2. Polynomial Interpolation

2.1. **Interpolation error.** We now turn to an analysis of the error $f(\bar{x}) - P_n(\bar{x})$, for $\bar{x} \neq x_0, \dots x_n$. For the moment, consider $\bar{x}$ fixed, and let P_{n+1} denote the polynomial of degree $\leq n+1$ interpolating $f(x)$ at $x_0, x_1, \dots, x_n$ and $\bar{x}$. Using the Newton form of the interpolating polynomial, we know that

$$P_{n+1}(x) = P_n(x) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (x - x_j).$$

Since $P_{n+1}(\bar{x}) = f(\bar{x})$, we have by the above formula that

$$f(\bar{x}) = P_n(\bar{x}) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j),$$

and so a representation of the error is given by

$$f(\bar{x}) - P_n(\bar{x}) = f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j). \tag{2.1}$$

We next find an equivalent expression for $f[x_0, \dots, x_n, \bar{x}]$, valid when f is sufficiently smooth.

Definition: Suppose r is a non-negative integer. Then f is a function in $C^r[a,b]$ if f and its first r derivatives are continuous on $[a,b]$. So $C^0[a,b]$ denotes the space of continuous functions on $[a,b]$ and we shall use $C^{-1}[a,b]$ to denote functions which may be discontinuous on $[a,b]$.

Lemma 1. *Let $f \in C^k[a,b]$ and $x_0, \dots, x_k$ be distinct points in $[a,b]$. The there exists a point $\xi \in (a,b)$ such that $f[x_0, x_1, \dots, x_k] = f^{(k)}(\xi)/k!$.*

Proof. Let $P_k(x)$ denote the polynomial of degree $\leq k$ interpolating f at $x_0, \dots, x_k$ and define $e_k(x) = f(x) - P_k(x)$. Observe first that $e_k(x)$ has at least $k+1$ distinct zeroes at the points $x_0, \dots, x_k$. Since f and therefore e_k is differentiable on (a,b), we can use Rolle's theorem to conclude that between each two adjacent zeroes of $e_k(x)$, there exists at least one zero of $e_k'(x)$. Hence $e_k'(x)$ has at least k zeroes in (a,b). Since f and therefore $e_k(x)$ is k times differentiable in (a,b), we can continue this argument to conclude that $e_k''(x)$ has at least $k-1$ zeroes in (a,b), and finally that $e_k^{(k)}$ has at least one zero in (a,b). If we denote that zero by the point ξ, then

$$0 = e_k^{(k)}(\xi) = f^{(k)}(\xi) - P_k^{(k)}(\xi).$$

Now by formula (1.2),

$$P_k(x) = \sum_{i=0}^{k} f[x_0, \dots, x_i] \prod_{i=0}^{i-1} (x - x_j) = f[x_0, \dots, x_k] x^k + \text{polynomial of degree } < k.$$

Hence $P_k^{(k)}(x) = f[x_0, \dots, x_k] k!$ for all x and so $f[x_0, \dots, x_k] = f^{(k)}(\xi)/k!$ for some $\xi \in (a,b)$. □

Combining Lemma (1) with the representation of the interpolation error given by formula (2.1), we get the following result.

Theorem 1. *Suppose that* $f \in C^{n+1}[a,b]$ *and that* $P_n(x)$ *is a polynomial of degree* $\leq n$ *that interpolates* f *at the* $n+1$ *distinct points* $x_0, \ldots, x_n \in (a,b)$. *Then for all* $x \in [a,b]$, *there exists a point* $\xi \in (a,b)$ *(depending on* x*) such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

Proof. If x is equal to any of the interpolation points x_j, then the equation holds since both sides are zero. If x is not equal to any of the interpolation points, we have from the representation of the error given by (2.1) with $\bar{x} = x$, that

$$f(x) - P_n(x) = f[x_0, \ldots, x_n, x] \prod_{j=0}^{n} (x - x_j).$$

Since the $n+2$ points $x_0, \ldots, x_n, x$ are all distinct, we can apply Lemma (1) to conclude that $f[x_0, \ldots, x_n, x] = f^{(n+1)}(\xi)/(n+1)!$ for some $\xi \in (a,b)$ (depending on x). Substituting this result gives the theorem. ☐

Note that since ξ is not known explicitly, this formula can not be used to find the actual error. This is not surprising, since f can take on any value at non-interpolation points. However, the theorem can be used to find an upper bound on the interpolation error if we have more information about the way the derivatives of f behave. The following results follow directly from the theorem.

Corollary 1. *Suppose the conditions of Theorem* (1) *are satisfied. If* $\max_{a \leq \xi \leq b} |f^{(n+1)}(\xi)| \leq M_{n+1}$, *then*

$$|f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |(x - x_0)(x - x_1) \cdots (x - x_n)|, \quad \text{for all } x \in [a,b] \tag{2.2}$$

and

$$\max_{a \leq x \leq b} |f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{a \leq x \leq b} |(x - x_0)(x - x_1) \cdots (x - x_n)|. \tag{2.3}$$

Let us now consider an application of these results to find a bound on the error in linear interpolation. Recall that the linear polynomial interpolating $f(x)$ at x_0 and x_1 is given by $P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$. If $x \in [x_0, x_1]$ and $\max_{x_0 \leq \xi \leq x_1} |f''(\xi)| \leq M_2$, then we have by (2.2) with $a = x_0$, $b = x_1$ that

$$|f(x) - P_1(x)| \leq \frac{M_2}{2} |(x - x_0)(x - x_1)|, \quad \text{for all } x \in [a,b]$$

and by (2.3) that

$$\max_{x_0 \leq x \leq x_1} |f(x) - P_1(x)| \leq \frac{M_2}{2} \max_{x_0 \leq x \leq x_1} |(x - x_0)(x - x_1)| \leq \frac{M_2}{8} (x_1 - x_0)^2,$$

since the maximum occurs at the midpoint $(x_0 + x_1)/2$.

2.2. **Divided differences for repeated points.** Recall

$$f[x_0, x_1, \dots, x_k] = \sum_{i=0}^{k} \frac{f(x_i)}{\prod_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)}.$$

Note that $f[y_0, \dots, y_k] = f[x_0, \dots, x_k]$ if $y_0, \dots, y_k$ is any reordering of $x_0, \dots, x_k$. So far, $f[x_0, \dots, x_k]$ has only been defined when the points $x_0, \dots, x_k$ are distinct. We now wish to extend the definition to include the case of repeated points.

Example: $k = 1$. $f[x_0, x_1] = [f(x_1) - f(x_0)]/(x_1 - x_0)$, $x_1 \neq x_0$. If $f \in C^1$, then by Taylor series, $f(x_1) = f(x_0) + f'(c)(x_1 - x_0)$ for some c between x_0 and x_1. Hence, $f[x_0, x_1] = f'(c) \to f'(x_0)$ as $x_1 \to x_0$. So we define $f[x_0, x_1] = f'(x_0)$ when $x_0 = x_1$. In general, define

$$f[x_0, \dots, x_k] = \frac{f^{(k)}(y)}{k!}, \quad \text{if } x_0 = x_1 = \cdots = x_k = y.$$

With this interpretation, we can still use the Newton formula

$$P_n(x) = \sum_{i=0}^{n} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

to describe the polynomial of degree $\leq n$ interpolating $f(x)$ at $x_0, \dots, x_n$, even when the points $x_0, \dots, x_n$ are not necessarily distinct. The error is still given by the formula

$$f(x) - P_n(x) = f[x_0, \dots, x_n, x] \prod_{j=0}^{n} (x - x_j),$$

where by the interpolating polynomial we now mean that if the point z appears $k+1$ times among $x_0, \dots, x_n$, then

$$P_n^{(j)}(z) = f^{(j)}(z), \quad j = 0, \dots, k.$$

To see why this is the proper generalization of the divided difference formula in the general case, note that the polynomial

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$$

satisfies $P_k^{(j)}(x_0) = f^{(j)}(x_0)$, $j = 0, \dots, k$, so $P_k(x)$ will be the interpolating polynomial when all the interpolation points are the same. The Newton formula in this case would be

$$P_k(x) = \sum_{j=0}^{k} f[x_0, \dots, x_0](x - x_0)^j,$$

where x_0 appears $j+1$ times in the expression $f[x_0, \dots, x_0]$ above. Hence, if we want the Newton formula to give the interpolating polynomial, we need to use the definition of divided differences given above for equally spaced points.

Furthermore, if $f \in C^{n+1}(a, b)$ and $x_0, \dots, x_n, x \in [a, b]$, then one can show that

$$f[x_0, \dots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some ξ satisfying $\min(x_0, \dots, x_n, x) \le \xi \le \max(x_0, \dots, x_n, x)$.

Example: $f(x) = \ln x$. Calculate $f(1.5)$ by cubic interpolation using the data: $f(1) = 0$, $f'(1) = 1$, $f(2) = 0.693147$, $f'(2) = 0.5$. Take $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 2$.

TABLE 2

Divided difference table

x_k	$f(x_k)$	f[,]	f[,,]	f[,,,]
1	0			
		1		
1	0		−.306853	
		.693147		.113706
2	.693147		−0.193147	
		.5		
2	.693147			

Using the divided difference table, we get

$$\begin{aligned} P_3(x) &= f(1) + f[1,1](x-1) + f[1,1,2](x-1)^2 + f[1,1,2,2](x-1)^2(x-2) \\ &= 0 + 1(x-1) + (-.306853)(x-1)^2 + (.113706)(x-1)^2(x-2). \end{aligned}$$

So $P_3(1.5) = .409074$.

From the error formula, we have $\ln(x) - P_3(x) = f^{(4)}(\xi)(x-1)^2(x-2)^2/4!$. Hence,

$$|\ln(1.5) - P_3(1.5)| \le \frac{1}{4!} \max_{1 \le \xi \le 2} \frac{6}{\xi^4} (.5)^4 = \frac{1}{64} = 0.015624$$

The actual error is .00361.