The Axiom of Choice

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Choice Functions

Proposition

There is a function $f : \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\} \to \mathbb{N}$ *such that*

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Proof.

$\min(A)$ □

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Problem

Is there a function $f : \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} \to \mathbb{R}$ *such that*

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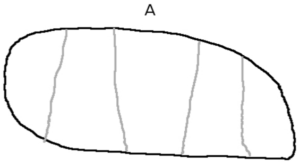

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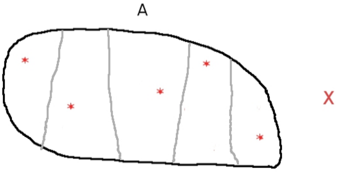

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Useful, but dangerous.

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Proof.

The relation

$$x - y \in \mathbb{Q}$$

is an *equivalence relation* on $\mathbb{R}$.
The sets A_r are the equivalence classes for this relation. □

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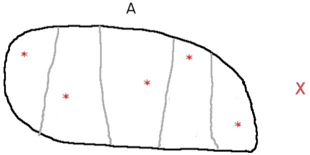

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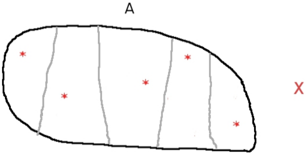

Can you find one?

$$X + q \ (q \in \mathbb{Q}) \text{ should partition } \mathbb{R}$$

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Proposition (Vitali (1905))

There is no such notion of length.

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Let X be a cross section for the Vitali partition *contained in* $[0, 1]$.
Claim: X cannot have a length.

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  $2 = \ell([0,2]) \geq \ell(X) + \ell(X) + \ell(X) + \ldots$

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  $1 = \ell([0, 1]) \leq 0 + 0 + 0 + \dots$

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Nonconstructivity - Philosophical Interlude

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Infinite dimensional vector spaces.

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Example: Independent subsets of V, with $\subseteq$. An *increasing sequence* of independent subsets of $\mathcal{I}$ would form a chain—and their union is an *upper bound* for the chain.

Maximal Elements of p.o.s.

Zorn's Lemma

Let $(I, \leq)$ be a p.o.s. in which every chain has an upper bound. Then I has a maximal element.

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Proof.

$\Longleftarrow$: To build a cross section X for a partition $(A_i)_{i \in I}$, let Ξ be the p.o.s. consisting of sets X satisfying

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ordered by inclusion.

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ordered by inclusion.
By Zorn's Lemma, there is a maximal $X \in \Xi$. It must be a cross section! □

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$\Longrightarrow$ *(sketch):*

If a p.o.s. I has no maximal element, use *AC* to find a function $f : I \to I$ such that

$$f(x) > x \text{ for } x \in I$$

Use f to build a *very long* increasing sequence (a chain)—which has no upper bound. $(\ldots)$. □

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Problem

Can $\mathbb{R}$ be well-ordered?

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Theorem (Zermelo)

Any set S can be well ordered.

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The Axiom of Choice is obviously true; the Well Ordering Principle is obviously false; and who can tell about Zorn's Lemma?—Jerry Bona

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Fréchet: An implication between two known truths is not a new result.
Lebesgue: An implication between two false statements is of no interest.

From *WO* to *AC*

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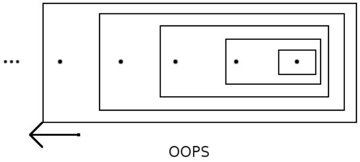

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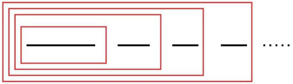
Second try: $X \subseteq Y$; $\leq_Y$ agrees with $\leq_X$ on X; and
$Y \setminus X > X$.



Better

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Fix a basis X for $\mathbb{R}$ as a vector space over $\mathbb{Q}$ with $1 \in X$. For $r \in \mathbb{R}$, let $f(r)$ be the coefficient of 1 in the expansion of r with respect to the basis X.

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These functions *exist,* but none has ever been *constructed.*

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Less vague:

Stan Wagon, *The Banach-Tarski Paradox*, Cambridge University Press, 1985 (paperback, 1993). ISBN 0-521-45704-1 (paperback).

Length, Area, Volume

Measures on subsets of $\mathbb{R}^n$: countably additive, invariant under translation and rotation, normalized on the unit cube. What about: finitely additive?

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For $n = 1, 2$ see Wagon. This also uses *AC*! □

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Lemma

Let $\mathcal{I}$ be the set of ideals in R which are not f.g. If $\mathcal{I}$ is nonempty, it contains a maximal element.

Famous Applications

Hahn-Banach: Extension of (bounded) linear functions from subspaces.

Tychonoff: A product of compact topological spaces is compact.

Major Contributors, References

Georg Cantor: Theory of ordinals, cardinals, well ordering (ca. 1883, work on sets of uniqueness for trigonometric functions).
Ernst Zermelo, Adolf Fraenkel: axioms for set theory (1908, 1910).
Felix Hausdorff, Stefan Banach, Alfred Tarski: Paradoxical decompositions (1914,1924).
Kazimierz Kuratowski, Max Zorn: Zorn's Lemma (1922,1935).
Paul Cohen: *AC* is not derivable from *ZF* (1963). (Cf. AMS Notices August 2010.)
Hugh Woodin: Under appropriate set theoretic hypotheses, all definable sets of reals are Lebesgue measurable (1984).

References

Thomas J. Jech, *The Axiom of Choice*, North-Holland 1973 and Dover Publications 2008. ISBN-13: 978-0-486-46624-8 (Dover paperback).

Kenneth Ross, *Informal Introduction to Set Theory*, 22 pages, http://math.uoregon.edu/people/ross/SetTheory.pdf

To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.—Bertrand Russell.