3.1. **Local convergence results.** In some cases, it can be difficult to verify the hypotheses of the convergence theorem. However, one may still be able to verify the hypotheses of the following theorem, that gives a *local convergence* result.

Theorem 2. *Suppose that (i) g and g' are continuous on the interval $I = [a, b]$ and (ii) the equation $x = g(x)$ has a solution $s \in (a, b)$ such that $|g'(s)| < 1$. Then there exists a number $d > 0$ such that the sequence $\{x_n\}$ determined by the iteration $x_{n+1} = g(x_n)$ converges to s for any initial guess x_0 satisfying $|x_0 - s| \leq d$.*

Note that since we do not know d explicitly, the theorem only says that fixed point iteration will converge if the initial guess x_0 is sufficiently close to the fixed point s.

Application to Newton's method: Let $g(x) = x - f(x)/f'(x)$. Suppose that $f(x) \in C^2[a, b]$, i.e., f, f', f'' are continuous on $[a, b]$ and that $f'(x) \neq 0$ for all $x \in [a, b]$. Suppose further that there is a point $s \in [a, b]$ such that $f(s) = 0$. (Note there can only be one such point since $f'(x) \neq 0$ so f is always increasing or always decreasing.) Then we can conclude that Newton's method will converge if the initial guess x_0 is sufficiently close to the root s. To see this, we apply the previous theorem. We first observe that the hypotheses on f imply that g and g' are continuous on the interval $I = [a, b]$. The root s of f is obviously a fixed point of g. Now

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Hence $g'(s) = 0$, and so hypothesis (iii) above is satisfied. We therefore conclude that Newton's method is locally convergent.

3.2. **Order of convergence.** One way to compare methods is to compare their speed of convergence.

Definition: Suppose that $\{x_n\}$ converges to x^*. We say that convergence is of order p if there exists a positive constant α such that

$$|x_{n+1} - x^*| \leq \alpha |x_n - x^*|^p, \quad \text{for all } n \geq n_0.$$

If $p = 1$ (linear convergence), we also require that $\alpha < 1$. Finally, we say the convergence is *superlinear* if there is a sequence $\{\alpha_n\}$ converging to zero such that

$$|x_{n+1} - x^*| \leq \alpha_n |x_n - x^*|, \quad \text{for all } n \geq n_0.$$

Theorem 3. *Suppose the sequence $\{x_n\}$ is obtained by the iteration $x_{n+1} = g(x_n)$ and that $\lim_{n\to\infty} x_n = x^*$. Suppose $g^{(k)}(x)$ is continuous at x^* for $k = 0, 1, \ldots, p$ and that*

$$x^* = g(x^*), \quad g'(x^*) = 0, \ldots, g^{(p-1)}(x^*) = 0, \quad g^{(p)}(x^*) \neq 0.$$

Then the convergence is of order p.

Proof. We obtain this result by expanding in a Taylor series, i.e., for some ξ_n between x^* and x_n,

$$x_{n+1} - x^* = g(x_n) - g(x^*) = g'(x^*)(x_n - x^*) + \frac{g''(x^*)}{2}(x_n - x^*)^2 + \cdots + \frac{g^{(p-1)}(x^*)}{(p-1)!}(x_n - x^*)^{p-1} + \frac{g^{(p)}(\xi_n)}{p!}(x_n - x^*)^p.$$

Applying the hypotheses of the theorem, we get

$$|x_{n+1} - x^*| = \frac{|g^{(p)}(\xi_n)|}{p!}|(x_n - x^*)^p|.$$

This is not quite the desired result, since ξ_n depends on n and is not a constant, so we cannot just take $\alpha = |g^{(p)}(\xi_n)|/p!$. However, since $g^{(p)}(x)$ is continuous at x^*, given $\epsilon > 0$ (say $\epsilon = 1$), we can find $\delta > 0$ such that $|g^{(p)}(\xi_n) - g^{(p)}(x^*)| < \epsilon$ for $|\xi_n - x^*| < \delta$. But x_n converges to x^* as $n \to \infty$. Hence $|x_n - x^*| < \delta$ for $n \geq n_0$. This implies $|\xi_n - x^*| < \delta$ for $n \geq n_0$. Combining these results, we get that $|g^{(p)}(\xi_n)| \leq \epsilon + |g^{(p)}(x^*)|$ for $n \geq n_0$. We now choose $\alpha = [\epsilon + |g^{(p)}(x^*)|]/p!$ to finish the proof. □

If we apply this result to Newton's method, then we have already seen that if $f(x) \in C^2$ and $f'(x^*) \neq 0$, i.e., x^* is a simple root, then $x^* = g(x^*)$ and $g'(x^*) = 0$. Since $g'(x) = f(x)f''(x)/[f'(x)]^2$, a simple computation shows that

$$g''(x^*) = f''(x^*)/f'(x^*).$$

Hence, in general, we get that Newton's method is quadratically convergent.

If $f(x)$ has a root x^* of multiplicity $m > 1$, then we may write $f(x) = (x - x^*)^m G(x)$, where $G(x^*) \neq 0$. Then

$$\begin{aligned} f'(x) &= (x - x^*)^{m-1}[(x - x^*)G'(x) + mG(x)], \\ f''(x) &= (x - x^*)^{m-2}[(x - x^*)^2 G''(x) + 2m(x - x^*)G'(x) + m(m-1)G(x)]. \end{aligned}$$

Hence,

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} = \frac{G(x)[(x - x^*)^2 G''(x) + 2m(x - x^*)G'(x) + m(m-1)G(x)]}{[(x - x^*)G'(x) + mG(x)]^2},$$

and so $g'(x^*) = (m-1)/m = 1 - 1/m$. Hence, the method is no longer quadratically convergent.