6.4. **Parametric curves.** We next study the approximation of curves which can not be expressed as a function of one coordinate variable in terms of the other, e.g., we cannot write $y = f(x)$. More generally, we consider parametric curves of the form $x = x(t)$, $y = y(t)$, where $t \in [a, b]$. A simple example is the unit circle $x^2 + y^2 = 1$, which may be written in parametric form as $x(t) = \cos(2\pi t)$, $y(t) = \sin(2\pi t)$, $t \in [0, 1]$.

The polynomial interpolation problem then becomes the following: Given points (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n), find polynomials $P_n(t)$ and $Q_n(t)$ such that

$$P_n(t_i) = x_i, \qquad Q_n(t_i) = y_i, \qquad i = 0, 1, \ldots, n.$$

Instead of approximating by polynomials, we can also develop piecewise polynomial approximations, using the methods we have already developed.

For some applications in computer graphics, piecewise Bézier curves are widely used to model smooth curves. Given points $\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, the Bézier curve of degree n is defined as:

$$\boldsymbol{B}(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \boldsymbol{P}_i, \quad t \in [0, 1].$$

So a Bézier curve is just a polynomial of degree $\le n$ written in a special form.

Examples: $n = 1$. Linear Bézier curve $\boldsymbol{B}(t) = (1-t)\boldsymbol{P}_0 + t\boldsymbol{P}_1$. This is just the straight line joining $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$.

$n = 2$. Quadratic Bézier curve

$$\boldsymbol{B}(t) = (1-t)^2\boldsymbol{P}_0 + 2(1-t)t\boldsymbol{P}_1 + t^2\boldsymbol{P}_2, \quad t \in [0, 1].$$

Note that $\boldsymbol{B}(0) = \boldsymbol{P}_0$ and $\boldsymbol{B}(1) = \boldsymbol{P}_2$, but $\boldsymbol{B}$ does not in general equal $\boldsymbol{P}_1$ for any t.

$n = 3$. Cubic Bézier curve

$$\boldsymbol{B}(t) = (1-t)^3\boldsymbol{P}_0 + 3(1-t)^2t\boldsymbol{P}_1 + 3(1-t)t^2\boldsymbol{P}_2 + t^3\boldsymbol{P}_3, \quad t \in [0, 1].$$

The points $\boldsymbol{P}_i$ are called the control points for the Bézier curve. In terms of our previous discussion, they are the degrees of freedom for this form of a polynomial. For all values of n, $\boldsymbol{P}_0 = \boldsymbol{B}(0)$ and $\boldsymbol{P}_n = \boldsymbol{B}(1)$, so these degrees of freedom have an immediate interpretation. We now give an interpretation of the other control points. Consider the case of a cubic Bézier curve. Then

$$\boldsymbol{B}'(t) = -3(1-t)^2\boldsymbol{P}_0 + 3(1-4t+3t^2)\boldsymbol{P}_1 + 3(2t-3t^2)\boldsymbol{P}_2 + 3t^2\boldsymbol{P}_3, \quad t \in [0, 1].$$

Hence,

$$\boldsymbol{B}'(0) = 3[\boldsymbol{P}_1 - \boldsymbol{P}_0], \qquad \boldsymbol{B}'(1) = 3[\boldsymbol{P}_3 - \boldsymbol{P}_2].$$

From a geometric point of view, we see that the control points $\boldsymbol{P}_0$ and $\boldsymbol{P}_3$ determine the end points of the curve. Since $\boldsymbol{B}'(0)$ is a tangent vector to the curve at $t = 0$, the equation $\boldsymbol{B}'(0) = 3[\boldsymbol{P}_1 - \boldsymbol{P}_0]$ says that this tangent vector is a multiple of the vector from the point $\boldsymbol{P}_0$ to the point $\boldsymbol{P}_1$. Thus moving $\boldsymbol{P}_1$ changes the slope of the curve at $\boldsymbol{P}_0$. Similarly, moving $\boldsymbol{P}_2$ changes the slope of the curve at $\boldsymbol{P}_3$. Previously, we have seen that we completely determine a cubic polynomial by choosing as degrees of freedom its value and the value of its derivative at both end points. In the Bézier form, we choose to control the derivatives at the end points by simply moving the control points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$.

To approximate a curve, we make use of piecewise Bézier curves, just as we approximated smooth functions by piecewise polynomials. Consider the points $a = t_0 < t_1 \cdots < t_n = b$ and suppose we wish to construct a piecewise cubic Bézier curve $\boldsymbol{B}(t)$ based on this partition, i.e., for $t \in [t_{i-1}, t_i]$, we set $\boldsymbol{B}(t) = \boldsymbol{B}_i(t)$, where

$$\boldsymbol{B}_i(t) = \frac{(t_i - t)^3}{(t_i - t_{i-1})^3}\boldsymbol{P}_{i-1} + 3\frac{(t_i - t)^2(t - t_{i-1})}{(t_i - t_{i-1})^3}\boldsymbol{Q}_{i-1} + 3\frac{(t_i - t)(t - t_{i-1})^2}{(t_i - t_{i-1})^3}\boldsymbol{R}_i + \frac{(t - t_{i-1})^3}{(t_i - t_{i-1})^3}\boldsymbol{P}_i.$$

Note that this formula is obtained from the previous formula by making the change of variable $t \to (t - t_{i-1})/(t_i - t_{i-1})$. Since the values of $\boldsymbol{B}(t)$ at the endpoints of the interval are degrees of freedom, it is easy to join two Bézier curves together, so that the resulting piecewise Bézier curve is continuous. We just use the same value for $\boldsymbol{B}$ at the common point, i.e., $\boldsymbol{B}_i(t_i) = P_i = \boldsymbol{B}_{i+1}(t_i)$. If we want the derivative at the common point to be continuous, we must have

$$\frac{3}{(t_i - t_{i-1})}(\boldsymbol{P}_i - \boldsymbol{R}_i) = \frac{3}{(t_{i+1} - t_i)}(\boldsymbol{Q}_i - \boldsymbol{P}_i).$$

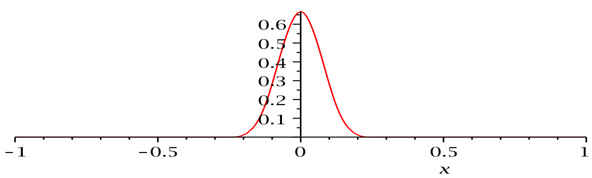
In the case when the t_i are equally spaced, this means that $\boldsymbol{P}_i = (\boldsymbol{Q}_i + \boldsymbol{R}_i)/2$, or equivalently that once the control points $\boldsymbol{R}_i$ and $\boldsymbol{P}_i$ are set, then $\boldsymbol{Q}_i = 2\boldsymbol{P}_i - \boldsymbol{R}_i$.

For some applications in computer graphics, even smoother curves are desirable, and then we want a piecewise cubic that is C^2. Considering the case of equally spaced points, we can use the B-spline basis defined previously to write any cubic spline $\boldsymbol{S}(t)$ satisfying $\boldsymbol{S}''(t_0) = \boldsymbol{S}''(t_n) = 0$ in the form

$$\boldsymbol{S}(t) = \sum_{i=0}^{n} \boldsymbol{P}_i(t_i) B_i(t),$$

where $B_i(t)$ is the cubic spline basis function centered at $t = t_i$. In this case the values $\boldsymbol{P}_i(t_i)$ are the control points (or degrees of freedom). An important aspect of this type of basis is the each basis function $B_i(t)$ is nonzero only on the four subintervals $[t_{i-2}, t_{i-1}]$, $[t_{i-i}, t_i]$, $[t_i, t_{i+1}]$, $[t_{i+1}, t_{i+2}]$. Thus, if we change the value of the control point $\boldsymbol{P}_i(t_i)$, we only change the value of $\boldsymbol{S}$ on these four subintervals.

The following pictures show first a graph of a B-spline basis function centered at $t = 0$ and then a graph of a cubic spline that is equal to one at $t = 0$, equal to zero at the other interpolation points $t_i = \pm i/8, i = 1, 2, \ldots, 8$. Note that the second graph is different from zero on more than four subintervals. The final plot is a blow up of the second graph on the interval $[.5, 1]$, showing that although the value of the interpolanting spline is small, it is not zero on any of the subintervals.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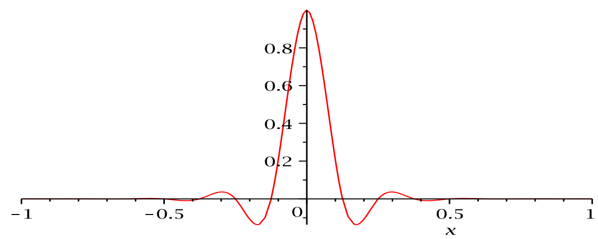
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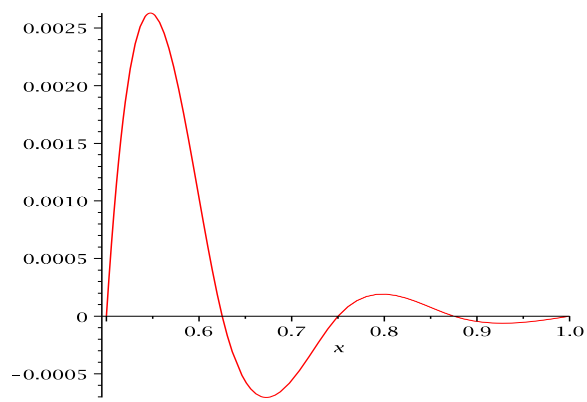
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