Math 373 Assignment 7 (Due Tuesday, November 4)

1. In this problem, we determine the two point Gaussian quadrature formula and error term for approximating $\int_0^1 f(x)\,dx$, i.e.,

$$\int_0^1 f(x)\,dx = H_0 f(x_0) + H_1 f(x_1) + E,$$

where H_0, H_1, x_0, x_1 are to be determined to make the resulting formula exact for f a polynomial of as high a degree as possible and E is the error term.

a) Using Lanczo's orthogonalization theorem, find the first three orthogonal polynomials Φ_0, Φ_1, Φ_2 with respect to the inner product

$$(f, g) = \int_0^1 f(x)g(x)\,dx.$$

b) Find the two roots x_0 and x_1 of $\Phi_2(x)$ and check that they are both real and lie in the interval $[0, 1]$.

c) Use the formulas developed in class to find the weights H_0 and H_1.

d) What does the error formula given in class give for the error in this particular case, i.e., determine the value of n and the corresponding constant γ_{n+1}.

e) What is the highest degree polynomial for which the resulting quadrature formula is exact?

2. In this problem, we develop a quadrature formula of the form

$$\int_a^b f(x)\,dx = H_0 f(a) + H_1 f(x_1) + E, \tag{1}$$

where H_0, H_1, and x_1 are to be determined to make the resulting formula exact for f a polynomial of as high a degree as possible. In the above, a is considered fixed and E denotes the error term.

a) Find quadratic polynomials $R_0(x)$, $R_1(x)$, and $R_2(x)$ depending on a and x_1 such that the quadratic polynomial $P(x)$ satisfying $P(a) = f(a)$, $P(x_1) = f(x_1)$, $P'(x_1) = f'(x_1)$ can be written in the form $f(a)R_0(x) + f(x_1)R_1(x) + f'(x_1)R_2(x)$. (Use the Newton form of the interpolating polynomial.)

b) By integrating $P(x)$, we get a formula of the form

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + E = H_0 f(a) + H_1 f(x_1) + H_2 f'(x_1) + E,$$

where E denotes the error. Express H_0, H_1 and H_2 in terms of R_0, R_1, R_2. Do not perform any integrations.

c) Show that if x_1 is chosen so that

$$\int_a^b (x-a)(x-x_1)\,dx = 0,$$

then one obtains a formula of the form (1) that is exact (i.e., $E = 0$) for quadratic polynomials.

d) Solve for x_1 in the special case $a = -1$, $b = 1$.

e) Again consider the special case $a = -1$, $b = 1$. By integrating the error formula for the polynomial approximation found in part(a), determine a formula for the error E in this quadrature formula (called a Radau quadrature formula) and simplify it as much as possible.

3. In this problem, we compare the use of Simpson's rule using an interval doubling strategy (the code quadsimp.m) and the use of an adaptive Simpson's rule, *Matlab's* `quad` function.

The command `[q, fcnt] = quad('f',a,b,tol)` will return in `q` an approximation to the integral of the function `f` given in the *Matlab* file `f.m` over the interval $[a, b]$, where `tol` is the absolute error tolerance. The variable `fcnt` returns the total number of function evaluations it took to compute the integral.

Use the function `quad` to produce approximations to the integral of the functions

$$f_1(x) = (1-4x(1-x))^{1/3}, \qquad f_2(x) = xe^{-x}.$$

over the interval $[0, 1]$, corresponding to each of the error tolerances `tol` $= 10^{-6}$, `tol` $= 10^{-12}$. For the same functions and error tolerances, use `quadsimp` to compute the integrals. Hand in the results, properly labeled, of these *Matlab* computations. Compute the final number of function evaluations needed for the `quadsimp` code (note this will be 1 more than twice the number of subintervals). Which one of these codes seems to do a better job in the sense of producing a given accuracy with fewer function evaluations?

Note that these are the same functions used in Assignment 6 and so you should have already copied to your home directory *m* files for these functions. Use `format long` in make sure you have enough accuracy.