Permutation groups of Finite Morley rank

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Theorem

Theorem (with Borovik)

If G is an algebraic group acting rationally, faithfully, and primitively on a variety X then the dimension of G can be bounded in terms of the dimension of X.

(Proved more broadly in the category of groups of finite Morley rank.)

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Conjecture

A simple group of finite Morley rank is algebraic.

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Let G be a simple group of finite Morley rank containing an involution. Then one of the following holds.

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Theorem (BBC: Nondegeneracy)

Let G be a connected group of finite Morley rank containing an involution. Then G contains an infinite 2-subgroup.

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Simplifying ingredient: fields are algebraically closed.
In finite group theory, the torus $\mathbb{F}_2^*$ reduces to a single element.
This breaks "generic" arguments.
Sporadic groups are only one manifestation of this.

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Wagner: in the multiplicative group of a field of finite Morley rank, in positive characteristic, torsion is dense (rigidity).

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"Black Box group theory" ...

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Either $d(ij)$ contains a unique involution k (commuting with i and j) or i, j are conjugate under the action of $d(ij)$. Furthermore one of these two possibilities holds *generically*. So the argument splits to two cases.

Case I: $d(ij)$ contains no involution

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Conclusion:

- Fiber ranks of ζ over points of $C(i)$ are constant;
- $\deg(G) \geq \deg(C(i))$

Hence G connected implies $C(i)$ connected.
We use a generically defined covariant map to transfer coarse structure between G and $C(i)$.

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Using centralizers of involutions (p odd).

Question: If G is a black box group, is $C(i)$ a black box group?

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In the favorable case $\zeta : G \to C(i)$ generically, $\zeta(g)$ picks elements of $C(i)$ randomly.

The uniform measure on G is carried to the uniform measure on $C(i)$.

Main Theorem

Theorem (Borovik/Cherlin)

If G is a group of finite Morley rank acting definably, faithfully, and primitively on a set X then the rank of G can be bounded in terms of the rank of X.

Imprimitive Example

$T = K^\times$ acting on V via $(t^i v_i)$, $W \leq V$ hyperplane in general position.

$\hat{G} = V \rtimes T$ acting on the coset space $X = W \backslash \hat{G}$ (faithfully).

$rk\,(X) = 2$, $rk\,(\hat{G}) = \dim(V) + 1$.

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If we wish to bound $rk\,(G)$ we must bound t. Conversely, in the primitive case, this is enough.

Bounds on t

(G, X) finite Morley rank, $r = rk\,(X)$ (fixed)

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Reduction to the simple case via Macpherson-Pillay form of O'Nan-Scott-Aschbacher.

Key Lemma

Lemma

T the definable closure of p-torus, acting faithfully on X. Then $rk\,(T/O_0(T)) \leq rk\,(X)$.

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Ingredient: Every involution belongs to some 2-torus (cf. torality theorem, Burdges-Cherlin).

Better bounds?

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Known in characteristic 0 with rational actions.
Not known in characteristic p with rational actions or in characteristic 0 with definable actions.

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Though we do not have an explicit classification of the simple groups of finite Morley rank, we can apply the theory as it stands in much the way that we would apply a full classification.

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Another “old chestnut”:

Theorem

If G is connected and satisfies the following equation generically:

$$x^{2^n} = 1$$

then indeed G is a 2-group of exponent at most 2^n.

Again, this seems to need the classification theory, in spite of its apparently elementary character.