13. Convergence of multistep methods

Theorem 15. *A necessary condition for the convergence of linear multistep method is that the method be consistent, i.e.,*

$$1 = \sum_{i=0}^{p} a_i, \qquad 1 = -\sum_{i=0}^{p} i a_i + \sum_{i=-1}^{p} b_i.$$

Proof. If the method is convergent, then it is convergent for the IVP $y' = 0$, $y(0) = 1$, whose exact solution is $y(x) = 1$. For this problem, the general linear multistep method becomes $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i}$. Let the starting values $y_0, \dots y_p$ be exact, i.e., equal to 1. Since the method is convergent, we must have that $y_n^h \to 1$ as $h \to 0$, $n \to \infty$, and $nh = x$. Hence, letting $n \to \infty$ in the expression $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i}$, we get $1 = \sum_{i=0}^{p} a_i$.

To establish the second equality, we consider the IVP $y' = 1$, $y(0) = 0$, whose exact solution is $y(x) = x$. The difference equation is now $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i$. Consider the sequence $y_n = nhA$, $n = 0, 1, \dots,$, where

$$A = \frac{\sum_{i=-1}^{p} b_i}{1 + \sum_{i=0}^{p} i a_i}.$$

We will first show that the sequence $\{y_n\}$ is a solution of the difference equation. To see this, we compute

$$\sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i = \sum_{i=0}^{p} a_i (n-i) hA + h \sum_{i=-1}^{p} b_i = \sum_{i=0}^{p} a_i (n-i) hA + hA(1 + \sum_{i=0}^{p} i a_i)$$
$$= hA + hAn \sum_{i=0}^{p} a_i = (n+1)hA = y_{n+1},$$

where we have used the first identity. We next observe that this sequence also satisfies the condition that $\lim_{h\to 0} y_n = 0$, $n = 1, 2, \dots p$. Since the method is convergent, $y_n^h \to x$ as $h \to 0$, $n \to \infty$, and $nh = x$, i.e., $nhA = x$ for $nh = x$. Hence $A = 1$, so the second equality is established. □

Definition: 1st and 2nd characteristic polynomial of a multistep method:

$$\rho(z) = z^{p+1} - \sum_{i=0}^{p} a_i z^{p-i}, \qquad \sigma(z) = \sum_{i=-1}^{p} b_i z^{p-i}.$$

The linear multistep method is consistent if $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$. The first identity is obvious. The second implies

$$(p+1) - \sum_{i=0}^{p} (p-i) a_i = \sum_{i=-1}^{p} b_i.$$

Hence

$$p(1-\sum_{i=0}^{p} a_i) + 1 - \sum_{i=0}^{p}(-i)a_i - \sum_{i=-1}^{p} b_i = 0.$$

Since the first term in the sum is zero, then so is the second.

Definition: The linear multistep method is said to be zero-stable (satisfy the root condition) if no root of the characteristic polynomial $\rho(z)$ has modulus greater than one and if every root of modulus one is simple.

Theorem 16. *A necessary condition for convergence of a linear multistep method is that it be zero-stable.*

Proof. We only give the proof in the case that the roots of $\rho(z)$ are real simple roots. If the method is convergent, then it is convergent for the IVP $y' = 0$, $y(0) = 0$, whose solution is $y(x) = 0$. For this problem, the method becomes $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i}$. If the method is convergent, then by (i), for any $x > 0$,

$$\lim y_n^h = 0, \quad n \to \infty, \quad h \to 0, \quad nh = x$$

for all solutions $\{y_n\}$ of the difference equation satisfying (ii) $\lim_{h\to 0} y_k(h) = 0$, $k = 0, \ldots, p$. We first show that all roots have modulus ≤ 1. Let $z = r$ be a real root of $\rho(z)$. Then $y_n = r^n$ is a solution of the difference equation and so is $y_n = hr^n$. Note that this second solution satisfies (ii). Hence, (i) must hold, i.e., $\lim_{n\to\infty} xr^n/n = 0$. Now

$$\lim_{n\to\infty} xr^n/n = x \lim_{n\to\infty} r^n/n = 0$$

if $0 \leq |r| \leq 1$. If $r > 1$, using L'Hospital's rule,

$$x \lim_{n\to\infty} r^n/n = x \lim_{n\to\infty} r^n \ln r/1 = \infty.$$

A similar result hold if $r < -1$. Hence for (i) to hold, we require $|r| \leq 1$. □

Theorem 17. *A necessary and sufficient condition for a linear multistep method to be convergent is that it be consistent and zero-stable.*

Proof. We have shown these conditions are necessary. The proof of sufficiency can be found in Henrici: Discrete Variable Methods in Ordinary Differential Equations. □