Metrically Homogeneous Graphs

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Leeds, July 20

Homogeneity

Definition

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2. A graph is metrically homogeneous iff it is homogeneous under the graph metric.

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Examples

- C_n;
- $T_{r,s}$: An r-tree of s-cliques [Macpherson];
- $\mathbb{U}_{\mathbb{Z}}^{\delta}$: the generic metrically homogeneous graph of diameter δ.

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Rescale the metric on each half: $\frac{1}{2}A$, $\frac{1}{2}B$ to get $T_{r,s}$ and $T_{s,r}$: vertices on each side represent cliques on the other. Homogeneity is inherited.

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The homogeneous metric spaces associated with metrically homogeneous graphs are the geodesic integral spaces: i.e., every geodesic triangle occurs (up to the diameter).

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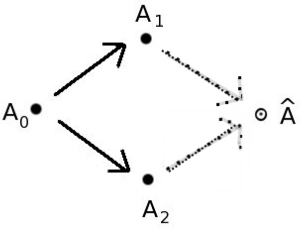
Sub(Γ): The category of f.g. structures embedding isomorphically in Γ.

Amalgamation Property:



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Theorem (Fraïssé Limit)

If $\mathcal{A}$ is a class of f.g. structures closed under isomorphism and substructure, with amalgamation and joint embedding, and with countably many isomorphism types, then there is a unique homogeneous structure $\Gamma = \lim \mathcal{A}$ with

$$\mathit{Sub}(\Gamma) = \mathcal{A}$$

Amalgamation of Metric Spaces

$A \cup \{a\}$, $A \cup \{b\}$:

$d^+(a, b)$: $\min_x(d_1(a, x) + d_2(b, x))$
$d^-(a, b)$: $\max_x |d_1(a, x) - d_2(b, x)|$

$d^- \leq d(a, b) \leq d^+$

$\mathbb{U}_{\mathbb{Z}}$: $\lim \mathcal{A}^{\delta}_{\mathbb{Z}}$

Some extreme cases

- Diameter ≤ 2: Lachlan/Woodrow 1980

  C_5, $K_3 \otimes K_3$

  $m \cdot K_n$ or its complement;

  Γ_n or its complement (omit K_n)

  The random graph, $\lim \mathcal{G}$.

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- Locally finite, diameter ≥ 3

  Finite, antipodal double of C_5 or $K_3 \otimes K_3$ (Cameron 1980)

  Infinite: $T_{r,s}$ with r, s finite (Macpherson 1982)

Smith's Theorem

- Imprimitive
  - Bipartite (and each half rescales to a metrically homogeneous graph of smaller diameter); or
  - Antipodal pairing $d(x, x') = \delta$: and $d(x', v) = \delta - d(x, v)$.

Not explicitly classified.

Henson type

Definition

A $(1,\delta)$ space is a metric space in which all distances are 1 or δ, in other words a union of cliques with maximal separation.

$\mathcal{A}_{\mathcal{S}}$: If $\mathcal{S}$ is a family of $(1,\delta)$-spaces, then $\mathcal{A}^{\delta}_{\mathcal{S}}$ is the family of metric spaces of diameter δ omitting $\mathcal{S}$.

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Amalgamation: $d^-(a, b) \leq i \leq d^+(a, b)$, and $1 < i < \delta$. Since $d^+ > 1$ and $d^- < \delta$, and $\delta \geq 3$, this is possible

3-constrained

Perimeter: $P = |(a,b,c)| = d(a,b) + d(a,c) + d(b,c)$

Definition

$\mathcal{A}^\delta_{\le C}$: $C = (C_0, C_1)$ and $P < C_i$ if $P \equiv 0 \mod 2$.
$\mathcal{A}^\delta_{K,\mathrm{odd}}$: $K = (K_1, K_2)$ and for $P = |(a,b,c)|$ odd

$$2K_1 + 1 \le P \le 2K_2 + 2i \text{ (i a side of the triangle)}$$

$\mathcal{A}^\delta_{K,C} = \mathcal{A}^\delta_{K,\mathrm{odd}} \cap \mathcal{A}^\delta_{\le C}$.

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$$\mathcal{A}^{\delta}_{K,C} = \mathcal{A}^{\delta}_{K,\mathrm{odd}} \cap \mathcal{A}^{\delta}_{\le C}.$$

Theorem

If $\mathcal{A}$ is an integral geodesic amalgamation class of metric spaces determined by contraints of order 3 then $\mathcal{A} = \mathcal{A}^{\delta}_{K,C}$ for some δ, K, C. Furthermore, the choices for δ, K, C which work are given by simple linear inequalities.

Admissible parameters δ, K, C

The families $\mathcal{A}^{\delta}_{K,C}$ are uniformly definable in Presburger arithmetic so it is reasonable to expect the choices of δ, K, C corresponding to the amalgamation property to be definable in Presburger arithmetic. This is the case. In terms of $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$ the conditions are:

- If $C \leq 2\delta + K_1$:
  - $C = 2K_1 + 2K_2 + 1$; $K_1 + K_2 \geq \delta$; $K_1 + 2K_2 \leq 2\delta - 1$.
  - If $C' > C + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$
- If $C > 2\delta + K_1$:
  - $K_1 + 2K_2 \geq 2\delta - 1$, $3K_2 \geq 2\delta$;
  - If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.
  - If $C' > C + 1$ then $C \geq 2\delta + K_2$.

Known Metrically Homogeneous Graphs

- $\delta \leq 2$;
- $T_{r,s}$;
- $\Gamma^{\delta}_{K,C;\mathcal{S}}$
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$\mathcal{A}^{\delta}_{K,C;\mathcal{S}} = \mathcal{A}^{\delta}_{K,C} \cap \mathcal{A}^{\delta}_{\mathcal{S}}$
$\Gamma^{\delta}_{K,C;\mathcal{S}} = \lim \mathcal{A}^{\delta}_{K,C;\mathcal{S}}$

$\Gamma^{\delta}_{a,n}$ is an antipodal graph omitting K_n
($\delta \ge 4$ if $n < \infty$)
But in the antipodal case, K_n corresponds to (K_i, K_j) with $i + j = n$ and separation $\delta - 1$, which is highly unusual.

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Conjecture

This is it.

Should we believe?

Yes!

1. If Γ_1 is exceptional then Γ is in the catalog.
2. Appears to hold in diameter 3 (ACM, in progress)
3. Covers the 3-constrained case.

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No!

3'. The completeness of the catalog for the 3-constrained case is a byproduct of its construction. What we actually need to prove is:

> *The triangle constraints in any amalgamation class are those of some* $\mathcal{A}^{\delta}_{K,C,\mathcal{S}}$.

—And the antipodal case looks slippery, but this is beside the point.

The Classification Problem

- $\mathcal{A}_\Delta = \mathcal{A}^\delta_{K,C}$
- If $\mathcal{A}_\Delta = \mathcal{A}^\delta_{K,C}$ then $\mathcal{A} = \mathcal{A}^\delta_{K,C;\mathcal{S}}$

One looks at the first stage toward explicit amalgamation arguments, in the second stage toward general inductive strategies.

Between the two one expects some critical amalgamation arguments involving structures of order 4; and a heavy use of induction to reduce each part to the full classification at all prior stages.

Lemma (Induction)

If $i < \delta$ and Γ_i contains an edge then either Γ_i is primitive, or Γ is antipodal and $i = \delta/2$.

Our Three Claims

- Assuming 3-constraint, $\mathcal{A} = \mathcal{A}^{\delta}_{K,C}$
- The inequalities are necessary.
- The inequalities are sufficient.

$\mathcal{A} \subseteq \mathcal{A}^{\delta}_{K,C}$

Definition

$$K_1 = \min(k : \exists(1, k, k))$$
$$K_2 = \max(k : \exists(1, k, k))$$
$$C_0, C_1 = \min(\neg\exists(a, b, c)) \quad (\text{and} \geq 2\delta)$$

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Warming up:

Lemma

$\mathcal{A} \subseteq \mathcal{A}^{\delta}_{K,C}$

Proof.

E.g.: if P is odd, $P > 2K_2 + 2i$, $i = d(a, b)$, then (a, b, c) is omitted—

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Proof.

E.g.: if P is odd, $P > 2K_2 + 2i$, $i = d(a, b)$, then (a, b, c) is omitted— Take a supposed counterexample with i minimal.

- If $i = 1$: $(1, k, k)$ with $k > K_2$, forbidden by the definition of K_2.

$\mathcal{A} \subseteq \mathcal{A}^{\delta}_{K,C}$

Lemma

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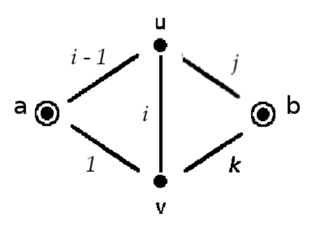
Proof.

- If $i > 1$: We may suppose $k > K_2$.



This forces $(i-1, j, k \pm 1)$ with perimeter $\geq 2K_2 + 2(i-1)$ and with $i - 1 < i$! □

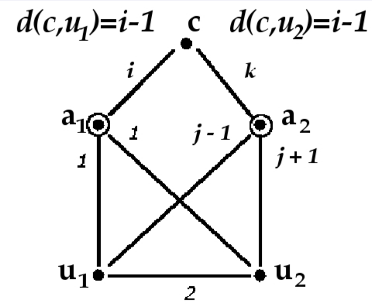
$\mathcal{A} \supseteq \mathcal{A}^{\delta}_{K,C}$

Induction from $P-2$ to P:

Lemma

Assuming 3-constraint:
$(i-1, j-1, k) \& (i-1, j+1, k) \implies (i, j, k)$

Proof.



What triangle types occur?

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$(i-1,j-1,k)\&(i-1,j+1,k) \implies (i,j,k)$

Proof.

What triangle types occur?

(c,u_1,u_2)	$(a_1,\cdot,\cdot)$	$(a_2,\cdot,\cdot)$
$(i-1,i-1,2)$	$(1,1,2)$	$(j-1,j+1,2)$
	$(i,1,i-1)$	$(i-1,j+1,k)$
	$(i,1,i-1)$	$(i-1,j-1,k)$

Geodesics, given, and one even of perimeter below 2δ.

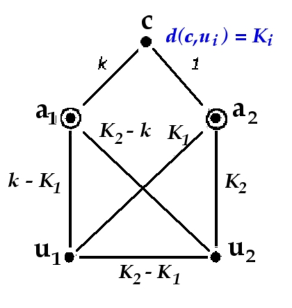
Triangles of small even perimeter follow by the same inductive argument. □

Between K_1 and K_2

Lemma

$K_1 \le k \le K_2 \implies (1, k, k)$

Proof.



Non-geodesic triangles: $(1, K_1, K_1)$ and $(1, K_2, K_2)$

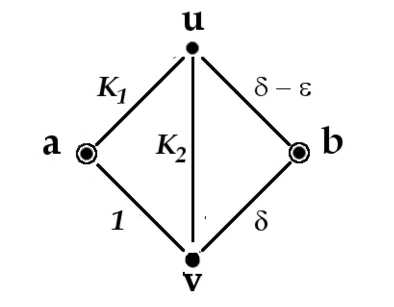
□

An inequality

Lemma

If $\min(C_0, C_1) > 2\delta + K_1$, *then* $K_1 + 2K_2 \geq 2\delta - 1$

Proof.



ε is the parity of K_1.

W.l.o.g. $K_2 < \delta$, so $d(a, b) = \delta - 1$ and we get

$(K_1, \delta - \varepsilon, \delta - 1)$

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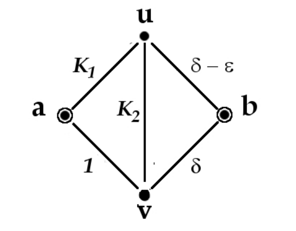

$(K_1, \delta - \varepsilon, \delta - 1)$

Odd perimeter, so $K_1 + 2\delta - (\varepsilon + 1) \leq 2K_2 + 2K_1$.
$K_1 + 2K_1 > 2\delta - (\varepsilon - 1)$. □

Amalgamation!

Example

Suppose $\mathcal{A} = \mathcal{A}^{\delta}_{K_1,K_2}$ where

$$K_1 + 2K_2 \geq 2\delta - 1$$
$$3K_2 \geq 2\delta$$

Amalgamation procedure:

- $d^- > K_1$: Use d^-;
- $d^+ < K_2$: Use d^+;
- $d^- \leq K_1 \leq K_2 \leq d^+$: Use K_2.

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In the third case, we must check for example that if a triangle of type (K_2, j, k) occurs with $d(a_1, x) = j$, $d(a_2, x) = k$, then

$$K_2 + j + k \leq 2K_2 + 2K_2$$

But $j + k \leq 2\delta \leq 3K_2$.

Problems

- Structure of $Aut(\Gamma^{\delta}_{K,C;\mathcal{S}})$
- Structural Ramsey theory for the linearly ordered variants
- Topological dynamics of the automorphism groups.