Math 373 Assignment 8 (Due Tuesday, November 11)

1. Determine whether each of the following functions satisfies a Lipschitz condition for $1 \le x \le 2$ and $-\infty < y < \infty$.

$$f(x,y) = \frac{1+y}{2+x}, \qquad f(x,y) = \frac{y^2}{2+x}.$$

If it does, find the Lipschitz constant L. If not, explain why it does not.
HINT: $y^2 - z^2 = (y-z)(y+z)$.

2. Find an approximation to $y(2)$ using Euler's method with step size $h = 1/2$ for the initial value problem $y' = (1+y)/(2+x)$, with $y(1) = 2$.

3. Let y be the solution of the initial value problem

$$y' = f(x,y), \qquad y(x_0) = y_0.$$

Let $M_2 = \max |y''(x)|$ and $e_n = y(x_n) - y_n$, where $\{y_n\}$ is the approximation produced by Euler's method with constant step size h. Suppose $d \le f_y(x,y) \le 0$ for all (x,y), and h is sufficiently small so that $1 + hd \ge 0$.

a. Following the procedure given in class, and using the Mean Value Theorem to write for some number z_n,

$$f(x_n, y(x_n)) - f(x_n, y_n) = f_y(x_n, z_n)(y(x_n) - y_n),$$

show that

$$e_{n+1} = [1 + hf_y(x_n, z_n)]e_n + (h^2/2)y''(\xi_n).$$

b. Using part (a) and the hypotheses, show that

$$|e_{n+1}| \le |e_n| + h^2 M_2/2.$$

c. If $e_0 = 0$, iterate the above equation to show that

$$|e_n| \le \frac{h}{2}(x_n - x_0)M_2.$$

b) For the problem

$$y' = -2y, \qquad y(0) = 1,$$

compare the above error bound with the one obtained in class when $x_n = 10$.

4. For the Initial Value Problem

$$y' = f(x,y) = xy, \qquad y(1) = 1,$$

find an explicit formula in terms of x and y for $T_3(x,y)$ in the Taylor algorithm of order 3.