Lab 3: The Pendulum

This Maple lab is closely based on earlier versions prepared by Professors R. Falk and R. Bumby of the Rutgers Mathematics department.

Introduction.   In this lab we use Maple to examine differential equations modeling free oscillations. In particular, we shall consider a linear model in part 1 and a nonlinear model of a pendulum in part 2, comparing the effect of damping in these models. For the linear equation, there are exact solutions that allow these properties to be studied via formulas. For the nonlinear equation, numerical solutions will be used.

Please obtain the seed file from the web page and save it in your directory on eden. Then prepare the Maple lab according to the instructions and hints in the introduction to Lab 0. Turn in only the printout of your Maple worksheet, after removing any extraneous material and any errors you have made.

0. Setup.   As usual, the seed file begins with commands which load the required Maple packages: `with(plots):` and `with(DEtools):`. The seed file also includes the definitions of some variables which will be used throughout the lab, as discussed in section **1** below.

1. The linear model.

Consider the motion of a pendulum, which consists of a mass m attached to one end of a rigid rod of length L. The other end of the rod is fixed at a point O and the rod is free to rotate, within a fixed vertical plane, about O. The position of the pendulum at time t is described by the angle $\theta(t)$ between the rod and the downward vertical direction, with the counterclockwise direction taken as positive (see Figure 9.2.2 on page 498 of the text). The differential equation governing the motion of the pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

It is expected that small changes in the equation will lead to small changes in the solution. Two modifications designed to approximate the equation by one that is more easily solved are to ignore damping by setting $c = 0$ or to replace $\sin\theta$ by θ to get a linear equation (or both). Note that θ is just the first term of the Taylor series for $\sin\theta$ about $\theta = 0$. As long as θ is *small*, so that the difference between $\sin\theta$ and θ is *very small*, the solutions to this linear equation should give a good approximation to those of the general equation. This project will show that, even when θ is not small, the *qualitative* aspects of the solution in the nonlinear case will be similar to the more easily determined properties of the linear case.

In a mathematical treatment, only the values of the coefficients and not their expression in terms of physical quantities will enter, so we write the general equation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\sin\theta = 0 \tag{G}$$

and its linear approximation as

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + K\theta = 0 \tag{L}$$

This study is expected to take place on earth, where the constant g, the acceleration due to gravity, is 9.81m/sec^2. For convenience we will take $K = g/L = 9$, corresponding a pendulum rod of length $L = 1.09$ meters. The examples that we consider will have values of the quantities m and c so that $\mu = c/(mL)$ takes on the values 1, 6, and 9. We also fix the initial conditions $\theta(0) = 1$, $\theta'(0) = 4$ corresponding to the pendulum being released 1 radian from its rest position, with a vigorous push away from that position. We will always graph our solutions over an interval from $t = 0$ to $t = 10$.

These general definitions are collected in the following statements that appear in the setup section of the seed file:

```
K:=9;
deG:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*sin(theta(t))= 0;
deL:=diff(theta(t),t,t) + mu*diff(theta(t),t)+K*theta(t)= 0;
Iv:=theta(0)=1, D(theta)(0)=4;
dom1:=t=0..10;
```

1a. Exact solution: undamped motion.   The `dsolve` command can be used to compute the exact solution of each linear model. Its use is not exactly the same as for first order equations, since we now have initial values of two quantities, $\theta(t)$ and its derivative, at $t = 0$. To solve equation (L) without damping and plot the solution, execute the following commands, which appear in the seed file.

```
soln1a:=dsolve({eval(deL,mu=0),Iv});
gr1a:=plot(rhs(soln1a),dom1,color=BLUE):
```

Note that the statement constructing the *plot structure* `gr1a` ends with a colon, so that the graph is not shown; you can take a look at the graph with the command `gr1a;`, but don't leave the graph or the command in your worksheet. Note also that the command constructing the graph includes a choice of color; in constructing further graphs in section **1b** below you should use a color other than blue for the graph of section **1b**, and two different colors for the graphs of sections **1c** and **1d**, so that it will be easy to distinguish them when they are plotted on the same set of axes.

1b. Damped motion.   Copy the two commands from section **1a** three times, modifying them to obtain solutions and plot structures for the linear damped equations for the three cases $\mu = 1, 6, 9$. Use different names for all these solutions and structures, and different colors for the graphs. As in **1a**, it is a good idea to look at the graphs you are producing, but *do not* leave them in the worksheet.

By looking at the solutions obtained, you should be able to determine, for each value of μ, whether the solution is underdamped, overdamped, or critically damped. Give your conclusions, with brief justifications, in the discussion section.

1c. Underdamping.   Select the graph built in (b) that illustrates an *underdamped* system and use the *display* command to combine it with the *undamped* graph from (a) in a single `display`. Be sure to give this graph a `title`. Then *compare* these two graphs in a *text discussion*, addressing the difference between the times when $\theta = 0$ for the two functions and the properties of the *critical points* (i.e., points where $d\theta/dt = 0$) on the two graphs.

1d. Overdamping.   Select the graphs in (b) that illustrate *critical damping*, and *overdamping* and combine them in a single `display`. Be sure to give this graph a `title`. Then *compare* these two graphs in a *text discussion*, addressing the properties of the *critical points* (where $d\theta/dt = 0$) on the two graphs and the rates at which the two functions approach zero. In particular, which decreases faster: the *critically damped* or the *overdamped* example?

1e. Overshoot.   Our solutions in the critically damped and overdamped cases approach the $\theta = 0$ axis without crossing it, but if the initial velocity is directed toward the equilibrium position and is sufficiently large the pendulum will "overshoot", passing the equilibrium position before settling back toward it. Experiment with this effect with the parameter μ found above to give overdamped motion: introduce a new initial condition (give it a new name), keeping $\theta(0) = 1$ but changing $\theta'(0)$ until you find value which produces an overshoot of about 0.1 radian. Include in the worksheet only the graph showing this overshoot and the commands needed to produce that graph. In the discussion section give explicitly the value of the initial velocity that you found.

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2. The nonlinear model.

For nonlinear equations, an *exact solution in elementary functions* cannot always be found. In such cases, the standard use of the `dsolve` command will not give a useful result. Instead, we use the `DEplot` command to plot a numerical solution. The `DEplot` command works for higher order equations, but it shows only a graph (there is no direction field since it is a *second derivative* that is determined by the equation; there will be unique solutions corresponding to initial values of both a *point* and a *direction*, which means that there are solutions in *all* directions through each point).

The seed file contains a definition of one possible set of options for the DEplot command, together with a command producing a plot structure, using those options, for the solution of the nonlinear equation with no damping, i.e., $\mu = 0$. The next command then combines the graphs for this case for the linear and nonlinear problems in a single `display`.

```
opt2a:=linecolor=BLACK,thickness=1;
gr2a:= DEplot(eval(deG,mu=0),theta(t),dom1,{[Iv],opt2a});
display({gr1a,gr2a});
```

2a. Undamped motion. For $\mu = 0$, both equations should have *periodic* solutions. *Compare the period* of the solutions of the two equations. Also, *compare the maximum value* of the solutions of the two equations.

In each of the remaining parts, construct a graph of a numerical solution to the nonlinear equation (G) with a value of μ which is the same as one used to construct a solution of (L) in part 1. For each choice of μ, your main worksheet should contain the two graphs, for the linear and nonlinear problems, combined in a single `display`. You should then *discuss* the differences between the graphs in *text*. Suggestions for that discussion are given in the individual sections below.

2b. Underdamped motion. These solutions repeatedly return to $\theta = 0$. In the linear case, the time between these returns is easily seen to be regular. Does this appear to also be true in the nonlinear case? Also, *compare* the time between returns for the two graphs.

2c. Critically damped motion. One important feature of this case is the *maximum value* of θ and the time at which it attained. *Compare* these between the solutions of the linear and nonlinear equations.

2d. Overdamped motion. Does the decay of the solution in the nonlinear case resemble the *exponential decay* of the linear case? *Compare* the rate of decay.

End of Lab 3