Metrically Homogeneous Graphs

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March 10, 2012, NYC MAMLS

Homogeneity

Definition

1. A metric space is homogeneous iff every isometry between finite parts is induced by a self-isometry of the whole.
2. A graph is metrically homogeneous iff it is homogeneous under the graph metric.

Problem (Cameron)

Classify the metrically homogeneous graphs.

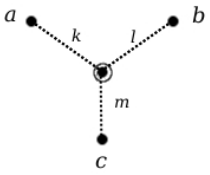
A census of infinite distance-transtive graphs
`http:www.maths.qmul.ac.uk/~pjc/preprints`

Special Cases

- Regular trees
- Diameter 2
- Urysohn graphs

Regular trees $T(r)$

Reconstruction of the convex closure from the metric.



$$P = |(a,b,c)| = 2(k+l+m);\ m = P/2 - d(a,b)$$

Semiregular trees and the Macpherson graphs

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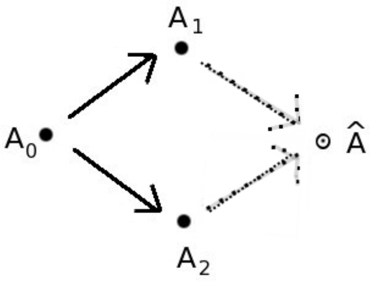
Obtain a metrically homogeneous graph on each half by rescaling—
$T_{r,s}$ and $T_{s,r}$ Macpherson graph: an r-tree of s-cliques, or vice versa.

Homogeneous Graphs

Theorem (Lachlan/Woodrow)

- C_5, $K_3 \otimes K_3$
- $m \cdot K_n$ *and its complement (m-partite).*
- *The Henson graphs* Γ_n *and their complements.*
- *The random graph*

Amalgamation Classes



Amalgamation Classes

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If Γ is a homogeneous structure then the category $\mathrm{Sub}(\Gamma)$ of f.g. substructures has the amalgamation property and joint embedding.

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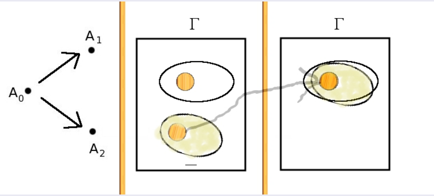
Proof.



□

There is a converse ...

The Fraïssé limit

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A set $\mathcal{A}$ of f.g. structures is an amalgamation class if

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Example

$(\mathbb{Q}, <)$ is the Fraïssé limit of the class $\mathcal{L}$ of finite linear orders.

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Example

The random graph Γ is the Fraïssé limit of the class $\mathcal{G}$ of all finite graphs.

The Henson Graphs

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Remark

The complement of a homogeneous graph is homogeneous

Lachlan/Woodrow

Theorem

An amalgamation class of graphs which contains arbitrarily large independent sets, a path of length 2, and its complement, as well as K_{n-1}, contains every finite graph not containing K_n.

Urysohn space

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$\mathbb{U}_{\mathbb{Q}}$ is the Fraïssé limit of the class of finite rational metric spaces.

Amalgamation of Metric Spaces

$$d^+(a,b) = \inf_x (d_1(a,x) + d_2(x,b))$$
$$d^-(a,b) = \sup_x |d_1(a,x) - d_2(x,b)|$$
$$d^-(a,b) \le r \le d^+(a,b)$$

Example

The set of all finite rational metric spaces of diameter at most 1 is an amalgamation class.

The Urysohn "sphere" $\mathbb{S}_{\mathbb{Q}}$.

Urysohn Graphs

$$\mathbb{U}_{\mathbb{Z}}$$

$\mathbb{Z}$-valued and 1-connected, hence metrically transitive.

Sphere variations $\mathbb{U}_{\mathbb{Z}}^{\delta}$: diameter δ.

Topology

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Topological dynamics: compact flows, minimal compact flows, universal minimal compact flows.

Fixed-point property (extreme amenability): Every compact flow has a fixed point.

Kechris, Pestov, Todorcevič

Theorem (Pestov)

$\mathrm{Aut}(\mathbb{Q}, <)$ *is extremely amenable.*

Proof.

Ramsey's theorem. □

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Theorem (KPT)

The universal ordered rational Urysohn space is extremely amenable.

Proof.

Structural Ramsey theory for finite rational valued metric spaces.
(Nešetřil, on demand.) □

Removing order

Theorem

The universal compact flow for $\mathbb{U}_{\mathbb{Q}}$ is the space of orders.

General Theory: Kechris/Pestov/Todorcevič

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Project. For natural homogeneous structures, find orders for which there is a Ramsey theorem.

Variations

- Henson Variations
- KMP Variations

Henson Variations

Diameter δ. $1 < \mathrm{d}^+$ and $\mathrm{d}^- < \delta$.
$(1, \delta)$-spaces
$\mathcal{S}$: a set of $(1, \delta)$-spaces.

Remark

Let $\mathcal{S}$ be a set of $(1, \delta)$ spaces, and $\mathcal{A}_{\mathcal{S}}$ the class of $\mathcal{S}$-free $\mathbb{Z}$-valued metric spaces of diameter at most δ. If $\delta > 2$, then $\mathcal{A}_{\mathcal{S}}$ is an amalgamation class.

KMP Variations

Theorem

Let $\mathcal{C}$ be any of the following:

- *The set $\mathcal{C}_K$ of odd cycles of length bounded by $2K + 1$.*
- *The set $\mathcal{C}_C$ of cycles of perimeter at least C.*

Then there is a universal $\mathcal{C}$-free graph.

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Proof.

Amalgamation in an augmented language, resembling a metric structure. □

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Theorem

Let $\mathcal{A}^{\delta}_{K,C}$ be the class of finite metric spaces of diameter at most δ such that

- *There is no triangle of odd perimeter less than $2K + 1$*
- *There is no triangle of perimeter greater than or equal to C.*

Then for $\delta \geq 3$ and subject to minor constraints, $\mathcal{A}^{\delta}_{K,C}$ is an amalgamation class.

$\mathcal{A}_K$

Amalgamation Procedure ($\mathcal{A}_K$)

- d^- if $d^- > K$;
- d^+ if $d^+ \leq K$;
- Else K.

First Try

- $\delta \leq 2$; or
- Finite; or
- $T_{r,s}$; or
- $\Gamma^{\delta}_{\Delta;\mathcal{S}}$

Here Δ stands for a set of constraints on triangles.

Conjecture (Leibniz/Candide)

Δ *means* (K, C)

First Correction

Antipodal Graphs

$C = 2\delta + 1$, i.e.:

- For each vertex v there is a unique paired vertex v' at distance δ.
- $d(u, v') = \delta - d(u, v)$

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There is a Henson variation involving cliques K_n
But then one must exclude pairs K_k, K_l at distance $\delta - 1$.
Strange . . .

Second Correction

Wrong about Δ.

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(27 amalgamation classes with 4 2-types, 18 can be interpreted as metric spaces, leading to 20 examples, 10 of them not of the above form.)

The right Δ

$C = (C_0, C_1)$: C_0 controls even perimeter; C_1 controls odd perimeter; and in the most common case they do indeed differ by 1, and we take $C = \min(C_0, C_1)$

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Theorem

If a metrically homogeneous graph of bounded diameter is determined by constraints of order 3, the associated amalgamation class is of the form

$$\mathcal{A}^{\delta}_{K,C}$$

with $K = (K_1, K_2)$, $C = (C_0, C_1)$ *as above.*

Second Try

- $\delta \leq 2$; or
- Finite; or
- $T_{r,s}$; or
- $\Gamma^{\delta}_{\Delta;\mathcal{S}}$; or even
- The antipodal variations $\mathcal{A}^{\delta}_{a;n} = \mathcal{A}^{\delta}_{1,2\delta+1;\mathcal{S}'_n}$

Here Δ stands for a set of constraints on triangles such that $\mathcal{A}^{\delta}_{\Delta}$ is an amalgamation class.
I.e.: $\Delta = (K, C)$ with $K = (K_1, K_2)$ and $C = (C_0, C_1)$ as described.

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Problem

But which of these classes actually is an amalgamation class?

Enter Presburger Arithmetic

$\mathcal{A}^{\delta}_{K,C}$ is given by a set of triples $T_{\delta,K,C} \subseteq \mathbb{N}^3$ with δ, K, C standing for 5 positive integer parameters.

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For fixed *n*, the *n*-amalgamation condition (amalgamation up to order *n*) is a definable property in the same language. Therefore the sets $A(n)$ of parameters (δ, K, C) for which $\mathcal{A}^\delta_{K,C}$ has the *n*-amalgamation property form a decreasing sequence of Presburger-definable sets; and we want the intersection.

Admissible Choices of Parameters

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The following are equivalent.

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We write down the exact conditions for 5-amalgamation
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…and then give an amalgamation procedure that works. □

Corollary

The necessary conditions on δ, K, C *involve congruences and inequalities, with terms linear in the data.*

The Conditions

$$\text{Let } C = \min(C_0, C_1)$$

If $C \leq 2\delta + K_1$:

- $C = 2K_1 + 2K_2 + 1$; $K_1 + K_2 \geq \delta$; $K_1 + 2K_2 \leq 2\delta - 1$.
- If $C' > C + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$

If $C > 2\delta + K_1$:

- $K_1 + 2K_2 \geq 2\delta - 1$, $3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.
- If $C' > C + 1$ then $C \geq 2\delta + K_2$.

Is that all?

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1. Right about the triangles
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Remark

Γ_1 *is a homogeneous graph; exceptional cases lead to* $T_{r,s}$.

A Problem in Topological Dynamics

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Strong conjecture: exactly when we get amalgamation.
Weak conjecture: some Presburger condition.

?