The Symmetric Group on 6 Points
Jan 25, Istanbul

I aim to discuss some properties of the group Sym(6) (S_6) in terms of the "geometry" of 6 points.

But first I will do this for the symmetric group on *four* points.

1. Sym(4)

We begin with the geometry of points and edges in the complete graph on 4 points, which we represent as follows.

$$\textcircled{4}\ \underline{\ 3-(12)-2\ }\ \textcircled{6}$$

We extend this by considering *perfect matchings* (pairs of disjoint edges).

$$\textcircled{4}\ \underline{\ 3-(12)-2\ }\ \textcircled{6}\ \underline{\ 1-(6)-2\ }\ \textcircled{3}$$

We have points, sets of points, sets of sets of points. We go one step farther and consider *factorizations*, which are disjoint sets of perfect matchings which cover the edges.

This gives us the following.

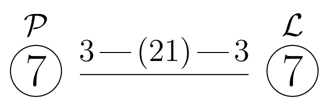
$$\overset{P}{\textcircled{4}}\ \underline{\ 3-(12)-2\ }\ \overset{E}{\textcircled{6}}\ \underline{\ 1-(6)-2\ }\ \overset{M}{\textcircled{3}}\ \underline{\ 1-(3)-3\ }\ \overset{F}{\textcircled{1}}$$

Here Sym(4) acts on everything. This gives us maps from Sym(4) to itself, Sym(6), Sym(3), and

Sym(1). The image of Sym(4) in Sym(6) is transitive so this is not one of the "standard" copies of Sym(4) that fixes two points.

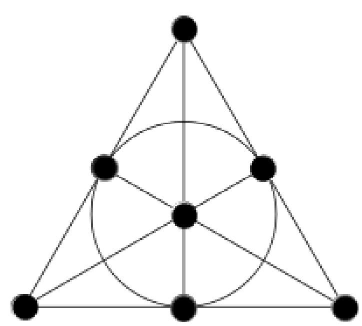
The map to Sym(3) is an *exceptional homomorphism* which captures something very special about Sym(4).

Another geometry has the points and matchings as its points, and the edges and factorizations as its lines. This becomes the following



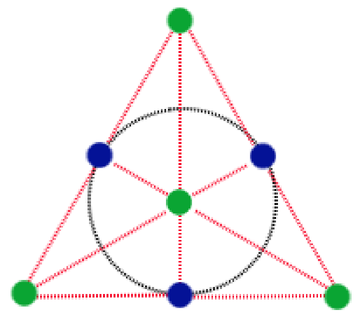
This is the *Fano plane*, a *projective plane* of order 7. The classical projective planes are built from 3-dimensional vector spaces by viewing 1-dimensional subspaces as points and 2-dimensional subspaces as lines.

The Fano plane is built from the field of order two, $\mathbb{F}_2$.



This geometry has as its automorphism group $GL(3,2)$; normally this should be $PGL(3,2)$, factoring out scalar matrices, but here there is nothing to factor out.

If we keep the four original points in view this becomes the following.



We see the action of $Sym(4)$ on four points, on the three points of a line, on the six lines which meet the four points, and on the unique line disjoint from them.

But the full group of symmetries has order $168 = 7 \cdot 24$, with Sym(4) appearing as the stabilizer of a line.

These four points are also what is called a *hyperoval*: a set of points meeting each line in zero or two points. Here the hyperovals are the complements of the lines, so the automorphism group of the geometry acts transitively on them.

Another point of view begins with linear algebra and the *permutation representation* of Sym(4) in four dimensions, acting by permuting a basis. This preserves the usual inner product

$$(u, v) = \sum u_i v_i$$

But it fixes the vector $v = (1, 1, 1, 1)$ and hence acts also on the 3-dimensional orthogonal complement

$$v^{\perp} : \sum u_i = 0$$

If we again work over the field $\mathbb{F}_2$ then (u, u) is the *parity* of u and $v^{\perp} = E$ is the subspace of even parity vectors. Thus

$$E = \{u \,|\, (u, u) = 0\}$$

On E the bilinear form (u, v) becomes *symplectic*: $(u.u) = 0$; but also degenerate: $E^{\perp} = \langle v \rangle$.

So we pass to $E/\langle v \rangle$ and now Sym(4) is acting on a 2-dimensional space with a symplectic inner product.

In this action, Sym(4) permutes the three non-zero vectors. These are

$$(1100)|(0011) \quad (1010)|(0101) \quad (1001)|(0110)$$

In other words - the three perfect matchings, again.

2. Sym(6)

Now we do the same thing with Sym(6).

First we look at the linear algebra.

Sym(6) acts on $V(6,2) = \mathbb{F}_2^6$ by permuting the standard basis and preserves the bilinear form

$$(u,v) = \sum u_i v_i$$

fixing the vector $v = (1,1,1,1,1,1)$ and thus acting also on

$$E = v^\perp$$

and

$$E/\langle v \rangle$$

which is now a 4-dimensional space with a non-degenerate symplectic form. Thus we have

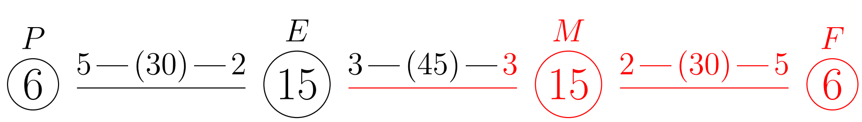
$$\mathrm{Sym}(6) \to \mathrm{Sp}(4,2)$$

It turns out that this is an isomorphism so now we have the *exceptional isomorphism*

$$\mathrm{Sym}(6) \simeq \mathrm{Sp}(4,2)$$

Now we turn to the combinatorics, looking at vertices, edges, matchings, and factorizations.

This time we arrive at the following.



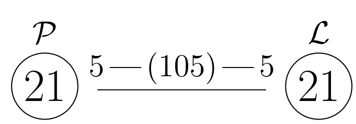
- Each edge lies in 3 matchings and each matching contains 3 edges so there are 15 of each.
- Each matching lies in 2 factorizations: namely, a given matching can be extended to a hexagon containing a particular edge in two ways, and then everything is determined. So the remaining numbers can be filled in.

This is very symmetrical.

In particular, it gives another action of Sym(6) on 6 points, the factorizations, which we may number I–VI rather than 1–6. This action is very different from the original action. If you take a transposition (12) fixing 4 points then it must act without fixed points on the factorizations (since it fixes at least one edge in each factor), and thus give something like $(III)(IIIIV)(VVI)$. If we identify I–VI with 1–6 then we are sending (12) to (12)(34)(56): changing the shape. This is an *outer automorphism* of Sym(6).

As before, we can build a projective plane by grouping the points and matchings as points, and the edges and factorizations as lines.

We get the following data.



This is now the projective plane over the field $\mathbb{F}_4$ with 4 elements, with 21 points and lines. Now the original 6 points form a hyperoval, which is disjoint from 6 lines.

This field is obtained from $\mathbb{F}_2$ the way $\mathbb{C}$ is obtained from $\mathbb{R}$: by adding a root to an irreducible quadratic polynomial.

$$\mathbb{F}_4 = \mathbb{F}_2[j] \qquad\qquad j^2 + j = 1$$

Just as in $\mathbb{C}$ we have complex conjugation, replacing i by $-i$, in $\mathbb{F}_4$ we have the symmetry $\sigma(a + bj) = a + b(j + 1)$.

The automorphism group of the geometry is called $\mathrm{P\Gamma L}(3, 4)$:

$$\mathrm{GL}(2, 4) \to \mathrm{PGL}(2, 4) \to \mathrm{PGL}(2, 4) \cdot \langle\sigma\rangle$$

Now Sym(6) acts on the points of the hyperoval, on the lines meeting the hyperoval (edges), the points off the hyperoval (matchings), and the lines disjoint from it (factorizations).

Furthermore there is a *dual plane* in which the lines are points and the points are line. The six lines disjoint from the hyperoval are a hyperoval in the dual plane. Thus the symmetry in the Sym(6) geometry reflects duality in $P(2, 4)$.

There are 168 hyperovals in $P(2, 4)$ and they are permuted transitively by $\mathrm{P\Gamma L}(3, 4)$. But there is a subgroup $\mathrm{PSL}(3, 4)$ of index 6, with quotient Sym(3). This subgroup has three orbits on ovals, each of order 56, and then the quotient Sym(3) permutes these three orbits.

We can get a new geometry on 22 points by adding a special point $*$ to the 21 lines, and taking the 56 hyperovals as the lines which do not contain $*$. Then whenever you remove a point, the lines through it give a projective geometry on the remaining 21 points. So we wind up with a transitive group on 22 points with PSL$(3,4)$ as point stabilizer. This is one of Mathieu's sporadic groups.

You can also put the 22 points, 77 blocks, and one more point b together to get the Higman-Sims graph on 100 points, whose automorphism group has another sporadic group, the Higman-Sims group, as a subgroup of index 2. Namely: the base point is adjacent to the 22 points, each of them is adjacent to the 21 blocks containing it, and disjoint blocks have edges between them.

To see that the automorphism group is transitive it suffices to check that if we take the special point $*$ as base point in place of b, it looks the same.

The neighbors of $*$ are the lines of $P(2,4)$ and the base point b, which becomes the "special point." The lines are the points of the dual plane and the original points are the lines of the dual plane. The hyperovals are connected to the lines disjoint from them, so they are again an orbit of hyperovals in the dual plane.

3. The Hoffman-Singleton graph

Another graph constructed from the Sym(6) geometry is the *Hoffman-Singleton graph.*

Here we start with the 6 points P and the 6 dual points F, and with two basepoints b_P and b_F adjacent to P and F respectively. The remaining points will be $P \times F$.

We join P and F naturally to $P \times F$, and then we need edges between pairs (p, f) and (p', f'); we require $p \neq p'$, $f \neq f'$, and the $(p, p') \in (f, f')$.

Each (p, f) has five neighbors of this type, and two neighbors in $P \cup F$. So every point has 7 neighbors except the basepoints. We join the basepoints as well and now the graph is regular of degree 7. It has a transitive automorphism group with Sym(7) as the stabilizer of a point. It is a *Moore graph:*

- Diameter 2.
- Triangle-free
- Unique geodesics.

Another example of a Moore graph is the pentagon.

Moore graphs are analyzed using *eigenvalue techniques.*

Namely we consider the *incidence matrix* A and we notice it satisfies the quadratic equation

$$A^2 + A = (k-1)I + J$$

where J is the all-1 matrix and k is the degree. Now J is rank 1 and has eigenvalues $(n, 0, \ldots, 0)$ so A has eigenvalue k with $k^2 + 1 = n$ and

$$\lambda^{\pm} = (-1 + \sqrt{4k-3})/2$$

The multiplicities $m^{\pm}$ satisfy

$$m^+ + m^- = n - 1 = k^2$$

$$k + m^+\lambda^+ + m^-\lambda^- = 0$$

from which one can deduce

$$(m^+ - m^-)\sqrt{4k-3} = k^2 - 2k$$

It follows that $m^+ \neq m^-$ for $k \neq 2$ and thus $\sqrt{4k-3}$ is an integer dividing $k^2 - 2k$, hence also dividing

$$16k^2 - 32k = (4k)^2 - 8(4k) \equiv 3^2 - 8 \cdot 3 = -15$$

and $\sqrt{4k-3} = 1, 3, 5, 15$ (or $\sqrt{5}$: pentagon).

Thus $k = 1$, 3, 7, or 57. The case $k = 7$ is the Hoffman-Singleton graph, the case $k = 3$ is the Petersen graph representing the action of Sym(5) on pairs, and the case $k = 57$ is a complete mystery, but probably does not exist.

This concludes our discussion of the geometry of 6 points.