Γ_n AS A CAYLEY GRAPH
A NOTE

GREGORY CHERLIN

Abstract. We discuss groups acting regularly on the Henson graphs Γ_n, and some related matters.

Contents

Introduction

Cameron's paper [Ca00] provides a body of material relating to the following problem, and includes a survey of prior work.

Problem. *For which pairs* (G, Γ) *consisting of a countably infinite group* G *and a homogeneous structure* Γ *is there an embedding of* G *into* $\mathrm{Aut}(\Gamma)$ *as a regular subgroup? Equivalently, when can we put a left* G*-invariant structure on* G *isomorphic to* Γ*?*

Many interesting open questions are raised as well, among them (on p. 751[1]) the question of whether the generic K_n-free graph Γ_n can be a Cayley graph for an infinite group when $n \geq 4$. It seems to me that Cameron's analysis would already answer this question positively in the case in which the group in question is the free group of infinite rank. In any case my aim is to look more closely at the constraints involved.

Henson showed in [Hen71] that there is a regular $\mathbb{Z}$-action (by automorphisms) on Γ_n only for $n \leq 3$. The proof shows that no regular abelian action exists for $n \geq 4$. Both of these results (existence and nonexistence) are extended, and systematized, in [Ca00].

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[1] *"It is not known whether Γ_p is a Cayley graph for any countable group if $p > 3$."*

I'll first give a direct construction for the case of the free group of infinite rank suggested by Cameron's analysis, and then consider the question more precisely.

1. A direct construction

This is more or less a practice run for §3.

Lemma 1.1. *Let F_ω be the free group on infinitely many generators and let Γ_n be the generic K_n-free graph. Then F_ω acts regularly on Γ_n as a group of automorphisms.*

Proof.

Construction. Given S a finite symmetric subset of $F_k^\# \subseteq F_\omega$, and U a finite subset of F_k, with U K_{n-1}-free in the induced graph defined by (F_k, S), take any $g \in F_{k+1} \setminus F_k$ and define

$$S' = S \cup g^{-1}U \cup U^{-1}g$$

Then we claim the following.

$$S' \cap F_k = S$$

$$gS' \cap F_k = U$$

$$\text{The graph associated to } (F_{k+1}, S') \text{ is } K_n\text{-free.}$$

The first two points are clear. The content of these two points is that the graph structures on F_k defined by S or by S' are the same, and that the graph structure on F_{k+1} defined by S' satisfies one instance of the extension property, with g having only U as its set of neighbors in F_k. This last point can be delicate even for the random graph, when other groups are considered.

Once we have the third point, we can then impose a graph structure on F_ω by stages so that the resulting graph has the full extension property corresponding to Γ_n, and is therefore isomorphic to Γ_n.

So we take up the final point. Suppose there is an n-clique K in the graph associated with (F_{k+1}, S'). Then we may suppose this n-clique contains the vertex g. Therefore $K \subseteq \{g\} \cup gS \cup U \cup gU^{-1}g$.

What edges may occur between two of the sets gS, U, $gU^{-1}g$? Edges (gs, u), $(gs, gu^{-1}g)$, $(u_1, gu_2^{-1}g)$ would correspond to elements

$$u^{-1}gs,\ s^{-1}u^{-1}g, \text{ or } u_1^{-1}gu_2^{-1}g \in S'$$

As none of these can lie in S, we come down to the equations

$$\begin{aligned}(gS,\, U)&: \quad u^{-1}gs = (g^{-1}u')^{\pm 1}\\ (gS,\, gU^{-1}g)&: \quad s^{-1}u^{-1}g = (g^{-1}u')^{\pm 1}\\ (U,\, gU^{-1}g)&: \quad u_1^{-1}gu_2^{-1}g = (g^{-1}u')^{\pm 1}\end{aligned}$$

The last is impossible, and so is the first, since $s \neq 1$. The second reduces to

$$s^{-1}u^{-1} = u'^{-1}$$

or $u' = us$.

So if K meets U then K is contained in $U \cup \{g\}$ and we have a contradiction. Thus K is contained in $\{g\} \cup gS \cup gU^{-1}g$; shifting, we may suppose

$$K \subseteq S \cup \{1\} \cup U^{-1}g$$

As the induced graph on F_k is K_n-free, K must meet $U^{-1}g$.

An edge within $U^{-1}g$ would require a relation $(g^{-1}u_1u_2^{-1}g) \in S'$, with $u_1 \neq u_2$, and this is impossible. So K must have the form

$$\{1, u^{-1}g\} \cup S_0$$

with $S_0 \subseteq S$. Consideration of the edges $(u^{-1}g, s)$ for $s \in S_0$, as above, gives

$$uS_0 \subseteq U$$

As S_0 is a clique it follows that $\{u\} \cup uS_0$ is an $(n-1)$-clique in U, and this is a contradiction. □

2. Constraints

As Henson's argument shows that no abelian group can act regularly on Γ_n for $n \geq 4$, and Cameron has extended this to the case of a regular normal action (i.e., both left and right invariant) by any group, one question that comes to mind is whether a nilpotent group can act regularly on Γ_n.

Apparently (according to the next section), it can. But first we notice the following.

Lemma 2.1. *Let G be a group with a subgroup H of finite index. Then G has no regular action on Γ_n which is H-normal, for any $n \geq 4$.*

In particular if the center of G has finite index there is no regular action of G on Γ_n.

Here we think of G as acting on the right, and the action is *H-normal* if it is left and right H-invariant, or in other words left invariant and invariant under conjugation by H. Note however that this refers to the action of H on G, not just on H. One may think of H-normality for a graph structure on H as a generalization of the condition that H be abelian; and H normality for a graph structure on G as a generalization of the condition that H be central.

Proof. Supposing there is a regular G action on Γ which is H-normal, we identify Γ with G. By the indivisibility of Γ_n the graph induced on H contains a copy of Γ_n. In particular the graph induced on the neighbors of 1 in H contains a copy of Γ_{n-1}.

We follow through the Henson construction.

We may take a clique $K \cup \{1\}$ of order $n-1$ in H.

Claim 1. There is an $h \in H$ adjacent to 1 such that $K \cup Kh$ is K_{n-1}-free.

Let $X = \{k_2^{-1}k_1 \mid k_1, k_2 \in K \text{ (distinct)}\}$. As K is a clique, X is contained in the set of neighbors of 1.

Let Δ_1 denote the set of neighbors of 1 in H and let Δ_1' be the union over $x \in X$ of the set of neighbors of x in Δ_1. The graph induced on Δ_1 contains a copy of Γ_{n-1}, while the graph induced on Δ_1' is a finite union of graphs omitting K_{n-2}. Therefore $\Delta_1 \setminus \Delta_1'$ is infinite.

An element $h \in \Delta_1 \setminus \Delta_1'$ will have the property that h is adjacent to 1 while the pairs $(k_1, k_2h) = k_2(k_2^{-1}k_1, h)$ are nonedges for $k_1 \neq k_2$. As $n \geq 4$, it follows that $K \cup Kh$ is K_{n-1}-free.

This proves the claim.

With h fixed as in our claim, there is then some $g \in \Gamma$ adjacent to $K \cup Kh$. We claim that

$$K \cup \{g, gh^{-1}\}$$

is an n-clique, which will give a contradiction.

We know that K is a clique, and by the choice of g and an application of right H-invariance, we have K adjacent to both g and gh^{-1}.

Finally $(g, gh^{-1}) = gh^{-1}(h, 1)$ is an edge, so we have arrived at a contradiction. □

I tried to prove that there is no regular action of G on Γ_n, for $n \geq 4$, whenever G has an abelian subgroup of finite index, but quickly became confused. In particular, I state the following.

Problem. *Can an infinite dihedral group operate regularly on a graph Γ_n with $n \geq 4$?*

This also suggests the following question.

Problem. *Can a K_n-free graph with transitive automorphism group contain Γ_n without being isomorphic to Γ_n?*

I would think this should be possible. But if not, then it would follow at once that no group with an abelian subgroup of finite index can act regularly on Γ_n for $n \geq 4$.

3. Nilpotent groups.

We know from Cameron's analysis that the sets

$$S(a_1, \dots, a_n) = \{x \in G \mid a_1^{-1}xa_2^{-1}x \cdot \cdots \cdot a_n^{-1}x = 1\}$$

play a special role in the theory of regular actions, at least for $n \leq 3$ (also the conjugacy sets $C(a, b) = \{x \mid a^x = b\}$ but these do not need to be made as visible).

Lemma 3.1. *Let G be a countable group and Γ a homogeneous symmetric binary structure whose finite substructures are closed under free amalgamation. Suppose that G satisfies the following conditions.*

- *The FC-center N of G has infinite index.*
- *G is not a finite union of sets of the following types.*
  - *Cosets of subgroups of infinite index;*
  - *Sets $S(a, b)$ (if all relations on Γ are symmetric we may omit the sets $S(a, a)$);*
  - *Sets $S(a, b, c)$.*

Then there is a regular action of G on Γ.

We first derive a corollary.

Corollary 3.2. *Let G be a countable group and Γ a homogeneous transitive symmetric binary structure whose finite substructures are closed under free amalgamation. Suppose that G satisfies the following conditions.*

- *The FC-center N of G has infinite index;*
- *In the quotient $\bar{G}/N$, the sets $S(\bar{a},\bar{b})$ and $S(\bar{a},\bar{b},\bar{c})$ are all finite.*

Then G acts regularly on Γ.

Proof. The sets $S(a,b)$ and $S(a,b,c)$ lie in the preimages of the corresponding sets $S(\bar{a},\bar{b})$ and $S(\bar{a},\bar{b},\bar{c})$ in $\bar{G}$, so all of the exceptional sets in the sense of the lemma are contained in finite unions of cosets of subgroups of infinite index. And by Neumann's Lemma, we cannot have a finite union of cosets of groups of infinite index covering G. □

Application. We have in particular a class of nilpotent groups acting regularly on all Henson graphs Γ_n (which we thought seemed unlikely). Namely, let G be nilpotent of class 2 and suppose the following conditions are satisfied.

- The FC-center of G is the center $Z(G)$;
- $G/Z(G)$ is infinite and contains no element of order 2 or 3

Then G acts regularly on all Γ_n.

Proof of Lemma 3.1. We make the usual construction by finite approximations with some care taken to avoid elements of N as codes for edges. The language L of Γ consists of a certain number of relations R which we may take to correspond to the nontrivial 2-types, but omitting one relation used for free amalgamation. Since we interpret the omitted relation as "no relation," we will assume the omitted relation is symmetric; but we do not assume the other relations are symmetric. (Also, we allow our groups to contain elements of order 2, so some symmetry condition is appropriate.)

A finite approximation to the action of G on Γ will be called a *G-fragment.* This consists of a finite structure

$$(\Delta, (\Delta_R)_{R\in L})$$

with the following properties.

(1) $1 \in \Delta = \Delta^{-1} \subseteq G$;
(2) Δ_R is a subset of Δ for $R \in L$;
(3) $\Delta_R \cap N = \emptyset$ for $R \in L$.

Recall that N denotes the FC-center of G.

We write Δ for both the G-fragment and its underlying set. We associate to a G-fragment Δ the corresponding G-invariant binary structure on G as follows.

$$G_\Delta = (G, (R_\Delta)_{R\in L});$$

$$R_\Delta(x,y) \iff x^{-1}y \in \Delta_R$$

We also consider the induced structure on Δ, which we denote by $\bar{\Delta}$.

$$\bar{\Delta} = G_\Delta \restriction \Delta$$

We say that a G-fragment Δ is Γ-*admissible* if

(4) $\bar{\Delta}$ embeds into Γ.

We will choose a sequence of Γ-admissible G-fragments Δ whose union gives G a G-invariant structure isomorphic with Γ.

Claim 1. If Δ is a Γ-admissible G-fragment, then G_Δ embeds into Γ.

Here we make use of the notion of a clique, by which we mean a clique with respect to the edge relation defined as the union of the relations R in L. As the finite substructures of Γ are closed under free amalgamation, if G_Δ does not embed in Γ then some finite clique C embeds in G_Δ, but not in Γ.

Translating, we may suppose that the identity element 1 belongs to C. Then as C is a clique and all sets Δ_R are contained in Δ, it follows that C is contained in Δ, and hence C embeds into $\bar{\Delta}$, contradicting the Γ-admissibility.

This proves the claim. From the claim it follows that whenever Δ is an admissible G-fragment, we can extend Δ to a larger admissible fragment containing any specified element of G. This deals with one of the constraints on our construction, namely that the G-fragments involved should eventually exhaust G.

Next we will state our main claim, an extension property for admissible G-fragments. First we establish notation for the main construction.

Construction. Let Δ be a G-fragment, and $B = \bar{\Delta} \cup \{b\}$ an extension of the structure $\bar{\Delta}$ by one additional vertex, where B also carries an L-structure. Let $h \in G$ be given. We then make the following definitions.

$$\Delta' = \Delta \cup (h^{-1}\Delta) \cup (\Delta^{-1}h)$$

$$\Delta_R^+ = \{a \in \Delta \mid R(b,a) \text{ holds in } B\} \qquad \Delta_R^- = \{a \in \Delta \mid R(a,b) \text{ holds in } B\}$$

$$\Delta'_R = \Delta_R \cup (h^{-1}\Delta_R^+) \cup ((\Delta_R^-)^{-1}h)$$

The point of course will be to choose the element h properly.

Claim 2. Let Δ be a Γ-admissible G-fragment and $B = \bar{\Delta} \cup \{b\}$ an extension of $\bar{\Delta}$ by one vertex. Suppose that B embeds into Γ. Then there is an element h such that the extension $(\Delta', (\Delta'_R)_{R\in L})$ has the following properties.

(2.1) Δ' is a G-fragment;
(2.2) $\bar{\Delta}' \restriction (\Delta \cup \{h\}) \cong B$ over $\bar{\Delta}$;
(2.3) Δ' is Γ-admissible.

Claims 1 and 2 taken together are sufficient to build a G-invariant structure on G which is isomorphic to Γ, by successive finite approximations. So it suffices to establish the second claim.

What we must show in each case is that the elements h violating one of our conditions (2.1–2.3) lie in a finite number of exceptional sets, that is cosets of subgroups of infinite index and subsets $S(a,b)$ or $S(a,b,c)$.

We deal with our three conditions (2.1–2.3) in order.

$$\Delta' \text{ is a } G\text{-fragment} \tag{2.1}$$

The content of this condition is that $\Delta'_R \cap N = \emptyset$ for all R. It will suffice to have

$$(h^{-1}\Delta \cup \Delta^{-1}h) \cap N = \emptyset$$

The exceptional h for this constraint lie in ΔN, a finite number of cosets of N, as required.

$$\bar{\Delta}' \restriction \Delta \cup \{h\} \cong B \text{ over } \bar{\Delta} \tag{2.2}$$

All relations present in B have been encoded into $\bar{\Delta}'$, so we concern ourselves with the converse. We first consider the structure induced by $G_{\Delta'}$ on Δ. Suppose $a_1, a_2 \in \Delta$ and a relation $R(a_1, a_2)$ holds in $G_{\Delta'}$ but not in $\bar{\Delta}$. Then

$$a_1^{-1}a_2 \in \Delta'_R \setminus \Delta_R \subseteq h^{-1}\Delta \cup \Delta^{-1}h$$

The set of exceptional h for which this occurs is finite.

Now we must consider the case where a relation $R(a, h)$ or $R(h, a)$ holds in $\bar{\Delta}'$, but the corresponding relation $R(a, b)$ or $R(b, a)$ does not hold in B.

If $R(a, h)$ holds then $a^{-1}h \in \Delta'_R$ and thus there are three possibilities.

$$\begin{aligned} &a^{-1}h \in \Delta_R \\ &a^{-1}h \in (\Delta_R^-)^{-1}h \\ &a^{-1}h \in h^{-1}\Delta_R^+ \end{aligned}$$

The first possibility corresponds to finitely many exceptional choices of h. The second possibility means that $a \in \Delta_R^-$, or in other words that $R(a, b)$ does hold in B. So we are left with the third possibility.

$$a^{-1}h = h^{-1}a_1$$

where $a_1 \in \Delta$. Thus $h \in S(a, a_1)$. So again h lies in one of a finite number of exceptional sets.

If $R(h, a)$ holds then similarly we come down to the case $h \in S(a_1, a)$.

Note: This analysis may be slightly refined when the relations on Γ are symmetric. Namely, we have $a^{-1}h = h^{-1}a_1$ with $a_1 \in \Delta_R^+$, so $R(b, a_1)$ holds in B and therefore $R(a_1, b)$ also holds. Thus we may suppose in this case that $a \neq a_1$; this significantly broadens the class of groups to which the analysis applies.

Our final condition is the following.

$$\bar{\Delta}' \text{ embeds into } \Gamma \tag{2.3}$$

By our assumptions on Γ, it suffices to check that every clique C in $\bar{\Delta}'$ embeds into Γ. It will be more convenient to prove directly that every clique C in $G_{\Delta'}$ embeds into Γ. Then we can translate the clique C so as to contain the element 1, and observe that as it is a clique, the elements of $C \setminus \{1\}$ all belong to the union

$$\bigcup_R \Delta'_R$$

and in particular lie in Δ'. Furthermore as Δ' is a G-fragment we conclude

$$C \cap N = \{1\}$$

Now we must extend our analysis of relations holding in $\Delta \cup \{h\}$ to relations holding in Δ'.

Relations holding between $h^{-1}\Delta$ and $\Delta^{-1}h$

Suppose the relation $R(h^{-1}a_1, a_2^{-1}h)$ holds, that is

$$a_1^{-1}ha_2^{-1}h \in \Delta'_R$$

We have the following possibilities, with $a \in \Delta$.

$$a_1^{-1}ha_2^{-1}h = a$$
$$a_1^{-1}ha_2^{-1}h = h^{-1}a$$
$$a_1^{-1}ha_2^{-1}h = a^{-1}h$$

In the first two cases h lies in one of the exceptional sets $S(a_1a, a_2)$ or $S(a_1, a_2, a)$. In the last case h lies in a finite set.

So (with h chosen appropriately) our clique C will not meet both $h^{-1}\Delta$ and $\Delta^{-1}h$.

This leaves us with two possibilities to analyze.

Case I. $C \subseteq \Delta \cup h^{-1}\Delta$.

We consider edges between Δ and $h^{-1}\Delta$. Suppose $(a_1, h^{-1}a_2)$ is such an edge, that is

$$a_1^{-1}h^{-1}a_2 \in \Delta'_R$$

Again we write out the possibilities, with $a \in \Delta$.

$$a_1^{-1}h^{-1}a_2 = a$$
$$a_1^{-1}h^{-1}a_2 = h^{-1}a$$
$$a_1^{-1}h^{-1}a_2 = a^{-1}h$$

The first and third possibilities involve finitely many exceptional sets. In the second case we have

$$(a_2a^{-1})^h = a_1$$

The elements h here lie in a coset of $C_G(a_1)$. Now if $a_1 \in C \cap \Delta$ and $a_1 \neq 1$, then $a_1 \notin N$, so $C_G(a_1)$ has infinite index in G and the relevant h lie in finitely many exceptional sets. So for appropriate h, since we cannot have $C \subseteq \Delta$ we must have

$$C \subseteq \{1\} \cup h^{-1}\Delta$$

Then translating by h we may take $C \subseteq \Delta \cup \{h\} \cong B$, and we have a contradiction.

Case II. $C \subseteq \Delta \cup \Delta^{-1}h$.

We consider the edges within $\Delta^{-1}h$. So suppose

$$(a_1^{-1}h)^{-1}(a_2^{-1}h) = (a_1a_2^{-1})^h \in \Delta'_R$$

This gives the following three possibilities with $a \in \Delta$.

$$(a_1a_2^{-1})^h = a$$
$$(a_1a_2^{-1})^h = h^{-1}a$$
$$(a_1a_2^{-1})^h = a^{-1}h$$

The second and third equations correspond to finitely many values of h, so we consider the first possibility. Since the left hand side is in Δ'_R, it follows that $a \notin N$, and as h is restricted to a coset of $C(a)$ this is again an exceptional set.

So we may suppose that the clique C contains a unique element $a^{-1}h$ with $a \in \Delta$. We now consider the edges $(a_1, a^{-1}h)$ in C, that is we suppose

$$a_1^{-1}a^{-1}h = (aa_1)^{-1}h \in \Delta'_R$$

The possibilities are as follows, with $a_2 \in \Delta$.

$$(aa_1)^{-1}h = a_2$$
$$(aa_1)^{-1}h = h^{-1}a_2$$
$$(aa_1)^{-1}h = a_2^{-1}h$$

This time the first two equations define exceptional sets, and the last possibility becomes

$$aa_1 = a_2$$

That is, $a(C \cap \Delta) \subseteq \Delta$. Thus translating by a, the clique aC has as its underlying set

$$a(C \cap \Delta) \cup \{h\}$$

which is included in $\Delta \cup \{h\}$ and thus embeds into B, a contradiction. □

4. Homogeneous Directed Graphs

We consider regular actions on digraphs of Henson type. Our conclusion is that this is much like the case of graphs of Henson type, but that a generalization to free amalgamation classes in general binary languages may be more subtle, in an interesting way.

This is another way to explore the meaning of Henson's argument.

Cameron generalizes the study of abelian actions to *normal actions*, where the graph structure is both left and right invariant (or, left invariant and invariant under conjugation, hence the name). Our convention is to take the left action as primary.

If one restricts attention to Henson digraphs with triangle constraints (i.e., one or both of the tournaments of order 3 is forbidden, and nothing else) then it seems like the analysis in [Ca00] applies, though I did not actually go through the equations again. On the other hand I will work through the other case in detail.

Lemma 4.1. *Let H be a countably infinite group and Γ the generic $\mathcal{T}$-free digraph, where $\mathcal{T}$ consists of an antichain of finite tournaments, at least one of which is of order at least 4. Then there is no regular normal action of H on Γ by isomorphisms.*

Proof. We suppose on the contrary we have an H-biinvariant structure isomorphic to Γ on H.

We take a forbidden tournament $T \in \mathcal{T}$ of order $n \geq 4$ and express it as

$$T = T_0 \cup \{a, b\}$$

with T_0 of order $n - 2$ and $a \to b$.

Take a copy of T_0 in Γ. We will look for elements $g, h \in H$ so that $T_0 \cup \{g, gh^{-1}\}$ is isomorphic to T over T_0 with g, gh^{-1} corresponding to a, b respectively.

The constraints on g, h are then the following.

- The type of g over T_0 is the type of a over T_0;
- The type of g over T_0h is the type of b over T_0; by right invariance we then have the type of gh^{-1} over T_0 equal to the type of b over T_0;
- We require $h \to 1$, so that by left invariance, $g \to gh^{-1}$.

So if we find such g, h then $T_0 \cup \{g, gh^{-1}\}$ will be a forbidden tournament and we arrive at a contradiction.

It is easy to rephrase all of this as a set of conditions on h, namely the first two sets of requirements on g should be consistent with the constraints of Γ, and the third constraint should apply.

This gives us the following requirements on h.

- T_0 and T_0h are disjoint;
- The required structure on $T_0 \cup T_0h \cup \{g\}$ does not contain a forbidden tournament;
- $h \to 1$.

We modify the second condition and aim at the following.

- T_0 and T_0h are disjoint;
- There are no arcs between T_0 and T_0h apart from the arcs (th, t) with $t \in T_0$;
- $h \to 1$.

We need to show that such an h exists, and that with such a choice of h, the corresponding configuration $T_0 \cup T_0h \cup \{g\}$ contains no forbidden tournament.

The condition that T_0 and T_0h be disjoint excludes only finitely many values of h. The other two conditions on h amount to the following, after applying appropriate left translations by elements of T_0.

- $(t_1^{-1}t_2, h)$ is never an arc;
- $(h, t_1^{-1}t_2)$ is an arc if and only if $t_1 = t_2$.

This is the description of a digraph structure on $T_0^{-1}T_0 \cup \{h\}$ for which any subtournament with more than two vertices is contained in $T_0^{-1}T_0 \subseteq H$, and hence embeds into Γ. Thus the conditions on the element h are consistent, and have infinitely many realizations in H. Therefore we can meet all three conditions.

Now we return to a consideration of the tournaments embedding into $T_0 \cup T_0 h \cup \{g\}$. Those which lie in $T_0 \cup \{g\}$ or $T_0 h \cup \{g\}$ embed into Γ since they are proper subtournaments of T, and $\mathcal{T}$ is an antichain.

Any others will be contained in a tournament of order 3 with vertices

$$\{t, th, g\}$$

with $t \in T_0$, $th \to t$, and the orientation of (t, g) and (th, g) determined by the orientations of (t, a) and (t, b) respectively.

As $th \to t$ and $a \to b$, one may see that an oriented 3-cycle (th, t, g) would correspond to an oriented 3-cycle (t, a, b) in T, and therefore a linearly ordered triple would also correspond to a linearly ordered triple. (Here we seem to be using some particular statement about tournaments of order 3.) So this is again a proper subtournament of T, and embeds in Γ. Thus we have all required conditions. □

5. Metric Spaces

We single out a few questions suggested by [Ca00] for the case of metric spaces.

Knowing that there are no insuperable obstructions to regular actions on Γ_n, one may ask the same question for the Henson variations on integer Urysohn space.

Problem 1. *Is there a nilpotent group which acts regularly on $\mathbb{Z}\mathbb{U}_{d,n}$, the K_n-free integer-valued Urysohn space of specified diameter d ($d = \infty$ allowed)?*

There are some constraints on actions on metric spaces so one probably wants the group in question to be very well behaved (e.g. free nilpotent of class 2). Then one could expect similar behavior in the metric and graph cases.

In a similar vein one may ask more generally the following.

Problem 2. *Which of the known metrically homogeneous graphs are Cayley graphs, and which admit $\mathbb{Z}$-actions?*

The analog of the Henson graphs for diameter $\delta \geq 3$ are the following: forbid a set $\mathcal{S}$ of $(1, \delta)$-spaces S (the metric on S takes values in $\{1, \delta\}$).

If these spaces have order at most 3 then we should be in the case analogous to Γ_3. If there is a minimal forbidden $(1, \delta)$-space S of order $n \geq 4$ then we expect to rule out regular normal actions. Something like this is discussed in [Ca00] but not in full generality. So let us trace it through.

Henson case, $|S| \geq 4$. We suppose we have such a regular normal action of H.

$S = S_0 \cup \{a, b\}$. We choose $d(a, b) = 1$ if possible, otherwise S is a δ-clique.

Let us first suppose that $d(a, b) = 1$.

Then we take h so that

- S_0, $S_0 h$ are disjoint;
- The distance from s_1 to $s_2 h$ is
  - $d(a, b)$ if $s_1 = s_2$;
  - 2 if $d(s_1, s_2) = 1$;

– $\delta - 1$ if $d(s_1, s_2) = \delta$.

We check the consistency. As usual this comes down to the second set of conditions, which we express in terms of the type of h over $S_0^{-1}S_0$.

- $d(1, h) = 1$;
- $d(s_2^{-1}s_1, h) = 2$ if $d(s_1, s_2) = 1$;
- $d(s_2^{-1}s_1, h) = \delta - 1$ if $d(s_1, s_2) = \delta$.

We need to check that this respects the triangle inequality in $S_0^{-1}S_0 \cup \{h\}$, for triangles containing h.

This is clear if the points $1, h$ both occur in the triangle, since in this case $d(s_2^{-1}s_1, h) = d(s_2^{-1}s_1, 1) \pm 1$.

Consider a triple $(h, s_2^{-1}s_1, s_4^{-1}s_3)$ with $s_1 \neq s_2$ and $s_3 \neq s_4$. Write $a = s_2^{-1}s_1$, $b = s_4^{-1}s_3$. If $d(s_1, s_2) = d(s_3, s_4)$ then $d(h, a) = d(h, b)$ and the triangle inequality holds for (h, a, b).

So we may may suppose $d(s_1, s_2) = 1$ and $d(s_3, s_4) = \delta$. So $d(1, a) = 1$ and $d(1, b) = \delta$. Therefore $d(a, b) \geq \delta - 1$. As $d(h, a) = 2$ and $d(h, b) = \delta - 1$ we again have the triangle inequality.

So a suitable element h may be chosen.

Next we require an element g satisfying the following.

- $d(g, s) = d(a, s)$ for $s \in S_0$;
- $d(g, sh)_= d(b, s)$ for $s \in S_0h$.

If this is achieved, then $S_0 \cup \{g, gh^{-1}\}$ is the desired copy of S, for a contradiction. So now we need to check the triangle inequality in $S_0 \cup S_0h \cup \{g\}$, and also the absence of forbidden $(1, \delta)$-subspaces. But there are no nontrivial $(1, \delta)$-subspaces except those embedding in $S_0 \cup \{g\}$ and $S_0h \cup \{g\}$, so we confine ourselves to the triangle inequality.

As $S_0 \cup \{g\}$ and $S_0h \cup \{g\}$ both embed into S, the only triangles of interest are those of the form

$$(g, s_1, s_2h)$$

with s_1, s_2 in S_0.

If $d(s_1, s_2) \leq 1$ then as we currently assume $d(a, b) = 1$ we get $d(g, s_1) = d(a, s_1) = d(b, s_2) = d(g, s_2)$ and the triangle inequality holds.

If $d(s_1, s_2) = \delta$ then $d(s_1, s_2h) = \delta - 1$ while $d(g, s_1)$ and $d(g, s_2)$ are not both equal to 1. Thus we have a triangle of type $(\delta - 1, \delta, \delta)$ or $(\delta - 1, \delta, 1)$, and the triangle inequality holds.

This completes the construction when $d(a, b) = 1$, and now we suppose that S is a δ-clique. In this case we choose h so that the distances realized between S_0 and S_0h are δ and $\delta - 1$ and proceed as before. Now all distances occurring are δ or $\delta - 1$. $\square$

Since the other examples come by mixing Henson constraints and triangle constraints, a reasonable guess is that if one has only triangle constraints then something like Cameron's analysis will apply (to $\mathbb{Z}$) and the rest of the time something like the above will eliminate regular normal actions. We have not looked farther. (We need to check that we do not accidentally introduce a forbidden triangle in the duplication process. The constraints in such case would severely limit the possibilities for S, but there is still something to be checked.)

Within this area the following seems very attractive.

Problem 3. *Which abelian groups act regularly on integer Urysohn space? In particular, which elementary abelian groups act regularly on integer Urysohn space?*

One can substitute one's preferred version of Urysohn space here but integer Urysohn space seems like the place to begin. A few idle thoughts—

Cameron and Vershik show that an elementary abelian 3-group cannot act regularly on integer Urysohn space (or on rational Urysohn space, or any dense subset of Urysohn space), but that an elementary abelian 2-group can act regularly. Both cases are handled by special means.

One way to look at their argument for $p = 3$ is as follows. Suppose the abelian group A of exponent n carries an A-invariant metric structure. This may be expressed as $d(a, b) = |a - b|$ where $|x| = d(x, 0)$. If $a_0, \ldots, a_{n-1}, b$ are in A then we have

$$\left|\sum_i a_i\right| \leq \sum |d(a_i, b)|$$

since $\sum_i a_i = \sum_i (a_i - b)$. In particular if $|a_i| = \alpha$ is constant and A contains a point b with $d(b, a_i) \approx \alpha/2$ for all i, we get, approximately,

$$\left|\sum a_i\right| \leq n\alpha/2$$

In the case $n = 3$ take $|a| = \alpha$; then $(0, a, 2a)$ forms an equilateral triangle with sides of length α and we can extend this by a point b satisfying

$$d(b, 0) = d(b, a) = \alpha; d(b, 2a) = 2\alpha$$

We then find that $(0, a, b)$ is also an equilateral triangle but

$$|0 + a + b| = |b - 2a| = 2\alpha > 3\alpha/2$$

For $n = 4$ if we work in rational Urysohn space and fix a_0, a_1, a_2, a_3, then writing α_{ij} for $d(a_i, a_j)$ and D for

$$\max(\alpha_{01} + \alpha_{23}, \alpha_{02} + \alpha_{13}, \alpha_{03} + \alpha_{12})$$

we can take b so that

$$\sum |d(b, a_i)| = D$$

and conclude

$$\left|\sum a_i\right| \leq 2D$$

which gives an auxiliary condition to be satisfied by the norm function $|a|$.

We also mentioned a couple of specific problems in §2, of a more ad hoc character.

References

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Department of Mathematics, Rutgers University