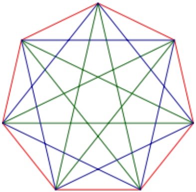
Twisted Isomorphism

Example

Graphs as metric spaces

C_5, $\mathcal{R}$, $\mathcal{H}_3$, as before (diameter 2)
C_6, C_7: diameter 3

C_7:



1 2 3

Twisted Isomorphism

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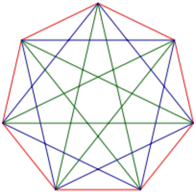

1 2 3 or 1 2 3

Twists: $\mathbb{Z}/3\mathbb{Z}$ (self-triality)

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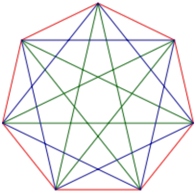
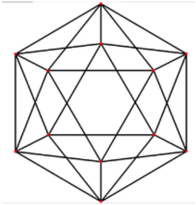
C_7: 

Ico: 

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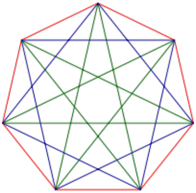

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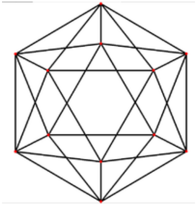

$1 \leftrightarrow 2$
Twists: $\mathbb{Z}/2\mathbb{Z}$

Which language?

Canonical Language

$G = \mathrm{Aut}(\Gamma)$

Relations R_O for G-orbits O

$\mathrm{Aut}(\Gamma_{can}) = \mathrm{Aut}(\Gamma)$

in fact: If $G \leq \mathrm{Sym}(\Gamma)$ is closed then $\mathrm{Aut}(\Gamma_{can}) = G$.

$$\mathrm{Aut}^*(\Gamma_{can}) = N_{\mathrm{Sym}(\Gamma)}(\mathrm{Aut}(\Gamma))$$

Proof.

$O^h = (O^h)^G$

$O = O^{hGh^{-1}}$

$G^h \leq \overline{G} = G$. □

Really?

The canonical language is infinite.
We can manage with less.

Definition

Γ is homogeneous for L if $\mathrm{Aut}(\Gamma)$-orbits on Γ^n coincide with L-isomorphism types.

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Example

C_n is homogeneous in the metric language, and homogeneous as a graph if the diameter is at most 2 ($n \leq 5$).

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Example

C_5 is a graph, but C_6 is a metric space.

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Example

C_5 is a graph, but C_6 is a metric space.

L_k: restriction to k variables. Use the least k that works (for us, $k = 2$).

If Γ is homogeneous for a finite language, and we use a suitable L_k, we get the same twisted automorphisms.

Two problems (Cameron, Tarzi)

- When does Aut^* split over Aut?
- When does Aut^* induce $\mathrm{Aut}(\mathrm{Aut})$?

$\mathcal{R}$ (Cameron, Tarzi)

$\mathrm{Aut}(L/\mathcal{R}) = \mathrm{Aut}^*(\mathcal{R})/\,\mathrm{Aut}(\mathcal{R}) \simeq \mathbb{Z}/2\mathbb{Z}$ (the group of twists).
Non-split: If $\alpha^2 = 1$ then $d(x, x^\alpha)$ is a fixed point for α, so $\alpha = 1$.

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Observation

$\mathrm{Aut}^*(\Gamma)$ induces continuous automorphisms of $\mathrm{Aut}(\Gamma)$.

Theorem (Automatic Continuity)

Any homomorphism from $\mathrm{Aut}(\mathcal{R})$ to a separable topological group is continuous.

Descriptive Set Theory+Combinatorics

Ample Generics and the Extension Property

Definition

A conjugacy class in G is generic if it is dense G_δ.
G has ample generics if G^n has a generic conjugacy class for each n.
G has the small index property if every subgroup of countable index is open.

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$$\text{Combinatorics} \implies \text{Ample generics}$$
$$\implies \text{S.I.P.} \implies \text{Automatic Continuity}$$

Strong Primitivity

Definition

Γ is strongly primitive if for all finite $A \subseteq \Gamma$ and all orbits Ω over A in $\Gamma \setminus A$,

$$\Omega \textit{ is infinite and primitive over } A$$

Cf. *extremely primitive* finite permutation groups (Mann, Burness, Praeger, Seress)

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Example

$\mathcal{R}$

Proposition (Cameron)

If Γ is strongly primitive then any proper open subgroup of $\mathrm{Aut}(\Gamma)$ is contained in the stabilizer of a non-empty finite set.

Corollary

$\mathrm{Aut}(\mathrm{Aut}(\mathcal{R}))$ preserves the family of setwise stabilizers of finite subsets.

$\mathrm{Aut}(\mathrm{Aut}(\mathcal{R})) = \mathrm{Aut}^*(\mathcal{R})$

Claim

$\mathrm{Aut}(\mathrm{Aut}(\mathcal{R}))$ *preserves the family of vertex stabilizers.*

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Count double cosets:

$H = (\mathrm{Aut}\,\Gamma)_{\{A\}}$.

Double cosets parametrize isomorphism types of pairs

$$(A_1, A_2) \quad A_1 \simeq A_2 \simeq A$$

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The number of isomorphism types is

$$\geq |A|^2 + |A| + 1 \quad (|A_1 \cap A_2|, |E \cap (A_1 \times A_2)| : |A| + (|A|^2 + 1))$$
$$= 3 \text{ if } |A| = 1$$

Vertex stabilizers minimize this number.

$\mathrm{Aut}(\mathrm{Aut}(\mathcal{R})) = \mathrm{Aut}^*(\mathcal{R})$

Combinatorics, descriptive set theory: automatic continuity

Strong primitivity: Recognize stabilizers of finite sets

Double cosets, Type counting: Recognize vertex stabilizers

$$\mathrm{Aut}(\mathrm{Aut}(\mathcal{R})) = \mathrm{Aut}^*(\mathcal{R}).$$

The m-random graph $\mathcal{R}_m$

Use an m-sided "coin" (die).
$L = \{1, \ldots, m\}$. $\mathcal{R}_m$ is L-homogeneous.
$\mathrm{Aut}(L/\mathcal{R}_m) = \mathrm{Sym}(m)$
$\mathcal{R} = \mathcal{R}_2$.

Questions

Splitting?
Induced automorphisms?

Splitting $\mathcal{R}_m$

Theorem (Cameron, Tarzi)

Aut^* *splits over* Aut *iff m is odd.*

The even case is illustrated by $\mathcal{R}$: an involution without fixed points cannot lift to a twisted automorphism of order 2.

Splitting $\mathcal{R}_m$

Theorem (Cameron, Tarzi)

Aut^* *splits over* Aut *iff m is odd.*

For m odd, the splitting argument depends on the robustness of the isomorphism type of $\mathcal{R}_m$.

Example

Modify the construction of $\mathcal{R}_m$ as follows:

- Allow any non-zero color probabilities $p_1, \ldots, p_m$ which sum to 1;
- Fix an infinite set of vertices, and only use the die when the edge contains one of the specified vertices; otherwise, use any method you like (including waiting till after all random choices are made)

Then with probability 1 the resulting structure is still $\mathcal{R}_m$.

Splitting $\mathcal{R}_m$. m odd

Lemma (Cameron, Tarzi)

There is a finite m-edge colored graph with a regular action of $\mathrm{Sym}(m)$ via correctly twisted automorphisms iff m is odd.

This comes down to choosing the color $\chi(\sigma)$ of $(1, \sigma)$ for $\sigma \neq 1$ subject to $\chi(\sigma)^{\sigma^{-1}} = \chi(\sigma^{-1})$, at which point one sees the relevance of the parity condition.

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Given a finite m-edge colored graph A as above, view $A \times \mathbb{N}$ as the disjoint union of copies of A, with $\mathrm{Sym}(m)$ acting naturally (fixing $\mathbb{N}$).

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Extend to a complete m-edge colored graph randomly on edges of type $[(a, i), (1, j)]$ with $i < j$, and use the action of $\mathrm{Sym}(m)$ to complete the definition. The result is $\mathcal{R}_m$ with the desired action of $\mathrm{Sym}(m)$.

The action of Aut^* on Aut

Everything done for $\mathcal{R}$ goes over to $\mathcal{R}_m$.

- Ample generics (Hrushovski, Herwig)
- Strong primitiivity
- Counting types of pairs (Cameron–Tarzi).

The known metrically homogeneous graphs

Fact (Cameron, Macpherson)

The locally finite metrically homogeneous graphs of diameter at least 3 are the following.

- C_n *for* $n \geq 6$.
- *The icosahedral graph*
- *The Johnson graph* $J(6, 3)$*: 3-tuples, edges when the intersection has order 2.*
- *Bipartite complement of perfect matching*
- *Macpherson's tree-like* $T_{m,n}$ *of infinite diameter, where* $m, n < \infty$.

One can also have $T_{m,n}$ with m or n infinite (still metrically homogeneous).

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One can also have $T_{m,n}$ with m or n infinite (still metrically homogeneous).
Anything not mentioned above is said to be of generic type.

Metrically homogeneous graphs of generic type

The random graph generalizes to the *generic K_n-free graph* (Henson). Metrically homogeneous analogs include the following.

Definition

$\Gamma^{\delta}_{K,C}$ is the generic metrically homogeneous graph of diameter δ with no metric triangle of perimeter greater than or equal to C, and no metric triangle of odd perimeter less than $2K+1$ (approximately bipartite).

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A modest generalization of this definition produces all known metrically homogeneous graphs of generic type (replacing K, C by K_1, K_2, C_0, C_1 and some clique-like constraints) . . .

Classification conjecture

Conjecture

The metrically homogeneous graphs of generic type are the graphs

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$$

The diameter 2 classification is classical (Lachlan/Woodrow, 1980).

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Diameter 3 has been done, joint with Amato and Macpherson (submitted).
In the diameter 3 case we took advantage of the twisted isomorphism

$$\Gamma^3_{C=8} \simeq^* \Gamma^3_{K=3}$$

afforded by the cycle (123).

$$\begin{array}{ccccc} (3,3,2) & \mapsto & (1,1,3) & \mapsto & (2,2,1) \\ (C=8) & & & & (K=3) \end{array}$$

Metric twists

Theorem (R. Coulson)

The possible twists of a metrically homogeneous graph of generic type are the permutations $\rho, \rho^{-1}, \tau_0, \tau_1$*, where* τ_ϵ *is the transposition* $(1, [\delta+\epsilon]-1)(3, [\delta+\epsilon]-3)\cdots$*, and*

$$\rho(i) = \begin{cases} 2i & i \le \delta/2 \\ 2(\delta - i) + 1 & i > \delta/2 \end{cases}$$

The map ρ *is a twisted isomorphism between*

$$\Gamma^\delta_{C=2\delta+2} \text{ and } \Gamma^\delta_{K_1=\delta}$$

The τ_ϵ *can act as twisted automorphisms, notably when*

$$K_1 = \left\lfloor \frac{\delta+\epsilon}{2} \right\rfloor \quad K_2 = \left\lceil \frac{\delta+\epsilon}{2} \right\rceil \quad C = 2\delta+1 \quad C' = C+1$$

Splitting

Theorem

For any known metrically homogeneous graph of generic type and diameter δ which allows a twist by τ_ϵ the twisted automorphism group splits over the automorphism group iff

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The splitting part may be stated more precisely.

Theorem

For any known metrically homogeneous graph of generic type and diameter δ which allows a twist by τ_ϵ, if $k, \delta + \epsilon - k$ are fixed points for τ_ϵ differing by at most 1, then there is a twisted automorphism α of order 2 affording τ_ϵ which satisfies

$$d(x, x^\alpha) \in \{k, \delta + \epsilon - k\} \text{ for all } x.$$

Central automorphisms

Definition

A *central automorphism* of a group G has the form

$$g^\alpha = g \cdot \zeta(g) \qquad \zeta : G \to Z(G)$$

Examples

Let Γ be metrically homogeneous, bipartite and *antipodal* (i.e., the relation $d(x, y) = \delta$ defines a bijection). The antipodal map π is in the center of $\mathrm{Aut}(\Gamma)$, and the setwise stabilizer of the two parts is a subgroup of index 2, so there is a central automorphism.

E.g. bipartite complement of perfect matching.

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The icosahedral graph, and the Johnson graph $J(6, 3)$ are also antipodal, and their automorphism groups have central automorphisms.

Twisted automorphisms and central automorphisms

For δ even, Γ bipartite antipodal, there is a a (non-triviallly) twisted automorphism which affords a central automorphism: namely $\alpha = (1, \pi)$ acting as 1 on one part and as π on the other.

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Theorem

Let Γ be a metrically homogeneous graph and $\alpha \in \mathrm{Aut}^*(\Gamma)$ a non-trivially twisted automorphism α of Γ inducing a non-trivial central automorphism of Γ. Then Γ is of antipodal type, bipartite, and of even diameter; if Γ is not an n-cycle, then α is $(1, \pi)$ or $(\pi, 1)$.

Generic type

Theorem(ish)

Let Γ be a known primitive metrically homogeneous graph of generic type.
Then $\mathrm{Aut}^*(\Gamma)$ induces the full automorphism group of $\mathrm{Aut}(\Gamma)$

Automatic Continuity

Combinatorial ingredients

Definition

$\mathcal{A}_p(\Gamma)$ is the set of finite $[\delta]$-labeled graphs which embed into Γ as subgraphs. These are called *partial* Γ*-spaces*.

Example

$\mathcal{A}_p(\Gamma^\delta)$ is the set of $[\delta]$-labeled graphs which can be extended to metric spaces with values in $[\delta]$.

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Remarks

Completion process: use the path metric, truncated to δ.
Obstructions: bad cycles where the path distance around the cycle is less than the label.

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Finitely many

Partial Γ-spaces

Theorem (2016–17)

Let Γ be a known primitive metrically homogeneous graph of generic type. Then the partial Γ-spaces are characterized by finitely many forbidden configurations.

R. Coulson for $C - 2\delta > K_1, \delta/2$, $K_2 \geq \delta - 1$;
Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný, Pawliuk in general.

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Corollary

Under these hypotheses, the universal minimal flow for $\mathrm{Aut}(\Gamma)$ *is metrizable.*

(Long story ....)

Generalized Ultrametric Spaces

Theorem (Sam Braunfeld)

For every finite distributive lattice Λ there is a generic Λ-ultrametric space $\mathbb{U}_\Lambda$.

$$d(x,y) \leq \sup(d(x,z), d(z,y))$$

Example

Λ linear order: ordinary ultrametric space.

Related work by van Thé, Conant, and others deals with other types of generalized metric spaces, retaining linear order but allowing other versions of the triangle inequality.

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Theorem (Braunfeld; Linear case known)

The universal minimal flow for $\mathrm{Aut}(\mathbb{U}_\Lambda$ is metrizable.

Uses techniques of Hubička and Nešetřil (2016 preprint). The relations $d(x, y) \leq \lambda$ are equivalence relations, so more sophisticated finiteness conditions are needed.

Questions

Problem

When does $\mathrm{Aut}^*(\mathbb{U}_\Lambda)$ *split over* $\mathrm{Aut}(\mathbb{U}_\Lambda)$*?*

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When does $\mathrm{Aut}^*(\mathbb{U}_\Lambda)$ *split over* $\mathrm{Aut}(\mathbb{U}_\Lambda)$*?*

—If Λ is a Boolean algebra then it splits; possibly always?

More generally: for homogeneous binary (symmetric?) structures, is there a simple characterization of splitting—and is it equivalent to splitting for all involutions of $\mathrm{Aut}(L/\Gamma)$?

This may be studied in particular in the finite setting: in that context, there is a conjectured classification of the finite primitive binary structures—see Wiscons; Gill, Hunt, Liebeck, Spiga, but no conjecture for the full classification.