Mathematics 251: Lab 3 MAXIMUM AND MINIMUM VALUES

In this lab, we use Maple to help visualize and compute the maximum and minimum values of a function of two variables.

Please turn in only the printout of your Maple worksheet, which should include the Maple commands you input and Maple's response. Your worksheet should also include explicit answers to all questions asked and labels on any plots included in your worksheet. These should be inserted by using the **text** feature of Maple. Also use the **text** feature of Maple to include your name and section number before each problem (do NOT write any of this material in by hand). Be sure to remove from the worksheet any extraneous material and any errors you have made.

1. In this problem we wish to find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = (y^3/9) + 3x^2y + 9x^2 + y^2 + xy + 9;$$

1a. First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \leq x \leq 1.5$, $-7 \leq y \leq 1$. Use the `axes=BOXED` option so you can see the values of f.

1b. Now use Maple's `diff` and `solve` commands to find all the critical points of f. This can be done by first assigning names such as `fx` and `fy` to the expressions for $\partial f/\partial x$ and $\partial f/\partial y$ obtained by the `diff` command, i.e.,

```
fx:=diff(f,x); fy:=diff(f,y);
```

and then using the `solve` command to obtain the values of x and y for which the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, are satisfied, i.e.,

```
solve({fx=0,fy=0},{x,y});
```

1c. Again using Maple's `diff` command, evaluate f_{xx} and $f_{xx}f_{yy} - (f_{xy})^2$ at each critical point to determine whether it is a local maximum, local minimum, or saddle point. Clearly label in your Maple worksheet the type of each critical point. Note that to evaluate an expression such as `fx` at the point $(1,2)$, one uses the `subs` command, i.e., `subs({x=1,y=2},fx);`

To evaluate both `fx` and `fy` at the point $(1,2)$, use `subs({x=1,y=2},{fx,fy});`

Sometimes the solutions obtained from the `solve` command might be quite long, in which case one would like to avoid retyping them. This may be done by assigning a name to the result, e.g.,

```
soln:= solve({fx=0,fy=0},{x,y});
```

In case there is more than one solution, use the syntax `soln[3]` to refer to the third one. Hence, to evaluate both `fx` and `fy` at `soln[3]`, type

```
subs(soln[3],{fx,fy});
```

1d. For each critical point, obtain a plot of f in a small region around the critical point, which clearly shows whether it is a local maximum, a local minimum, or a saddle point.

Be sure to include a text label of each plot giving the coordinates and type of the critical point. Your classification should agree both with the results of (1c) and the appearance of the graph.

2. Now consider the function $f(x, y) = y(1 - 10xy)\exp(-x^2 - y^2)$. In this case, we would like to find the absolute minimum and absolute maximum of f. Before continuing with Maple, don't forget to use the **`restart`** command; otherwise many of your variables may already be defined.

2a. First obtain a rough idea of what this function looks like, by plotting it over the region $-5 \le x \le 5$, $-5 \le y \le 5$. This should show that the function is close to zero except for a small neighborhood of the origin. To get a better look, plot the function again using the region $-2 \le x \le 2$, $-2 \le y \le 2$.

2b. What happens if you use Maple's **`diff`** and **`solve`** commands to try to find the critical points of f?

2c. Now use the **`fsolve`** command to find the critical points which lie in the region $-2 \le x \le 2$, $-2 \le y \le 2$. There are exactly four of them, as indicated by the plot in part (a). Also find the values of the function f at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. **It is not necessary to use a second derivative test in this problem or in part 2d below.**

The **`fsolve`** command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. What happens if you use the command in the form

```
fsolve({fx=0,fy=0},{x,y});
```

where **`fx`** and **`fy`** are the names assigned to the expressions obtained for the partial derivatives f_x and f_y, respectively? To improve the chance of convergence, add a region in which you expect the root to lie. You can obtain such a region by using the second of the plots obtained in part (a). For example, one of the critical points can be found by using the following form of the **`fsolve`** command.

```
p1:= fsolve({fx=0,fy=0},{x,y}, x =-1..0, y=0..1);
```

Note that as in Problem 1, by assigning the name **`p1`** to the solution of the **`fsolve`** command, you can then evaluate the function $f(x, y)$ at the values of x and y given by Maple in response to this command by typing `evalf(subs(p1,f));`
assuming that **`f`** is the name assigned to the expression for the function f. The command **`evalf`** is used to convert the result of the **`subs`** command to a numerical value.

2d. From your answer in 2c and the plot in 2a, determine the absolute minimum and absolute maximum values of the function f?

The results of this problem will be used in the next one, so do **not** use the **`restart`** command before beginning Problem 3.

Problem 3 is an optional bonus problem.

3. In this problem, we find the absolute minimum and absolute maximum of the function f of Problem 2, but this time restricted to the semicircular region R given by $y \geq 0$, $x^2 + y^2 \leq 1.5$. We know from the general theory developed in class (or in the book) that the absolute minimum and maximum of f occur either at critical points of f which lie in the interior of R or on the boundary of the region R. As in Problem 2, it is **not** necessary to use the second derivative test in this problem.

3a. Obtain a plot of f over the region R.

3b. Determine which of the critical points found in Problem 2 lie in the region R and evaluate f at these points. Note that a simple way to check whether the point given by `p1` is inside the circle $x^2 + y^2 = 1.5$ is use the `subs` command to evaluate the expression $x^2 + y^2 - 1.5$ at `p1`. If this value is less than zero and if the y component of `p1` is ≥ 0, then the point `p1` lies in the region R. Use the `text` feature of Maple to clearly state your conclusions.

3c. The boundary of R consists of two pieces. The first is the line segment $y = 0$, $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. What is the value of f on this line segment? Note that you don't need Maple to find it.

3d. The second piece of the boundary of R is the semicircle $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$, $y = \sqrt{1.5 - x^2}$. To find the maximum and minimum of f on the semicircle, set $g(x) = f(x, \sqrt{1.5 - x^2})$ and find the maximum and minimum of g over the interval $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. This can be done by first using the `plot` command to obtain a plot of the function g over the interval $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. From the plot you should be able to see whether the maximum or minimum of g occurs at one of the end points of the interval or at a place where $g'(x) = 0$. To find more precise values where $g'(x) = 0$, use the `fsolve` command. This command performs better if you supply ranges for x containing $xmax$ and $xmin$, the values of x you are seeking at which the maximum and minimum occur. These ranges are easily obtained by looking at your plot of $g(x)$. Once $xmin$ and $xmax$ are obtained, find $ymax$ and $ymin$, the corresponding values of y on the semicircle, and evaluate f at the points $(xmax, ymax)$ and $(xmin, ymin)$.

3e. Now compare the values of f at the critical points inside R (from part (b)), on the line segment $y = 0$ (from part (c)), and at the points $(xmax, ymax)$ and $(xmin, ymin)$ (from part (d)). Clearly state the maximum and minimum values of f over the region R and the coordinates of the points at which they occur.