4.8. **Hermite interpolation.** The polynomial interpolation problem we have discussed so far only involves interpolation of the values of a function $f$ at distinct points. It is also possible to seek a polynomial that interpolates $f$ and some of its derivatives at a set of points. One example of this that we have already seen is the Taylor polynomial of degree $\leq n$ about the point $x_0$, i.e.,

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

This has the property that

$$T_n(x_0) = f(x_0), \qquad T_n'(x_0) = f'(x_0), \qquad T_n^{(n)}(x_0) = f^{(n)}(x_0),$$

i.e., $T_n$ interpolates $f$ and its derivatives up to order $n$ at the point $x_0$.

A more general interpolation problem is to find the polynomial of least degree such that

$$\frac{d^kP}{dx^k}(x_i) = \frac{d^kf}{dx^k}(x_i), \qquad i = 0, 1, \cdots, n, \quad k = 0, 1, \cdots, m_i.$$

Note that we allow the possibility of interpolating a different number $(m_i)$ derivatives of $f$ at different interpolation points.

For this case $m_i = 1$, there is a formula analogous to the Lagrange interpolation formula discussed previously.

**Theorem 5.** *If $f \in C^1[a, b]$ and $x_0, x_1, \cdots x_n$ are distinct points in $[a, b]$, then the unique polynomial of least degree agreeing with $f$ and $f'$ at $x_0, x_1, \cdots x_n$ is the Hermite polynomial of degree $\leq 2n + 1$ given by:*

$$H_{2n+1}(x) = \sum_{k=0}^{n} H_{k,n}(x)f(x_k) + \sum_{k=0}^{n} \hat{H}_{k,n}(x)f'(x_k),$$

*where*

$$H_{k,n}(x) = [1 - 2(x - x_k)L_{k,n}'(x_k)]L_{k,n}^2(x), \qquad \hat{H}_{k,n}(x) = (x - x_k)L_{k,n}^2(x),$$

*and for $k = 0, 1, \ldots, n$:*

$$L_{k,n}(x) = \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}, \quad n \geq 1, \qquad L_{0,0}(x) = 1.$$

*Moreover, if $f \in C^{2n+2}[a, b]$, then there is a point $\xi(x) \in (a, b)$, such that*

$$f(x) - H_{2n+1}(x) = \frac{f^{2n+2}(\xi(x))}{(2n + 2)!}(x - x_0)^2 \cdots (x - x_n)^2.$$

Recall that the key property satisfied by $L_{k,n}(x)$ was $L_{k,n}(x_i) = 1$ if $k = i$ and $L_{k,n}(x_i) = 0$ if $k \neq i$. The analogous properties for $H_{k,n}(x)$ and $\hat{H}_{k,n}(x)$ are:

$$H_{k,n}(x_i) = 1,\ k = i, \quad H_{k,n}(x_i) = 0,\ k \neq i, \qquad H_{k,n}'(x_i) = 0,\ i = 0, \ldots, n,$$
$$\hat{H}_{k,n}'(x_i) = 1,\ k = i, \quad \hat{H}_{k,n}'(x_i) = 0,\ k \neq i, \qquad \hat{H}_{k,n}(x_i) = 0,\ i = 0, \ldots, n.$$

4.9. **Divided differences for repeated points.** Recall

$$f[x_0, x_1, \dots, x_k] = \sum_{i=0}^{k} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}.$$

Note that $f[y_0, \dots, y_k] = f[x_0, \dots, x_k]$ if $y_0, \dots, y_k$ is any reordering of $x_0, \dots, x_k$. So far, $f[x_0, \dots, x_k]$ has only been defined when the points $x_0, \dots, x_k$ are distinct. We now wish to extend the definition to include the case of repeated points.

Example: $k = 1$. $f[x_0, x_1] = [f(x_1) - f(x_0)]/(x_1 - x_0)$, $x_1 \neq x_0$. If $f \in C^1$, then $\lim_{x_0, x_1 \to y} f[x_0, x_1] = f'(y)$. So we define $f[x_0, x_1] = f'(x_0)$ when $x_0 = x_1$. In general, define

$$f[x_0, \dots, x_k] = \frac{f^{(k)}(y)}{k!}, \quad \text{if } x_0 = x_1 = \cdots = x_k = y.$$

With this interpretation, we can still use the Newton formula

$$P_n(x) = \sum_{i=0}^{n} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

to describe the polynomial of degree $\leq n$ interpolating $f(x)$ at $x_0, \dots, x_n$, even when the points $x_0, \dots, x_n$ are not necessarily distinct. The error is still given by the formula

$$f(x) - P_n(x) = f[x_0, \dots, x_n, x] \prod_{j=0}^{n} (x - x_j),$$

where by the interpolating polynomial we now mean that if the point $z$ appears $k+1$ times among $x_0, \dots, x_n$, then

$$P_n^{(j)}(z) = f^{(j)}(z), \quad j = 0, \dots, k.$$

Furthermore, if $f \in C^{n+1}(a, b)$ and $x_0, \dots, x_n, x \in [a, b]$, then one can show that

$$f[x_0, \dots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi$ satisfying $\min(x_0, \dots, x_n, x) \leq \xi \leq \max(x_0, \dots, x_n, x)$.

Example: $f(x) = \ln x$. Calculate $f(1.5)$ by cubic interpolation using the data: $f(1) = 0$, $f'(1) = 1$, $f(2) = 0.693147$, $f'(2) = 0.5$. Take $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 2$.

TABLE 2

*Divided difference table*

| $x_k$ | $f(x_k)$ | f[,] | f[,,] | f[,,,] |
|---|---|---|---|---|
| 1 | 0 | | | |
| | | 1 | | |
| 1 | 0 | | −.306853 | |
| | | .693147 | | .113706 |
| 2 | .693147 | | −0.193147 | |
| | | .5 | | |
| 2 | .693147 | | | |

Using the divided difference table, we get

$$\begin{aligned} P_3(x) &= f(1) + f[1,1](x-1) + f[1,1,2](x-1)^2 + f[1,1,2,2](x-1)^2(x-2) \\ &= 0 + 1(x-1) + (-.306853)(x-1)^2 + (.113706)(x-1)^2(x-2). \end{aligned}$$

So $P_3(1.5) = .409074$.

From the error formula, we have $\ln(x) - P_3(x) = f^{(4)}(\xi)(x-1)^2(x-2)^2/4!$. Hence,

$$|\ln(1.5) - P_3(1.5)| \le \frac{1}{4!} \max_{1\le\xi\le2} \frac{6}{\xi^4}(.5)^4 = \frac{1}{64} = 0.015624$$

The actual error is .00361.

4.10. **Runge example.** One might infer from the error formula for polynomial interpolation that as one adds more and more interpolation points, one gets a better and better approximation. This fact is not true in general and depends on how the points are added.

Example: Runge $f(x) = 1/(1+x^2)$, $x \in [-5,5]$. Set $x_j = -5 + j\Delta x$, $j = 0, 1, \dots, n$, $\Delta x = 10/n$. For each $n$, there is a unique polynomial $P_n(x)$ of degree $\le n$ satisfying $P_n(x_j) = f(x_j)$. However, $|f(x) - P_n(x)|$ will become arbitrarily large at points in $[-5,5]$ as $n$ becomes large. One can show that for $n = 2r$,

$$f(x) - P_n(x) = \prod_{j=0}^{n} x(x-x_j) \frac{f(x)(-1)^{r+1}}{\prod_{j=0}^{r}(1+x_j^2)}.$$

For $n = 2$,

$$|f(x) - P_2(x)| \le \frac{|x^2(x+5)(x-5)}{26(1+x^2)} \le 1,$$

by looking at the graphs. For $n = 10$, $x = -4.5$, $|f(x) - P_n(x)| = 1.53166$, so the maximum error is not getting smaller as $n$ increases.