6. Cubic spline approximation

6.1. **Cubic spline interpolation.** We consider the problem of finding a C^2 piecewise cubic function $S(x)$ that satisfies $S(x_i) = f(x_i)$, $i = 0, \dots, n$ plus two additional conditions. These are usually taken to be either $S''(x_0) = f''(x_0)$ and $S''(x_n) = f''(x_n)$ or $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$. We will consider the first set of conditions.

We will obtain $S(x)$ by first obtaining $S''(x)$ and then integrating. Since $S''(x)$ is a continuous piecewise linear function, it is uniquely determined by the values $S''(x_i)$, $i = 0, \dots, n$. On the subinterval $[x_{i-1}, x_i]$, we can write it in the form

$$S''(x) = S_i''(x) = \frac{x_i - x}{h_i} S''(x_{i-1}) + \frac{x - x_{i-1}}{h_i} S''(x_i),$$

where $h_i = x_i - x_{i-1}$. Integrate twice on each subinterval to get

$$S_i(x) = \frac{h_i^2}{6} \left[\frac{x_i - x}{h_i}\right]^3 S''(x_{i-1}) + \frac{h_i^2}{6} \left[\frac{x - x_{i-1}}{h_i}\right]^3 S''(x_i) + A_i \frac{x_i - x}{h_i} + B_i \frac{x - x_{i-1}}{h_i}.$$

Note: Integrating twice introduces an arbitrary linear function that we represent as above.

Now $\quad f(x_{i-1}) = S_i(x_{i-1}) = \frac{h_i^2}{6} S''(x_{i-1}) + A_i, \qquad f(x_i) = S_i(x_i) = \frac{h_i^2}{6} S''(x_i) + B_i.$

Hence,

$$S_i(x) = \frac{h_i^2}{6} \left[\frac{x_i - x}{h_i}\right]^3 S''(x_{i-1}) + \frac{h_i^2}{6} \left[\frac{x - x_{i-1}}{h_i}\right]^3 S''(x_i) + [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})] \frac{x_i - x}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)] \frac{x - x_{i-1}}{h_i}.$$

We next determine the values of $S''(x_i)$ by the conditions that S' is continuous at each x_i, i.e., $S_i'(x_i) = S_{i+1}'(x_i)$, $i = 1, \dots, n-1$. Now

$$S_i'(x) = -\frac{h_i}{2} \left[\frac{x_i - x}{h_i}\right]^2 S''(x_{i-1}) + \frac{h_i}{2} \left[\frac{x - x_{i-1}}{h_i}\right]^2 S''(x_i) - [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})] \frac{1}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)] \frac{1}{h_i}.$$

Then

$$S_i'(x_i) = \frac{h_i}{3} S''(x_i) + \frac{h_i}{6} S''(x_{i-1}) + \frac{1}{h_i}[f(x_i) - f(x_{i-1})],$$

$$S_{i+1}'(x_i) = -\frac{h_{i+1}}{3} S''(x_i) - \frac{h_{i+1}}{6} S''(x_{i+1}) + \frac{1}{h_{i+1}}[f(x_{i+1}) - f(x_i)].$$

Equating these quantities to insure continuity, we get:

$$\frac{h_i}{6} S''(x_{i-1}) + \frac{h_i + h_{i+1}}{3} S''(x_i) + \frac{h_{i+1}}{6} S''(x_{i+1}) = f[x_i, x_{i+1}] - f[x_{i-1}, x_i].$$

Thus, the $n-1$ quantities $S''(x_1), \dots, S''(x_{n-1})$ are determined by solving the linear system

$$\begin{pmatrix} (h_1+h_2)/3 & h_2/6 & \cdots & \cdots \\ h_2/6 & (h_2+h_3)/3 & h_3/6 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & h_{n-1}/6 & (h_{n-1}+h_n)/3 \end{pmatrix} \begin{pmatrix} S''(x_1) \\ S''(x_2) \\ \cdots \\ S''(x_{n-1}) \end{pmatrix}$$
$$= \begin{pmatrix} f[x_1,x_2]-f[x_0,x_1]-h_1 f''(x_0)/6 \\ f[x_2,x_3]-f[x_1,x_2] \\ \cdots \\ f[x_{n-1},x_n]-f[x_{n-2},x_{n-1}]-h_n f''(x_n)/6 \end{pmatrix}$$

To check that the matrix is nonsingular, we can use the following result.

Theorem 6. *Let $A = (a_{ij})$ be an $N \times N$ matrix. If $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$, $i = 1, 2, \dots, N$, then A is nonsingular.*

Proof. Suppose A is singular. Then there is a vector $x \neq 0$ such that $\sum_{j=1}^{N} a_{ij} x_j = 0$ for $i = 1, \dots, N$. Let k satisfy $|x_k| \geq |x_j|$, $j \neq k$. Note $|x_k| \neq 0$. Since $a_{kk} x_k = -\sum_{j \neq k} a_{kj} x_j$, $|a_{kk}||x_k| \leq \sum_{j \neq k} |a_{kj}||x_j|$. Hence

$$|a_{kk}| \leq \sum_{j \neq k} |a_{kj}|(|x_j|/|x_k|) \leq \sum_{j \neq k} |a_{kj}|.$$

Contradiction. □

6.2. **Cubic spline basis functions.** When the mesh points are equally spaced at a distance h apart, we can define basis functions for the space of cubic splines in the following simple way. We first define the function $B(x) \in C^2$ to satisfy: $B(x) = 0$ for $x < -2$ and $x > 2$,

$$B(-2) = 0, \quad B'(-2) = 0, \quad B''(-2) = 0,$$
$$B(2) = 0, \quad B'(2) = 0, \quad B''(2) = 0,$$

and the condition $B(-1) + B(0) + B(1) = 1$. Note: a cubic spline on 4 subintervals has 7 degrees of freedom. Can show that $B(x)$ is given by:

$$B(x) = (x+2)^3/6, \ -2 < x < -1, \quad B(x) = -x^3/2 - x^2 + 2/3, \ -1 < x < 0,$$
$$B(x) = x^3/2 - x^2 + 2/3, \ 0 < x < 1, \quad B(x) = -(x-2)^3/6, \ 1 < x < 2,$$

and $B(x) = 0$ for $x < -2$ and $x > 2$. To see this, note that $B(x)$ will have the form $A(2+x)^3$ for $-2 < x < -1$ and $C(2-x)^3$ for $1 < x < 2$. Hence

$$B(-1) = A, \quad B'(-1) = 3A, \quad B''(-1) = 6A,$$
$$B(1) = C, \quad B'(1) = -3C, \quad B''(1) = 6C,$$

and so

$$B(x) = A + 3A(x+1) + 3A(x+1)^2 + D(x+1)^3, \quad -1 < x < 0,$$
$$B(x) = C - 3C(x-1) + 3C(x-1)^2 + E(x-1)^3, \quad 0 < x < 1.$$

Then since $B \in C^2$, the following two expressions for each of the quantities $B(0)$, $B'(0)$ and $B''(0)$ must be equal.

$$B(0) = 7A + D, \quad B'(0) = 9A + 3D, \quad B''(0) = 6A + 6D,$$
$$B(0) = 7C - E, \quad B'(0) = -9C + 3E, \quad B''(0) = 6C - 6E.$$

Solving these equations, we find: $C = A$, $D = -3A$, $E = 3A$. Finally, since $B(-1) + B(0) + B(1) = 1$, we find $A = 1/6$.

A graph of $B(x)$ is given below. Then we define the B-spline $B_i(x) = B([x - x_i]/h)$, $i = 0, \ldots n$. In general, we need $n + 3$ basis functions, so we add two additional basis functions $B_{-1}(x)$, which is non-zero only for $x_0 \le x \le x_2$ and $B_{n+1}(x)$, which is non-zero only for $x_{n-2} \le x \le x_n$, defined by the properties:

$$B_{-1}(x_0) = 0,\ B''_{-1}(x_0) = 1,\ B_{-1}(x_2) = 0,\ B'_{-1}(x_2) = 0,\ B''_{-1}(x_2) = 0,$$
$$B_{n+1}(x_n) = 0,\ B''_{n+1}(x_n) = 1,\ B_{n+1}(x_{n-2}) = 0,\ B'_{n+1}(x_{n-2}) = 0,\ B''_{n+1}(x_{n-2}) = 0,$$

and the additional property that $B_{-1}(x)$ and $B_{n+1}(x)$ belong to $C^2[x_0, x_n]$.

Somewhat more complicated expressions (using ideas similar to those applied in the case of the piecewise cubic Hermite basis functions) are needed to define the B-splines when the mesh points are not equally spaced.

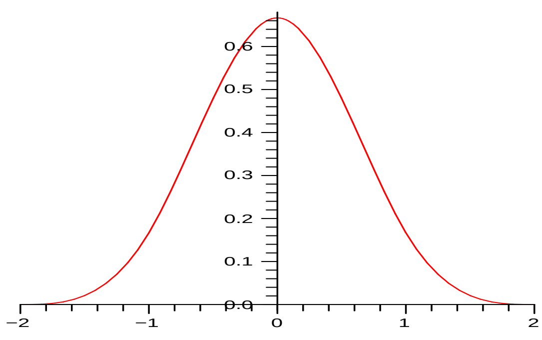

FIGURE 3. B-Spline

6.3. **Error in cubic spline interpolation.** One can derive the following error estimates for cubic spline interpolation. Again, we consider only the case when $S''(a) = f''(a)$ and $S''(b) = f''(b)$.

Theorem 7. *There exists constants C_0, C_1, and C_2, independent of f, such that*

$$\max_{a \le x \le b} |f(x) - S(x)| \le C_0 h^4 M_4, \qquad \max_{a \le x \le b} |f'(x) - S'(x)| \le C_1 h^3 M_4,$$
$$\max_{a \le x \le b} |f''(x) - S''(x)| \le C_2 h^2 M_4,$$

where $M_4 = \max_{a \le x \le b} |f^4(x)|$.