4.6. **Divided differences.** Defining the divided difference $f[x_0] = f(x_0)$, we can generate the polynomials $P_n(x)$ recursively. Beginning with $P_0(x) = f(x_0) = f[x_0]$, we obtain

$$P_1(x) = P_0(x) + f[x_0, x_1](x - x_0) = f[x_0] + f[x_0, x_1](x - x_0).$$

Then

$$\begin{aligned} P_2(x) &= P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1). \end{aligned}$$

In general, we get the formula:

$$P_n(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j), \tag{4.2}$$

where we define $\prod_{j=0}^{-1}(x - x_j) = 1$.

Formula (4.2) is known as the Newton form of the interpolating polynomial.

In order to use formula (4.2), we must of course be able to evaluate the divided difference $f[x_0, \ldots, x_i]$. Using (4.1), we have that

$$f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

and

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_0)}.$$

Observing that

$$\frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} = \frac{f(x_1)}{(x_1 - x_0)(x_0 - x_2)} - \frac{f(x_1)}{(x_1 - x_2)(x_0 - x_2)},$$

we can rewrite

$$f[x_0, x_1, x_2] = \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{x_1 - x_0)} \right\} / (x_2 - x_0) = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

The reason for writing $f[x_0, x_1, x_2]$ in this form is that it indicates an easy way of generating divided differences recursively. We can show in general that

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

This formula allows us to generate all the divided differences needed for the Newton formula in a simple manner by using a divided difference table, rather than using formula (4.1). We illustrate such a table in the case $n = 4$.

The divided differences in the table are calculated a column at a time using the formula

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

The coefficients needed for the Newton formula are then found at the beginning of each column.

Table 1

Divided difference table

x_k	$f(x_k)$	f[,]	f[,,]	f[,,,]	f[,,,,]
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$f(x_2)$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$f(x_3)$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$f(x_4)$				

Note how entries are added to the table each time a a new data point is added. For example, if we started with the entries in the table involving only the points x_0, x_1, x_2, x_3, and added the point x_4, we would successively generate $f[x_3, x_4]$. $f[x_2, x_3, x_4]$, $f[x_1, x_2, x_3, x_4]$, and finally $f[x_0, x_1, x_2, x_3, x_4]$, the additional divided difference needed for the construction of $P_4(x)$.

We shall return frequently to the idea of degrees of freedom of a function f. These are quantities that uniquely determine the function f. In all our applications, these will be values of f or its derivatives at specific points, or possibly moments of f, i.e., quantities of the form $\int_a^b x^r f(x)\,dx$ for some integers $f \geq 0$. Note that a function may be uniquely determined by several sets of degrees of freedom, and the choice of which ones to use and how to represent the function will depend on the application. For example, we can also represent any polynomial of degree $\leq n$ by its Taylor series expansion about a point x_0, i.e.,

$$P_n(x) = \sum_{j=0}^{n} \frac{P^{(j)}(x_0)}{j!}(x - x_0)^j.$$

In this representation, we see that $P_n(x)$ is uniquely determined by the quantities $P^{(j)}(x_0)$, i.e., its derivatives up to order n at the point x_0.

4.7. **Interpolation error.** We now turn to an analysis of the error $f(\bar{x}) - P_n(\bar{x})$, for $\bar{x} \neq x_0, \dots x_n$. For the moment, consider $\bar{x}$ fixed, and let P_{n+1} denote the polynomial of degree $\leq n+1$ interpolating $f(x)$ at $x_0, x_1, \dots, x_n$ and $\bar{x}$. Using the Newton form of the interpolating polynomial, we know that

$$P_{n+1}(x) = P_n(x) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (x - x_j).$$

Since $P_{n+1}(\bar{x}) = f(\bar{x})$, we have by the above formula that

$$f(\bar{x}) = P_n(\bar{x}) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j),$$

and so a representation of the error is given by

$$f(\bar{x}) - P_n(\bar{x}) = f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j). \tag{4.3}$$

We next find an equivalent expression for $f[x_0, \dots, x_n, \bar{x}]$, valid when f is sufficiently smooth.

Definition: Suppose r is a non-negative integer. Then f is a function in $C^r[a,b]$ if f and its first r derivatives are continuous on $[a,b]$. So $C^0[a,b]$ denotes the space of continuous functions on $[a,b]$ and we shall use $C^{-1}[a,b]$ to denote functions which may be discontinuous on $[a,b]$.

Lemma 1. *Let $f \in C^k[a,b]$ and $x_0, \dots, x_k$ be distinct points in $[a,b]$. The there exists a point $\xi \in (a,b)$ such that $f[x_0, x_1, \dots, x_k] = f^{(k)}(\xi)/k!$.*

Proof. Let $P_k(x)$ denote the polynomial of degree $\leq k$ interpolating f at $x_0, \dots, x_k$ and define $e_k(x) = f(x) - P_k(x)$. Observe first that $e_k(x)$ has at least $k+1$ distinct zeroes at the points $x_0, \dots, x_k$. Since f and therefore e_k is differentiable on $(a.b)$, we can use Rolle's theorem to conclude that between each two adjacent zeroes of $e_k(x)$, there exists at least one zero of $e_k'(x)$. Hence $e_k'(x)$ has at least k zeroes in (a,b). Since f and therefore $e_k(x)$ is k times differentiable in (a,b), we can continue this argument to conclude that $e_k''(x)$ has at least $k-1$ zeroes in (a,b), and finally that $e_k^{(k)}$ has at least one zero in (a,b). If we denote that zero by the point ξ, then

$$0 = e_k^{(k)}(\xi) = f^{(k)}(\xi) - P_k^{(k)}(\xi).$$

Now by formula (4.2),

$$P_k(x) = \sum_{i=0}^{k} f[x_0, \dots, x_i] \prod_{i=0}^{i-1} (x - x_j) = f[x_0, \dots, x_k] x^k + \text{polynomial of degree } < k.$$

Hence $P_k^{(k)}(x) = f[x_0, \dots, x_k] k!$ for all x and so $f[x_0, \dots, x_k] = f^{(k)}(\xi)/k!$ for some $\xi \in (a,b)$. ☐

Combining Lemma (1) with the representation of the interpolation error given by formula (4.3), we get the following result.

Theorem 4. *Suppose that $f \in C^{n+1}[a,b]$ and that $P_n(x)$ is a polynomial of degree $\leq n$ that interpolates f at the $n+1$ distinct points $x_0, \dots, x_n \in (a,b)$. Then for all $x \in [a,b]$, there exists a point $\xi \in (a,b)$ (depending on x) such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

Proof. If x is equal to any of the interpolation points x_j, then the equation holds since both sides are zero. If x is not equal to any of the interpolation points, we have from the representation of the error given by (4.3) with $\bar{x} = x$, that

$$f(x) - P_n(x) = f[x_0, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j).$$

Since the $n+2$ points $x_0, \ldots, x_n, x$ are all distinct, we can apply Lemma (1) to conclude that $f[x_0, \ldots, x_n, x] = f^{(n+1)}(\xi)/(n+1)!$ for some $\xi \in (a, b)$ (depending on x). Substituting this result gives the theorem. □

Note that since ξ is not known explicitly, this formula can not be used to find the actual error. This is not surprising, since f can take on any value at non-interpolation points. However, the theorem can be used to find an upper bound on the interpolation error if we have more information about the way the derivatives of f behave. The following results follow directly from the theorem.

Corollary 3. *Suppose the conditions of Theorem* (4) *are satisfied. If* $\max_{a \le \xi \le b} |f^{(n+1)}(\xi)| \le M_{n+1}$, *then*

$$|f(x) - P_n(x)| \le \frac{M_{n+1}}{(n+1)!} |(x - x_0)(x - x_1) \cdots (x - x_n)|, \quad \text{for all } x \in [a, b] \tag{4.4}$$

and

$$\max_{a \le x \le b} |f(x) - P_n(x)| \le \frac{M_{n+1}}{(n+1)!} \max_{a \le x \le b} |(x - x_0)(x - x_1) \cdots (x - x_n)|. \tag{4.5}$$

Let us now consider an application of these results to find a bound on the error in linear interpolation. Recall that the linear polynomial interpolating $f(x)$ at x_0 and x_1 is given by $P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$. If $x \in [x_0, x_1]$ and $\max_{x_0 \le \xi \le x_1} |f''(\xi)| \le M_2$, then we have by (4.4) with $a = x_0$, $b = x_1$ that

$$|f(x) - P_1(x)| \le \frac{M_2}{2} |(x - x_0)(x - x_1)|, \quad \text{for all } x \in [a, b]$$

and by (4.5) that

$$\max_{x_0 \le x \le x_1} |f(x) - P_1(x)| \le \frac{M_2}{2} \max_{x_0 \le x \le x_1} |(x - x_0)(x - x_1)| \le \frac{M_2}{8} (x_1 - x_0)^2,$$

since the maximum occurs at the midpoint $(x_0 + x_1)/2$.