12.2. **Taylor series methods.** Consider the Taylor series of $y(x)$, the solution of the IVP, about $x = x_n$:

$$y(x_n + h) = y(x_n) + y'(x_n) + \frac{h^2}{2}y^{(2)}(x_n) + \cdots + \frac{h^k}{k!}y^{(k)}(x_n) + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(\xi).$$

Now $y'(x) = f(x, y(x))$, so

$$\begin{aligned} y''(x) = f'(x, y(x)) = \frac{d}{dx}f(x, y(x)) &= f_x(x, y(x)) + f_y(x, y(x))\frac{dy}{dx} \\ &= f_x(x, y(x)) + f_y(x, y(x))f(x, y(x)). \end{aligned}$$

In general,

$$y^{(k)}(x) = f^{(k-1)}(x, y(x)) = \frac{d}{dx}f^{(k-2)}(x, y(x)) = f_x^{(k-2)}(x, y(x)) + f_y^{(k-2)}(x, y(x))\frac{dy}{dx}.$$

Hence, if $y(x_n)$ were known, we could compute an approximation to $y(x_n + h)$ by using the truncated Taylor series:

$$y(x_n + h) \approx y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2}f'(x_n, y(x_n)) + \cdots + \frac{h^k}{k!}f^{(k-1)}(x_n, y(x_n)),$$

i.e., if we denote by y_n the approximation to $y(x_n)$, we can define the Taylor algorithm of order k as the sequence of computations

$$y_{n+1} = y_n + hT_k(x_n, y_n), \qquad n = 0, 1, \ldots,$$

where

$$T_k(x, y) = f(x, y) + \frac{h}{2}f'(x, y) + \cdots + \frac{h^{k-1}}{k!}f^{(k-1)}(x, y).$$

Note: Euler's method is the Taylor algorithm of order 1.

Example: We wish to solve the IVP $y' = 1/x^3 - y/x - y^2$, $y(1) = 2$ by the Taylor algorithm of order 2. Now

$$f(x, y) = \frac{1}{x^3} - \frac{y}{x} - y^2$$

and

$$\begin{aligned} f'(x, y) = f_x(x, y) + f_y(x, y)f(x, y) &= -\frac{3}{x^4} + \frac{y}{x^2} - \left[\frac{1}{x} + 2y\right]f \\ &= -\frac{3}{x^4} + \frac{y}{x^2} - \left[\frac{1}{x} + 2y\right]\left[\frac{1}{x^2} - \frac{y}{x} - y^2\right]. \end{aligned}$$

Then

$$y_{n+1} = y_n + hf(x_n.y_n) + \frac{h^2}{2}f'(x_n, y_n),$$

where $f'(x, y)$ defined above.

Note we can also compute $f'(x,y)$ directly, remembering that y is a function of x, i.e.,

$$\begin{aligned} f'(x,y) = \frac{d}{dx}\left[\frac{1}{x^2} - \frac{y}{x} - y^2\right] = -\frac{3}{x^4} - \frac{xy'-y}{x^2} - 2yy' &= -\frac{3}{x^4} + \frac{y}{x^2} - \left[\frac{1}{x} + 2y\right] y' \\ &= -\frac{3}{x^4} + \frac{y}{x^2} - \left[\frac{1}{x} + 2y\right]\left[\frac{1}{x^2} - \frac{y}{x} - y^2\right]. \end{aligned}$$

Definition: The local truncation error for the Taylor series method of order k is defined by:

$$y(x_{n+1}) - y(x_n) - hT_k(x_n, y(x_n)) = \frac{h^{k+1}}{(k+1)!} y^{(k+1)}(\xi_n).$$

The local truncation of Euler's method is $h^2 y^{(2)}(\xi_n)/2$.

The Taylor algorithm of order k is an example of a one-step method, i.e, the value of y_{n+1} only depends on one past value, y_n. One-step methods have the form

$$y_{n+1} = y_n + h\Phi(x_n, y_n), \qquad n = 0, 1, \dots,$$

Analogously to the Taylor series methods, we define the Local Truncation Error of such methods to be

$$LTE = y(x_{n+1}) - y(x_n) - h\Phi(x_n, y(x_n)).$$

Then we have the following result giving a bound on the global error.

Theorem 14. *If $|\Phi(x,u) - \Phi(x,v)| \le \mathcal{L}|u-v|$ for $a \le x \le b$, $0 < h < h_0$ and all u, v and if the local truncation is $O(h^{p+1})$, then for any $x_n = x_0 + nh \in [x_0, b]$,*

$$|y(x_n) - y_n| \le C\frac{h^p}{\mathcal{L}}(e^{\mathcal{L}(x_n - x_0)} - 1).$$

The proof of this result is essentially identical to the proof of the error bound for Euler's method.

Although Taylor series methods become increasingly more accurate as k increases, their major disadvantage is that they require calculation of high derivatives of the function f.