Model Theory and Algebraic Groups

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Topics

- Model Theory of $\mathbb{C}$
- Diophantine problems in commutative algebraic groups
- Definably compact groups in expansions of $\mathbb{R}$
- Finite dimensional simple groups

Two properties of ACF_0

- The Nullstellensatz
- Structure theory

The Nullstellensatz (1893)

Theorem

TFAE

1. $V_K(f_1, \ldots, f_n)$ *has a point.*
2. $V_{K'}(f_1, \ldots, f_n)$ *has a point (some* $K' \supseteq K$*)*
3. $1 \notin (f_1, \ldots, f_n)$

$2 \iff 3$: $K' = K[X]/\mathfrak{m}$
$1 \iff 2$: this could be a definition . . . (Abraham Robinson)
"Existentially closed" fields

Existentially closed . . .

Ordered fields: Artin-Schreier (Hilbert's 17th problem) 1927
p-adically closed fields: Ax-Kochen 1965—Integrality-satz
Differentially closed fields: Seidenberg 1956/Robinson 1959
Separably closed fields: Ershov 1967
Existentially closed difference fields (ACFA): 1990's

- "Applied" model theory

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- "Applied" model theory

Question: Can one do "geometry" over any theory?

Structure Theory

Steinitz 1910
Transcendence basis, uniqueness of algebraic closure.

Corollary

Any ACF_0 of cardinality c is isomorphic to $\mathbb{C}$.

κ-categoricity (for κ uncountable)
$\kappa = \aleph_0$? $(\mathbb{Q}, <)$: something else entirely
Morley 1963 (answering Łoś): κ categoricity for one uncountable κ implies κ-categoricity for all

- “Pure” model theory

Dimension

Lemma (Morley)

If M is a model of an uncountably categorical theory then M has a well behaved notion of dimension for definable sets.

- Terminology: Morley rank, rk

$$\mathrm{rk} : \bigcup_n \mathrm{Def}(M^n) \to \text{ordinals}$$

ACF: $\mathrm{rk}(X) = \dim(\bar{X})$

Bonus: dimensions are actually finite (Baldwin, Zilber).

Worlds Collide I

Theorem (Lindstrøm)

If a theory is κ-categorical and closed under unions of increasing chains, then its infinite models are existentially closed.

Proof.

Suppose not.
Build M_1 and M_2 both of cardinality κ, one existentially closed and the other not.
(M_1): trivial
(M_2): cardinality shifting (Löwenheim-Skolem)

□

Worlds Collide II

Lenore Blum (1968): Differentially closed fields have Morley rank ω.
Application: uniqueness of differential closure (via Shelah).

Angus Macintyre (1971): fields with Morley rank are algebraically closed

Carol Wood (1979): separably closed fields are "stable" (local Morley rank)

Worlds Collide III

Hrushovski 1996: Geometric Mordell-Lang in all characteristics, with uniformities, via the model theory of abelian groups of finite Morley rank.

Mordell/Lang

Mordell/Faltings: Finiteness of rational points in genus ≥ 2.

$C \hookrightarrow J(C)$ Jacobian (abelian variety), $\dim(J) = \text{genus}(C)$
elliptic curve

Mordell/Weil: finite generation of rational points on an abelian variety.

Mordell/Lang: $C \cap \Gamma$ for Γ of finite rank.

Geometric Mordell/Lang

An analog of Mordell/Lang for K a function field over K_0

Theorem

$X \subseteq A$ K/K_0 function field, $X \cap \Gamma$ Zariski dense.

Then either Stab(X) is infinite or X comes from K_0.

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Theorem

$X \subseteq A$ K/K_0 function field, $X \cap \Gamma$ Zariski dense. Γ a subgroup of finite rank defined over the algebraic closure of K.
Then either Stab(X) is infinite or there is a bijective morphism $X \leftrightarrow X_0$ onto a variety X_0 defined over K_0.

Groups of finite Morley rank?

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Manin, Buium, ... use additional structure
K embeds into a differentially closed field $\hat{K}$ with K_0 as the constant field.
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Why not $\Gamma = A$?
$\mathrm{rk}\,(\hat{K}) = \omega$, so now finite rank is a *finiteness condition* on Γ
Now study $X \cap \hat{\Gamma}$.
Lost: the apparatus of algebraic geometry.
Kept: the theory of dimension.

Zariski Geometries

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Applies to finite dimensional sets in differentially closed fields by quantifier elimination (theory of prolongations)

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• Plug in various expansions of *ACF*
More: Scanlon, BSL 7 (2001)
Drinfeld modules, André-Oort

o-minimality

Definition

A structure with an ordering is *o-minimal* if every definable subset is a finite union of intervals.

Ref: van den Dries 1998: **Tame Topology and o-minimal structures**
Wilkie 1996: Reals with exponentiation
Speissegger 2000: the "Pfaffian closure" (Hovanski)

Groups definable in *o*-minimal structures

Generalized infinitesimals: $G^{\circ\circ}$
The intersection of the ∞-definable subgroups of *bounded index.*
Existence is highly nontrivial; granted existence, the quotient is a compact topological group in the "logic topology" (definable $\to$ closed).

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Restrict to: *definably compact groups*

Pillay's conjectures

Theorem

$G/G^{\circ\circ}$ is a compact Lie group, of the same real dimension as the formal dimension of G.

1. A descending chain condition for ∞-definable subgroups;
2. Control of dimension (?)

Three ingredients

- Structure theory (simple case)
- Topology (abelian case)
- Model theory (mixed case)

Simple Groups

Theorem (PPS 2000-2002)

Let G be a definably simple group in an o-minimal structure. Then G is a model of the same theory as some definably simple Lie group.

In the noncompact case $G^{\circ\circ} = G$ which is sad, but in the compact case $G^{\circ\circ}$ is the infinitesimal neighborhood of the identity and $G/G^{\circ\circ}$ is the corresponding compact Lie group.

Abelian Groups

$\bar{A} = A/A^{\circ\circ}$.

1. $\bar{A}[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{\dim \bar{G}}$ (clear)
2. $A[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{\dim G}$ (cohomology)
3. $A[m] \simeq \bar{A}[m]$ (model theory)

A little model theory

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Somewhere between: $\dim(G \setminus X) < \dim G$ and $\dim X = \dim G$.
Pathology: $\mathbb{R} = (\infty, 0] \cup [0, \infty)$ the union of two nongeneric sets.
Peterzil-Pillay: not in definably compact groups.

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$x = 2x + 1$, $\chi(0,1) = -1$.

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$\chi = \pm 1 \implies$ torsion-free

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Proof.

Show $p | \chi(\{a : a^p = 1\})$.
$X = \{(a_1, \ldots, a_p) : \prod_i a_i = 1\}$

$$\chi(\{a : a^p = 1\}) = \chi(X \cap \Delta) = \chi(X) - \chi(X \setminus \Delta)$$

and both terms at the end are divisible by p. □

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Cor. No elementary abelian p-groups.
Proof: By Lagrange $\chi(A) = 0$ so by Cauchy there are elements of all prime orders.

Composition Series

The general group is neither simple nor abelian. How can we climb up a composition series?

Hrushovski, Peterzil, Pillay: Groups, measures, and the NIP. Generalized stable group theory.

Algebraicity Conjecture

Conjecture

A simple group of finite Morley rank is algebraic.

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Let G be a simple group of finite Morley rank containing an infinite elementary abelian 2-subgroup. Then G is algebraic.

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Let G be a simple group of finite Morley rank containing an infinite elementary abelian 2-subgroup. Then G is algebraic.

Parabolic subgroup: contains $N(S)$ for S a Sylow 2-subgroup.
Thin (1 minimal parabolic): strong embedding.
Quasi-thin (2 minimal parabolics): amalgam method.
Generic type (many minimal parabolics): Niles' theorem.
Generation: $C(G, T)$ theorem.

Something more geometric

Theorem (BBC)

A connected group of finite Morley rank containing an involution has an infinite Sylow 2-subgroup.

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“Semisimple torsion”: Altınel, Burdges, Deloro, Frécon
Burdges-Deloro: The Weyl group in a minimal simple group is cyclic.

Carter Subgroups

Definition: Connected, almost self-normalizing, and nilpotent.

Theorem (Frécon-Jaligot 2005)

Carter subgroups exist.

(via Burdges unipotence theory.)

Genericity and Generosity

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Lemma

A point belonging to finitely many conjugates of a given Q belongs to a unique one.

Proof.

X: the intersection of the conjugates. $N(X)$ acts on the set of such conjugates, so $N^\circ(X)$ normalizes each one; as they are Carters, $N^\circ(X)$ is contained in each one, hence in X, and so $X = Q$. □

Conjugacy of Carter subgroups

Theorem (Frécon, in press)

Carter subgroups of K^*-groups are conjugate.

. . . a tour de force

Conclusion

- One can do a surprising amount of geometry equipped with a rudimentary notion of dimension, particularly when inside a group.

And, this is sometimes useful.