Math 373 Assignment 5 (Due Thursday, October 16.)

1. Let V_2 denote the space of continuous, piecewise quadratic functions defined on a partition $\mathcal{P}$: $a = x_0 < x_1 < \cdots < x_N = b$, where $x_i = a + ih$ and $h = (b-a)/N$. We showed that the dimension of V_2 is $M \equiv 2N + 1$. We wish to construct a set of basis functions $\phi_0, \cdots, \phi_N$ and $\psi_1, \cdots, \psi_N$ in the space V_2, with the property that each ϕ_i is different from zero only on an interval of length at most $2h$ and each ψ_i is different from zero only on an interval of length at most h.

Determine $\phi_i(x) \in V_2$ and $\psi_i(x) \in V_2$ so that any $Q \in V_2$ can be written in the form:

$$Q(x) = \sum_{i=0}^{N} \phi_i(x) Q(x_i) + \sum_{i=1}^{N} \psi_i(x) Q(x_{i-1/2}).$$

2a. Determine a C^1 piecewise quadratic function $Q(x)$ satisfying the following properties.

(i) $Q(x)$ is a quadratic on each of the following three subintervals: $[-1, 0]$, [0,1], [1,2]. (Denote these quadratics by Q_0, Q_1, and Q_2, respectively.) (ii) Q and $Q^{'}$ are continuous on $[-1, 2]$. (iii) $Q(-1) = Q'(-1) = Q(2) = Q'(2) = 0$, (iv) $Q(-1/2) + Q(1/2) + Q(3/2) = 1$.

Hint:

(i) First show that $Q_0(x) = A(x+1)^2$ and $Q_2(x) = C(2-x)^2$, for some constants A and C.

(ii) Next, use a Taylor series expansion to show that $Q_1(x) = A + 2Ax + Bx^2$ for some constant B.

(iii) Use the remaining conditions to determine A, B, and C.

2b. What are the values of $Q(0)$, $Q(1/2)$, and $Q(1)$?

2c. Draw a graph of the function $Q(x)$ on the interval $[-1, 2]$.

3. The following on-line demos help explain the difference between interpolation by single polynomials and cubic spline interpolation.

3a. Go the web page http://www.math.ucla.edu/˜baker/java/hoefer/Lagrange.htm Click once on the Add point icon so that there are 6 interpolation points, to be fit by a single polynomial of degree ≤ 5. Click on the 5th interpolation point from the left (second from the right) and move it up. Does the graph change on the left most subinterval?

3b. Go the web page http://www.math.ucla.edu/˜baker/java/hoefer/Spline.htm which shows 6 interpolation points to be fit by a cubic spline. Click on the 5th interpolation point from the left (second from the right) and move it up. Does the graph change on the left most subinterval?

You can also compare parts(a) and (b) on the same page by going to the web page http://www.math.ucla.edu/~baker/java/hoefer/TwoDemos.htm Don't forget to add one point to the bottom interpolation, so that they are both using the same number of interpolation points.

4. Go to the web page http://www.math.ucla.edu/~baker/java/hoefer/Bezier.htm Follow the instructions to create the four control points and the corresponding Bezier cubic polynomial. Now click on each of the control points to see the effect on the curve.

What happens when you move the left and right most points?

What happens when you move the two interior points?

5. Let $P_2(x)$ be the polynomial of degree ≤ 2 interpolating $f(x)$ at $x = x_0$, x_1, and x_2.

a) Using the error formula

$$f(x) - P_2(x) = f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2),$$

the fact that

$$(d/dx)f[x_0, \ldots, x_n, x] = f[x_0, \ldots, x_n, x, x],$$

and the recursion formula for divided differences,

$$f[x_0, x_1, x_2, x, x] = \{f[x_1, x_2, x, x] - f[x_0, x_1, x_2, x]\}/(x - x_0),$$

show that

$$\begin{aligned} f'(x) - P_2'(x) &= f[x_1, x_2, x, x](x - x_1)(x - x_2) \\ &\quad + f[x_0, x_1, x_2, x](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x](x - x_0)(x - x_2). \end{aligned}$$

b) If $x_2 - x_1 = x_1 - x_0 = h$ and $f \in C^3[x_0, x_2]$, show that for all $x \in [x_0, x_2]$

$$|f'(x) - P_2'(x)| \leq \frac{5}{6}h^2 M_3, \qquad \text{where} \qquad M_3 = \max_{x_0 \leq x \leq x_2} |f'''(x)|.$$

Hint: Bound each term in the error separately and add the results.