Math 373 Assignment 4 (Due Thursday, October 2.)

1. Using the Newton form of the interpolating polynomial, determine a cubic polynomial $P(x)$ satisfying

$$P(1) = 1, \qquad P(3) = 5, \qquad P'(3) = 4, \qquad P''(3) = 14.$$

2. Consider the problem of determining a polynomial $P_2(x)$ of degree ≤ 2 satisfying

$$P_2(x_0) = f(x_0), \qquad P_2'(x_1) = f'(x_1), \qquad P_2(x_2) = f(x_2),$$

where f, x_0, x_1, x_2 are given and $x_0 < x_1 < x_2$. Does this problem have a solution for all given f in $C^1[x_0, x_2]$ and all x_0, x_1, x_2 ? If so, find it; if not, give an example for which there is no solution. Note that this problem does not fit the theory developed in class.

3a. Let $I = \{-1, -1/2, 0, 1/2, 1\}$ and $f(x) = 1/(1+x^2)$. Making use of the following divided difference table,

Divided difference table

x	f(x)	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
-1	.5				
		.6			
-.5	.8		-.2		
		.4		-.4	
0	1		-.8		.4
		-.4		.4	
.5	.8		-.2		
		-.6			
1	.5				

find the value of each of the following at $x = 3/4$.

i) The polynomial $P(x)$ of degree ≤ 4 interpolating $f(x)$ on the set I.

ii) The piecewise linear function $L(x)$ defined on a mesh of width $1/2$ interpolating $f(x)$ on I.

iii) The piecewise quadratic function $Q(x)$ defined on a mesh of width 1 interpolating $f(x)$ on I.

3b. Using the error formula, find the smallest bound on the quantity $\max_{x\in[-1,1]} |f(x) - L(x)|$ where $L(x)$ is the approximation given in (ii). As a computational check,

$$f'''(x) = -\frac{24x(x^2-1)}{(1+x^2)^4}.$$

4. Let $a = x_0 < x_1 < \cdots < x_N = b$, where $x_i = a + ih$ and $h = (b-a)/N$. Note that $x_i - x_{i-1} = h$ for all values of i in this range. Let $Q(x)$ be a piecewise quadratic function defined by $Q(x) = Q_i(x)$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, N$, where $Q_i(x)$ is a quadratic polynomial satisfying:

$$Q_i(x_{i-1}) = f(x_{i-1}), \qquad Q_i(x_{i-1/2}) = f(x_{i-1/2}), \qquad Q_i(x_i) = f(x_i),$$

where $x_{i-1/2} = (x_{i-1} + x_i)/2$.

a) Give an explicit formula for $Q_i(x)$.

b) Will $Q(x)$ be a continuous function if $f(x)$ is a continuous function?

c) Show that

$$\max_{x \in [x_{i-1}, x_i]} |(x - x_{i-1})(x - x_{i-1/2})(x - x_i)| = \frac{h^3}{12\sqrt{3}}.$$

HINT: Let $y = x - x_{i-1/2}$. Then

$$\max_{x \in [x_{i-1}, x_i]} |(x - x_{i-1})(x - x_{i-1/2})(x - x_i)| = \max_{y \in [-h/2, h/2]} |(y - h/2)y(y + h/2)|.$$

d) If $|f^{(3)}(x)| \leq M_{3,i}$ for all $x \in [x_{i-1}, x_i]$, find an upper bound on $|f(x) - Q_i(x)|$ valid for all $x \in [x_{i-1}, x_i)]$ which depends only on $M_{3,i}$, and h (but does not depend on x).

e) Let $M_3 = \max_{1 \leq i \leq N} M_{3,i}$. Show that for all $x \in [a, b]$,

$$|f(x) - Q(x)| \leq \frac{M_3}{3!} \frac{h^3}{12\sqrt{3}}.$$

f) If $f(x) = 24e^{3(x-1)}$ and $[a, b] = [0, 1]$, determine a value of h that will guarantee an error of $\leq 10^{-6}$ for all $x \in [0, 1]$.