5. Piecewise Polynomial Approximation

Consider a partition $\mathcal{P}$ of an interval $[a, b]$ by points $x_0, \ldots, x_n$, i.e., $a = x_0 < x_1 < \ldots < x_n = b$.

Definition: We say $Q(x)$ is a C^r piecewise polynomial of degree $\leq k$ with respect to the partition $\mathcal{P}$ if $Q \in C^r[a, b]$ and Q has the form $Q(x) = Q_j(x)$ for $x \in (x_{j-1}, x_j)$, $j = 1, \ldots n$, where $Q_j(x)$ is a polynomial of degree $\leq k$ for each value of j.

Note that since $Q(x) \in C^r$ and the Q_j are polynomials, its first r derivatives are continuous and its $r + 1$st derivative is defined everywhere except possibly at the points x_j.

Examples of piecewise polynomials:

$k = 0$: Piecewise constants (dimension $= n$). Degrees of freedom are: values of Q on each subinterval.

$k = 1$: Discontinuous piecewise linears (dimension $= 2n$). Degrees of freedom are: 2 values of Q on each subinterval. Continuous piecewise linears (dimension $= n + 1$). Degrees of freedom are: values of Q at points x_j. Note, in this case, we choose the mesh points x_j to insure continuity of Q, i.e., to have $Q_j(x_j) = Q_{j+1}(x_j)$.

$k = 2$: Discontinuous piecewise quadratics (dimension $= 3n$). Degrees of freedom are: 3 values of Q on each subinterval. Continuous piecewise quadratics (dimension $= 2n + 1$). Degrees of freedom are: values of Q at x_j and at one interior point in each subinterval. C^1 piecewise quadratics (dimension $= 2n + 1 - (n - 1) = n + 2$. Degrees of freedom are: values of Q at x_j and $Q'(x_0)$.

$k = 3$: C^{-1}, C^0, C^1, C^2 piecewise cubics. C^{-1}: (dimension $= 4n$). Degrees of freedom are: 4 values of Q on each subinterval. C^0: (dimension $= 3n + 1$). Degrees of freedom are: values of Q at x_j and at 2 interior points in each subinterval. C^1: (dimension $= 2n + 2$). Degrees of freedom are: values of Q and Q' at x_j. Note that this choice will guarantee that Q and Q' will be continuous across mesh points. C^2: (dimension $= 2n + 2 - (n - 1) = n + 3$). Degrees of freedom are: values of Q at x_j plus 2 additional conditions.

Note: A C^r piecewise polynomial of degree $\leq r$ is a global polynomial of degree $\leq r$. If $r = k - 1$, Q is called a spline function.

Table 3

Number of Degrees of Freedom of a
Piecewise Polynomial of Degree $\leq n$

n	C^{-1}	C^0	C^1	C^2
0	n	1		
1	$2n$	$n + 1$	2	
2	$3n$	$2n + 1$	$n + 2$	3
3	$4n$	$3n + 1$	$2n + 2$	$n + 3$

5.1. **Piecewise linear approximation.** Consider in more detail the case of continuous, piecewise linear approximation. Define the continuous, piecewise linear interpolant of a function f as the continuous, piecewise linear function $L(x)$ satisfying $L(x_j) = f(x_j)$, $j = 0, \dots n$. Using the Lagrange form of the interpolating polynomial, can write this as:

$$L(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i), \quad x \in [x_{i-1}, x_i].$$

A useful basis for the space of continuous, piecewise linear functions is the set $\{\psi_i\}_{i=0}^n$, where

$$\begin{aligned} \psi_i(x) &= 0, \quad x \notin [x_{i-1}, x_{i+1}], \\ &= (x - x_{i-1})/(x_i - x_{i-1}), \quad x \in [x_{i-1}, x_i], \\ &= (x_{i+1} - x)/(x_{i+1} - x_i), \quad x \in [x_i, x_{i+1}]. \end{aligned}$$

The basis function $\psi_i(x)$ is called a hat function. Note that $\psi_i(x_j) = 0$ for $i \neq j$ and $= 1$ for $i = j$. Hence, we can write

$$L(x) = \sum_{i=0}^{n} \psi_i(x) L(x_i) = \sum_{i=0}^{n} \psi_i(x) f(x_i).$$

In this form, the degrees of freedom for $L(x)$ are the values $L(x_i)$, and thus the solution of the interpolation problem is simple.

When the points x_j are equally spaced, we get a simplification. Let

$$\begin{aligned} \phi(x) &= 0, \quad x \geq 1, \text{ and } x \leq -1, \\ &= 1 - x, \quad 0 \leq x \leq 1, \\ &= 1 + x, \quad -1 \leq x \leq 0. \end{aligned}$$

Then $\psi_i(x) = \phi([x - x_i]/h)$, where $h = x_{i+1} - x_i$.

How good an approximation is the continuous piecewise linear interpolant? On each subinterval, $L(x)$ is just the linear interpolating polynomial. Hence, using the error formula, we have for $x \in [x_{i-1}, x_i]$,

$$|f(x) - L(x)| \leq M_{2,i}(x_i - x_{i-1})^2/8,$$

where $M_{2,i} = \max_{x_{i-1} \leq \xi \leq x_i} |f''(\xi)|$. Hence, for all $x \in [a, b]$,

$$|f(x) - L(x)| \leq M_2 \max_{i=1,n} (x_i - x_{i-1})^2/8 \leq M_2 h^2/8,$$

where $M_2 = \max_{i=1,n} M_{2,i} = \max_{x_0 \leq \xi \leq x_n} |f''(\xi)|$ and $h = \max_{i=1,n} |x_i - x_{i-1}|$.

Hence, if $f \in C^2[a, b]$, then taking more subintervals and letting the subinterval size approach zero, we can make the error as small as desired.

5.2. **Piecewise cubic Hermite approximation.** Next consider the piecewise cubic Hermite interpolant, i.e., a C^1 piecewise cubic $H(x)$ satisfying

$$H(x_j) = f(x_j), \quad H'(x_j) = f'(x_j), \quad j = 0, \dots n.$$

On the subinterval $[x_{i-1}, x_i]$, H is just the cubic polynomial satisfying:

$$H(x_{i-1}) = f(x_{i-1}), \quad H(x_i) = f(x_i), \quad H'(x_{i-1}) = f'(x_{i-1}), \quad H'(x_i) = f'(x_i),$$

and so for $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} H(x) = f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + f[x_{i-1}, x_{i-1}, x_i](x - x_{i-1})^2 \\ + f[x_{i-1}, x_{i-1}, x_i, x_i](x - x_{i-1})^2(x - x_i). \end{aligned}$$

On the interval $[x_{i-1}, x_i]$, the error

$$\begin{aligned} |f(x) - H(x)| &= \frac{|f^{(4)}(\xi_i)|}{4!}(x - x_{i-1})^2(x - x_i)^2 \\ &= \frac{M_{4,i}}{4!}\frac{(x_i - x_{i-1})^4}{16} \leq M_4 h^4/384, \end{aligned}$$

where $M_{4,i} = \max_{x_{i-1} \leq \xi \leq x_i} |f^{(4)}(\xi)|$, $M_4 = \max_{i=1,n} M_{4,i} = \max_{x_0 \leq \xi \leq x_n} |f^{(4)}(\xi)|$ and $h = \max_{i=1,n} |x_i - x_{i-1}|$.

It is also useful to have a representation of H that uses the degrees of freedom $H(x_j)$ and $H'(x_j)$, $j = 0, 1, \dots, n$. Since the dimension of the space of piecewise C^1 cubics is $2n+2$, we need to find basis functions $\phi_i(x)$, $\psi_i(x)$, $i = 0, 1, \dots, n$ that are C^1 piecewise cubics so that

$$H(x) = \sum_{i=0}^{n} \phi_i(x) H(x_i) + \sum_{i=0}^{n} \psi_i(x) H'(x_i).$$

We do this by finding $\phi_i(x)$ and $\psi_i(x)$ satisfying:

$$\begin{aligned} &\phi_i(x_i) = 1, \ \phi_i(x_j) = 0, \ j \neq i, \ \phi_i'(x_j) = 0, \text{ for all } j, \\ &\psi_i(x_j) = 0, \text{ for all } j, \ \psi_i'(x_i) = 1, \ \psi_i'(x_j) = 0, \ j \neq i. \end{aligned}$$

We see that this implies that $\phi_i(x) = 0$ and $\psi_i(x) = 0$ for $x \leq x_{i-1}$ and $x \geq x_{i+1}$. On the subintervals (x_{i-1}, x_i) and (x_{i1}, x_{i+1}), we have:

$$\begin{aligned} \phi_i(x) &= \left(\frac{x - x_{i-1}}{h_i}\right)^2 \left(1 - 2\frac{x - x_i}{h_i}\right), \quad x_{i-1} < x < x_i, \\ \phi_i(x) &= \left(\frac{x_{i+1} - x}{h_{i+1}}\right)^2 \left(1 + 2\frac{x - x_i}{h_{i+1}}\right), \quad x_i < x < x_{x+1}, \\ \psi_i(x) &= (x - x_i)\left(\frac{x - x_{i-1}}{h_i}\right)^2, \quad x_{i-1} < x < x_i, \\ \psi_i(x) &= (x - x_i)\left(\frac{x_{i+1} - x}{h_{i+1}}\right)^2, \quad x_i < x < x_{i+1}, \end{aligned}$$

We can reduce the work in finding these functions by noting that since $\phi_i(x_{i-1})$ and $\phi_i'(x_{i-1}) = 0$, $(x - x_{i-1})^2$ must be a factor of $\phi_i(x)$ on the subinterval (x_{i-1}, x_i) and by the defining conditions, $\psi_i(x)$ must be a multiple of $(x - x_i)(x - x_{i-1})^2$ on that subinterval. Similar formulas hold on the subinterval (x_i, x_{i+1}).

When the mesh points are equally spaced at a distance h apart, we can define basis functions for the space of piecewise cubic Hermite functions in the following simple way. We first define the functions

$$\phi(x) = (x+1)^2(1-2x), \quad -1 < x < 0, \qquad \phi(x) = (1-x)^2(1+2x), \quad 0 < x < 1,$$
$$\psi(x) = x(x+1)^2, \quad -1 < x < 0, \qquad \psi(x) = x(1-x)^2, \qquad 0 < x < 1,$$

with $\phi(x) = 0$ and $\psi(x) = 0$ for $x \leq -1$ and $x \geq 1$. Then, the $2n+2$ basis functions are given by

$$\phi_i(x) = \phi([x - x_i]/h), \qquad \psi_i(x) = h\psi([x - x_i]/h), \qquad i = 0, \ldots, n.$$

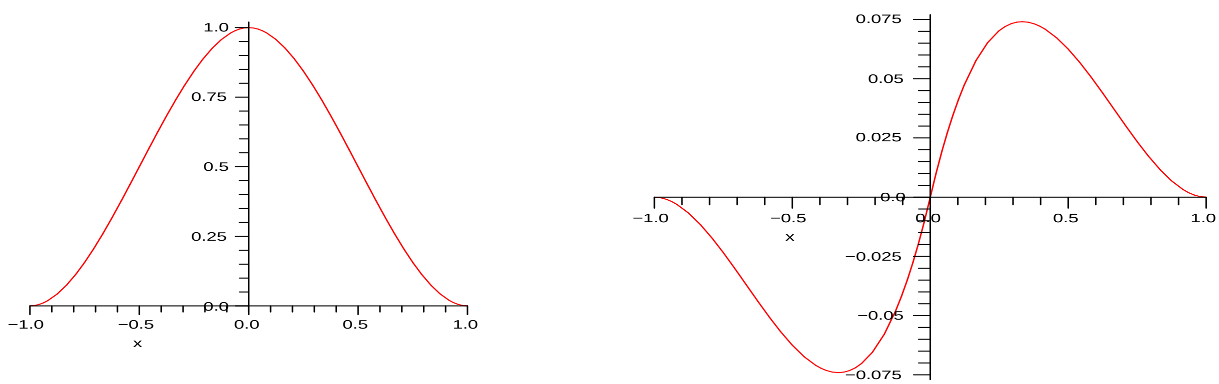

FIGURE 2. The functions $\phi(x)$ and $\psi(x)$