1. Brief Review of Calculus

To be successful in this course in Numerical Analysis, a student needs to understand and use a substantial amount of material from the first three semesters of calculus and a course in differential equations. In Section 1.1 of the textbook is a compilation of some of the most fundamental ideas from these courses. For the first topic to be covered, the solution of nonlinear equations, we will make use of the following results.

Definition: The sequence $\{x_n\}_{n=1}^{\infty}$ of real or complex numbers converges to s (has a limit s) if, for any $\epsilon > 0$, there exists a positive number $N(\epsilon)$ such that $|x_n - s| < \epsilon$ whenever $n > N(\epsilon)$. We write this as:

$$\lim_{n\to\infty} x_n = s, \quad \text{or} \quad x_n \to s \quad \text{as} \quad n \to \infty.$$

Definition: $C[a, b]$ – the set of continuous functions defined on the interval $[a, b]$.

Definition: $C^n[a, b]$ – the set of functions defined on the interval $[a, b]$ which are continuous and whose first n derivatives are also continuous.

Intermediate Value Theorem: If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$.

Example 1: $f(x) = x, 0 \le x < 1; f(x) = 1 + x, 1 \le x \le 2$. If $K = 1.5$, there is no value of c such that $f(c) = K$. But f is not continuous, so the theorem does not apply.

Example 2: $f(x) = x^3, -1 \le x \le 1$.

Mean Value Theorem: If $f \in C[a, b]$ and f is differentiable in (a, b), then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We shall use this theorem in the form: $f(b) - f(a) = f'(c)(b - a)$.

Example: $f(x) = x^3$ on $[0, 2]$. $c = 2/\sqrt{3}$.

Extreme Value Theorem: If $f \in C[a, b]$, then there are numbers $c_1, c_2 \in [a, b]$ such that $f(c_1) \le f(x) \le f(c_2)$ for all $x \in [a, b]$. In addition, if f is differentiable on (a, b), then the numbers c_1 and c_2 occur either at the endpoints of $[a, b]$ or where $f' = 0$.

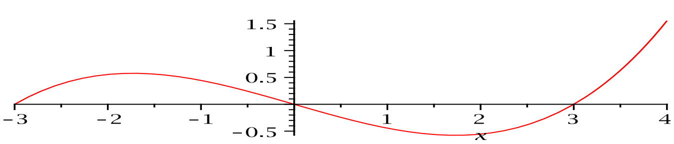

Figure 1. $f(x) = x^3/18 - x/2$ on $[-3, 4]$

Note that the minimum of f occurs at a point where $f' = 0$, but the maximum occurs at $x = 4$.

Taylor's Theorem: Suppose that $f \in C^n[a,b]$, that f^{n+1} exists on $[a,b]$, and $x_0 \in [a,b]$. Then for every $x \in [a,b]$, there exists a number $\xi(x)$ lying between x and x_0, such that $f(x) = P_n(x) + R_n(x)$, where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \\ R_n(x) &= \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}. \end{aligned}$$

Since $\xi(x)$ is not known, we only know a form for the error term $R_n(x) = f(x) - P_n(x)$, rather than a formula which we can evaluate. Typically, we seek an upper bound on the error, i.e.,

$$|R_n(x)| = \frac{|f^{(n+1)}(\xi(x))|}{(n+1)!}|(x - x_0)^{n+1}| \leq \frac{|(x - x_0)^{n+1}|}{(n+1)!} \max_{x \in [a,b]} |f^{(n+1)}(x)|.$$

Although in many cases, determining $\max_{x \in [a,b]} |f^{(n+1)}(x)|$ is not possible, the formula still gives us information about the behavior of the error. In particular, we see it depends on how close x is to the point x_0 and the size of $|f^{(n+1)}(x)|$.

2. Solution of Nonlinear Equations

We consider the numerical approximation of the roots of a single nonlinear equation $f(x) = 0$ (e.g., $f(x) = x - e^{-x} = 0$) and of a system of nonlinear equation $\boldsymbol{F}(\boldsymbol{x}) = 0$, where $\boldsymbol{F} = (F_1, \ldots, F_n)$ and $\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{F}_i(x_1, \ldots, x_n)$.

The methods will be iterative and we consider the issues (i) under what conditions does the iteration converge (to a root) and (ii) how fast does the iteration converge.

In some cases, we will be able to show that the iteration defining the method converges for all initial guesses in some specific range. In general, we will settle for a *local convergence result* which says that the iteration converges if the starting guess is *sufficiently close* to the root.