1 Metrically Homogeneous Graphs

The Classification Problem

Γ connected, with graph metric d.
Γ is *metrically homogeneous* if the metric space (Γ, d) is (ultra)homogeneous.

(Cameron 1998) Classify the countable metrically homogeneous graphs.
Contexts: infinite distance transitive graphs, homogeneous graphs, homogeneous metric spaces

1.1 Finite Distance Transitive Graphs

Finite Distance Transitive Graphs

distance transitivity = metric homogeneity for pairs

Smith's Theorem:
• Imprimitive case: Bipartite or Antipodal (or a cycle)
Antipodal: maximal distance δ
• Reduction to the primitive case (halving, folding)

1.2 Homogeneous Graphs

Classification of Homogeneous Graphs

Metrically homogeneous diameter ≤ 2 = Homogeneous.
(The metric *is* the graph)

Fraïssé Constructions: Henson graphs H_n, H_n^c
Lachlan-Woodrow 1980 The homogeneous graphs are

- $m \cdot K_n$ and its complement;
- The pentagon and the line graph of $K_{3,3}$ (3×3 grid)
- The Henson graphs and their complements (including the Rado graph)

Method: Induction on Amalgamation Classes
Claim: If $\mathcal{A}$ is an amalgamation class of finite graphs containing all graphs of order 3, I_∞, and K_n, then $\mathcal{A}$ contains every K_{n+1}-free graph.
Proof by induction on the order $|A|$ where A is K_{n+1}-free
This doesn't work directly, but a stronger statement can be proved by induction.

Induction via Amalgamation

$\mathcal{A}'$ is the set of finite graphs G such that any 1-point extension of G lies in $\mathcal{A}$.
Inductive claim: Every finite graph belongs to $\mathcal{A}'$.

Not making much progress yet, but ...

1-complete: complete. 0-complete: co-complete.
$\mathcal{A}^p$ is the set of finite graphs G such that any finite p-complete graph extension of G belongs to $\mathcal{A}$.

$\mathcal{A}^p \subseteq \mathcal{A}'$
$\mathcal{A}^p$ *is an amalgamation class*

Target: The generators of $\mathcal{A}$ all lie in one $\mathcal{A}^p$, for some p.

Lachlan's Ramsey Argument

How to get into $\mathcal{A}^p$:

$$\text{1-point extensions of a large direct sum } \oplus A_i$$
$$\Longrightarrow$$
$$p\text{-extensions of one of the } A_i.$$

If A_i is itself a direct sum of generators, we get a fixed value of p.

First used for tournaments: Lachlan 1984, cf. Cherlin 1988

1.3 Homogeneous Metric Spaces

Homogeneous Metric Spaces

Rational-valued Urysohn space.
$\mathbb{Z}$-valued Urysohn space is a metrically homogeneous space.
Or $\mathbb{Z} \cap [0, \delta]$-valued.
S-valued: Van Thé AMS Memoir 2010

A metrically homogeneous graph of diameter δ is:

A $\mathbb{Z}$-valued homogeneous metric space with bound δ, and all triangles $(1, i, i+1)$ allowed (connectivity).

2 A Catalog

2.1 Special Cases

Special Cases

- Diameter ≤ 2 (Lachlan/Woodrow 1980)
- Locally finite (Cameron, Macpherson)
- Γ_1-exceptional
- Imprimitive (Smith's Theorem)

The Locally Finite Case

Finite of diameter at least 3 and vertex degree at least 3: Antipodal double covers of certain finite homogeneous graphs (Cameron 1980)

Infinite, Locally Finite: Tree-like $T_{r,s}$ (Macpherson 1982)

Construction:

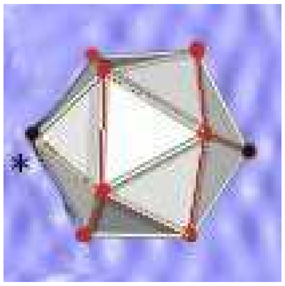


Figure 1: Antipodal Double cover of C_5

The graphs $T_{r,s}$

The trees $T(r,s)$: Alternately r-branching and s-branching.
Bipartite, metrically homogeneous if the two halves of the partition are kept fixed.

The graph obtained by "halving" on the r-branching side is $T_{r,s}$.
Each vertex lies at the center of a bouquet of r s-cliques.

Another point of view: the graph on the neighbors of a fixed vertex:
$\Gamma_1 : r \cdot K_{s-1}$.

From this point of view, we may also take r or s to be infinite!

Γ_1

$\Gamma_i = \Gamma_i(v)$: Distance i, with the induced metric.

Remark 1. *If distance* 1 *occurs, then the connected components of* Γ_i *are metrically homogeneous.*

In particular Γ_1 is a homogeneous graph.

Exceptional Cases: finite, imprimitive, or H_n^c.
The finite case is Cameron+Macpherson, the imprimitive case leads back to $T_{r,s}$ with r or s infinite, and H_n^c does not occur for $n > 2$ (Cherlin 2011)
In other words, the nonexceptional cases are

- I_∞
- Henson graphs H_n including Rado's graph.

Imprimitive Graphs

"Smith's Theorem" (Amato/Macpherson, Cherlin):
Part I: Bipartite or antipodal, and in the antipodal case with classes of order 2 and the metric antipodal law for the pairing:

$$d(x,y') = \delta - d(x,y)$$

Hence no triangles of diameter greater than 2δ:

$$d(x,z) \leq d(x,y') + d(y',z) = 2\delta - d(x,y) - d(x,z)$$

Part II: The bipartite case reduces by halving to a case in which Γ_1 is the Rado graph.

On the other hand, *the antipodal case does not reduce:* while distance transitivity is inherited after “folding,” metric homogeneity is not.

There is also a bipartite antipodal case.

2.2 Generic Cases

Some Amalgamation Classes

Within $\mathcal{A}^\delta$: finite integral metric spaces with bound δ:

- $\mathcal{A}^\delta_{K,\text{even}}$: No odd cycles below $2K+1$.
- $\mathcal{A}^\delta_{C,\text{bounded}}$: Perimeter at most C.
- $(1,\delta)$-constraints.

The first two classes are given (implicitly) in Komjath/Mekler/Pach 1988 as examples of constraints admitting a universal graph, which is constructed by amalgamation.

The last is a generalization of Henson's construction. A $(1,\delta)$-space is a space in which only the distances 1 and δ occur (a vacuous condition if $\delta = 2$).

Any set $\mathcal{S}$ of $(1,\delta)$-constraints may be imposed.

Mixing: $\mathcal{A}^\delta_{K,C;\mathcal{S}}$

Expectations ca. 2008

- The generic case is $\mathcal{A}^\delta_{\Delta,\mathcal{S}}$ with Δ some set of forbidden triangles ...
- and Δ is a mix of parity constraints K and size constraints C.

Not quite ...

Variations on a theme

More examples

- $C = (C_0, C_1)$: C_0 controls large even parity, C_1 controls large odd parity
- $K = (K_1, K_2)$: K_1 controls odd cycles at the bottom, K_2 controls odd cycles midrange.
  - (i,j,k): $P = i+j+k$
  - For P odd, forbid

$$P < 2K_1 + 1 \tag{1}$$
$$P > 2K_2 + i \tag{2}$$

Triangle Constraints

Theorem 1. *If $\mathcal{A}$ is a geodesic amalgamation class of finite integral metric spaces with diameter δ, determined by triangles, then $\mathcal{A}$ is one of the classes*

$$\mathcal{A}^{\delta}_{K,C;\mathcal{S}}$$

with $K = (K_1, K_2)$ and $C = (C_0, C_1)$.

But not all such classes work ....

Definability in Presburger Arithmetic

The classes $\mathcal{A}^{\delta}_{K,C}$ are uniformly definable in Presburger arithmetic from the parameters $K_1, K_2, C_0, C_1, \delta$.

The k-amalgamation property is amalgamation for diagrams of order at most k.

With constraints of order 3, one expects k-amalgamation for some low k to imply amalgamation. (In the event, $k = 5$.)

Observation 1. *k-amalgamation is a definable property in Presburger arithmetic for the classes $\mathcal{A}^{\delta}_{K,C}$.*

Therefore it should be expressible using inequalities and congruence conditions on linear combinations of the parameters.

Acceptable Parameters

- $\delta \geq 3$.
- $1 \leq K_1 \leq K_2 \leq \delta$ or $K_1 = \infty$ and $K_2 = 0$;
- $2\delta + 1 \leq C_{\min} < C_{\max} \leq 3\delta + 2$, with one even and one odd.

Conditions for amalgamation (or 5-amalgamation):

Conditions on K, C

- If $K_1 = \infty$:

$$K_2 = 0, C_1 = 2\delta + 1,$$

- If $K_1 < \infty$ and $C \leq 2\delta + K_1$:

$$C = 2K_1 + 2K_2 + 1, K_1 + K_2 \geq \delta, \text{ and } K_1 + 2K_2 \leq 2\delta - 1$$
$$\text{If } C' > C + 1 \text{ then } K_1 = K_2 \text{ and } 3K_2 = 2\delta - 1.$$

- If $K_1 < \infty$, and $C > 2\delta + K_1$:

$$\begin{array}{l} K_1 + 2K_2 \geq 2\delta - 1 \text{ and } 3K_2 \geq 2\delta. \\ \text{If } K_1 + 2K_2 = 2\delta - 1 \text{ then } C \geq 2\delta + K_1 + 2. \\ \text{If } C' > C + 1 \text{ then } C \geq 2\delta + K_2. \end{array}$$

Notes:
$C = \min(C_0, C_1)$, $C' = \max(C_0, C_1)$
$C' > C + 1$ means we need both C_0 and C_1.

Conditions on $\mathcal{S}$

- If $K_1 = \infty$:

$$\mathcal{S} \text{ is } \begin{cases} \text{empty} & \text{if } \delta \text{ is odd, or } C_0 \leq 3\delta \\ \text{a set of } \delta\text{-cliques} & \text{if } \delta \text{ is even, } C_0 = 3\delta + 2 \end{cases}$$

- If $K_1 < \infty$ and $C \leq 2\delta + K_1$:

$$\text{If } K_1 = 1 \text{ then } \mathcal{S} \text{ is empty.}$$

- If $K_1 < \infty$, and $C > 2\delta + K_1$:

$$\begin{array}{l} \text{If } K_2 = \delta \text{ then } \mathcal{S} \text{ cannot contain a triangle of type } (1, \delta, \delta). \\ \text{If } K_1 = \delta \text{ then } \mathcal{S} \text{ is empty.} \\ \text{If } C = 2\delta + 2, \text{ then } \mathcal{S} \text{ is empty.} \end{array}$$

2.3 Proofs

Antipodal Variations

- $\mathcal{A}^\delta_a = \mathcal{A}^\delta_{1,\delta-1;\,2\delta+2,2\delta+1;\,\emptyset}$ is the set of finite integral metric spaces in which no triangle has perimeter greater than 2δ.
- $\mathcal{A}^\delta_{a,n}$ is the subset of $\mathcal{A}^\delta_a$ containing no subspace of the form $I_2^{\delta-1}[K_k, K_\ell]$ with $k+\ell = n$; here $I_2^{\delta-1}$ denotes a pair of vertices at distance $\delta-1$ and $I_2^{\delta-1}[K_k, K_\ell]$ stands for the corresponding composition, namely a graph of the form $K_k \cup K_\ell$ with K_k, K_ℓ cliques (at distance 1), and $d(x, y) = \delta - 1$ for $x \in K_k$, $y \in K_\ell$. In particular, with $k = n$, $\ell = 0$, this means K_n does not occur.

Necessity: Amalgamation diagrams

Lemma 2. *Let $\mathcal{A}$ be an amalgamation class of diameter δ determined by triangle constraints with associated parameters K_1, K_2, C, C'. Then*

$$C > \min(2\delta + K_1, 2K_1 + 2K_2)$$

We suppose

$$C \leq 2\delta + K_1$$

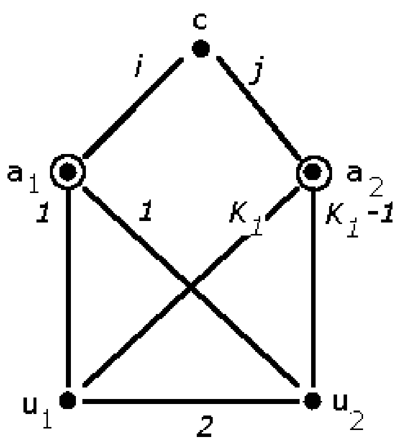
and we show that

$$C > 2K_1 + 2K_2$$

Set $j = \lfloor \frac{C-K_1}{2} \rfloor$, and $i = (C - K_1) - j$. Then $1 < j \leq i \leq \delta$.

$C > \min(2\delta + K_1, 2K_1 + 2K_2)$

In the following amalgamation, vertices u_1, u_2 force $d(a_1, a_2) = K_1$ and $|a_1 a_2 c| = C$:



$$d(c, u_1) = d(c, u_2) = i - 1$$

So omit ca_2u_1 or ca_2u_2, with $P \geq 2K_1 + 1, \ldots$

Proofs of amalgamation

Three amalgamation strategies:

- $d^-(a,b) = \max(d(a,x) - d(a,b))$
- $d^+(a,b) = \inf d(a,x) + d(x,b)$
- $\tilde{d}(a,b) = \inf[C - (d(a,x) + d(a,b))]$

Amalgamation for $\mathcal{A}^\delta_{K,C}$

- If $C \leq 2\delta + K_1$:
  - If $d^-(a_1, a_2) \geq K_1$ then take $d(a_1, a_2) = d^-(a_1, a_2)$.
    Otherwise:
  - If $C' = C + 1$ then:
    - If $d^+(a_1, a_2) \leq K_2$ then take $d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2))$
    - If $d^-(a_1, a_2) < K_1$ and $K_2 < d^+(a_1, a_2)$ then take $d(a_1, a_2) = \tilde{d}(a_1, a_2)$ if $\tilde{d}(a_1, a_2) \leq K_2$ and $d(a_1, a_2) = K_1$ otherwise.
  - if $C' > C + 1$ then:
    - If $d^+(a_1, a_2) < K_2$ then take $d(a_1, a_2) = d^+(a_1, a_2)$;
    - If $d^-(a_1, a_2) < K_2 \leq d^+(a_1, a_2)$ then take

$$d(a_1, a_2) = \begin{cases} K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta \\ K_2 & \text{otherwise} \end{cases}$$

- If $C > 2\delta + K_1$:
- If $C > 2\delta + K_1$:
  - If $d^-(a_1, a_2) > K_1$ then take $d(a_1, a_2) = d^-(a_1, a_2)$;
    Otherwise:
  - If $C' = C + 1$ then:
    - If $d^+(a_1, a_2) \leq K_1$ then take $d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2))$;
    - If $d^+(a_1, a_2) > K_1$ then take

$$d(a_1, a_2) = \begin{cases} K_1 + 1 & \text{if there is } v \in A_0 \text{ with} \\ & d(a_1, v) = d(a_2, v) = \delta, \\ & \text{and } K_1 + 2K_2 = 2\delta - 1 \\ K_1 & \text{otherwise} \end{cases}$$

  - If $C' > C + 1$ then:
    - If $d^+(a_1, a_2) < K_2$ then take $d(a_1, a_2) = d^+(a_1, a_2)$;
    - If $d^+(a_1, a_2) \geq K_2$ then take $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$.

3 Conclusion

Completeness?
Good points:

- All cases with exceptional Γ_1

- $\delta \leq 3$, probably (Amato/Cherlin/Macpherson)
- Exact as far as triangle constraints are concerned
- Smith's Theorem

Weak points

- Smith's Theorem
  - Bipartite to be completed inductively
  - Antipodal description may be incomplete
- Induction to Γ_i is not always available

In fact, for antipodal graphs omitting K_n, triangles and $(1, \delta)$-constraints do not suffice.

That class was found on an ad hoc basis. (And is invisible in diameter 3.)

Toward a classification theorem

Strategy?

- (Step 0) Prepare diameter 4 and Γ_2 generally? (Prudent)
- (Step 1) Characterize triangles occurring in amalgamation classes
- (Step 2) Show that if the triangle constraints are as expected, then Γ_i has the expected constraints.
- (Step 3) Assuming the first two conditions, characterize Γ.

(Works in diameter 3)

... With Lachlan's Ramsey method in reserve.

Furthermore

No need to wait for a classification:

- Ramsey theory for these homogeneous metric spaces
- Topological dynamics
- Other aspects of the automorphism group (normal subgroups, subgroups of small index)