13.10. **Additional types of stability.**

Definition: A method for which the region of absolute stability is equal to the entire left hand plane is said to to *A-stable.*

For such methods, there will no restriction on the step size to achieve stability. However, A-stability is a severe restriction as shown by the following results.

Theorem 16. *(Dahlquist) (i) An explicit linear multistep method cannot be A-stable.*
(ii) The order of an A-stable implicit linear multistep method cannot exceed 2.
(iii) The second order A-stable implicit linear multistep method with smallest error constant is the trapezoidal rule.

This restriction on order is quite severe since it means that one must take a very small stepsize to achieve reasonable accuracy. However, there are a number of other definitions that are less demanding, but sufficiently restrictive so that stability problems do not occur.

Definition: A numerical method is said to be $A(\alpha)$-stable, $\alpha \in (0, \pi/2)$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda : \pi - \alpha < \arg h\lambda < \pi + \alpha$. The method is $A(0)$-stable it is it $A(\alpha)$-stable for some $\alpha \in (0, \pi/2)$.

Note: For a given λ, with $\mathrm{Re}\lambda < 0$, the point $h\lambda$ either lies inside W_α for all positive h or lies outside W_α for all positive h. Hence, if one can determine in advance that all the eigenvalues lie in the wedge W_α, then an $A(\alpha)$-stable method can be used without any restriction on the stepsize. In particular, an $A(0)$-stable method can be used if the eigenvalues are known to be real, e.g., if the Jacobian is symmetric.

Theorem 17. *(i) An explicit linear multistep method cannot be $A(0)$-stable.*
(ii) There is only one $A(0)$-stable k-step method whose order exceeds k, (the Trapezoidal rule).
(iii) For all $\alpha \in [0, \pi/2)$, there exist $A(\alpha)$-stable linear k-step methods of order r for which $k = r = 3$ and $k = r = 4$.

13.11. **Stiff differential equations.**

Roughly speaking, a stiff system of ordinary differential equations is one in which solution components of interest are slowly varying, but other components are rapidly decaying. Although the rapidly decaying components are only a small perturbation of the true solution, they can place a severe stability restriction on the step size used to compute the numerical approximation.

More specifically, consider the $m \times m$ linear system of ordinary differential equations: $Y' = AY + \Phi(x)$, where we assume that A has distinct eigenvalues λ_j and corresponding eigenvectors V_j, $j = 1, \ldots, m$.

Definition: This linear system is said to be *stiff* if
(i) $\mathrm{Re}\lambda_j < 0$, $j = 1, \ldots, m$
(ii) $\max_j |\mathrm{Re}\lambda_j| >> \min_j |\mathrm{Re}\lambda_j|$.

The ratio $\max_j |\text{Re}\lambda_j|/\min_j |\text{Re}\lambda_j|$ is called the stiffness ratio (as much as 10^6 in some practical problems).

The general solution of this system has the form

$$Y(x) = \sum_{j=1}^{m} c_j e^{\lambda_j x} V_j + \Psi(x),$$

where $\Psi(x)$ is a particular solution. Since $\text{Re}\lambda_j < 0$ for $j = 1, \ldots, m$, $\sum_{j=1}^{m} c_j e^{\lambda_j x} V_j \to 0$ as $x \to \infty$. This part of the solution is called the transient solution, while $\Psi(x)$ is the steady-state solution. Next suppose λ_μ and λ_ν are eigenvalues of A satisfying

$$|\text{Re}\lambda_\mu| \geq |\text{Re}\lambda_j| \geq |\text{Re}\lambda_\nu|, \quad j = 1, \ldots, m.$$

To approximate the steady-state solution, we must numerically integrate until the slowest decaying exponent $e^{\lambda_\nu x}$ is negligible. However, if $|\text{Re}\lambda_\mu|$ is large, then stability forces the use of a very small stepsize h in order that $h\lambda_\mu$ lies in the region of absolute stability. If $|\text{Re}\lambda_\mu| >> |\text{Re}\lambda_\nu|$, then one must integrate over a long interval using a stepsize which is everywhere excessively small relative to the size of the interval – this is the problem of stiffness.

The following multistep methods, developed by W. Gear, are called backward differentiation formulas. The methods of order 1-6 (but not 7-15) satisfy another type of stability definition call *stiffly stable*. The starting point of the methods is the equation $y'(x_{n+1}) = f(x_{n+1}, y(x_{n+1}))$. We then replace y by the polynomial interpolating y at the points x_{n+1}, x_n, …, x_{n-p+1}, i.e,

$$y(x) \approx \sum_{k=0}^{p} y[x_{n+1}, x_n, \ldots, x_{n+1-k}] \prod_{i=0}^{k-1} (x - x_{n+1-i}).$$

To compute an approximation to $y'(x_{n+1})$, we observe that

$$\begin{aligned}(d/dx) \prod_{i=0}^{k-1} (x - x_{n+1-i}) &= (d/dx)[(x - x_{n+1}) \prod_{i=1}^{k-1} (x - x_{n+1-i})] \\ &= (x - x_{n+1})(d/dx) \prod_{i=1}^{k-1} (x - x_{n+1-i}) + \prod_{i=1}^{k-1} (x - x_{n+1-i}).\end{aligned}$$

Evaluating this last expression at x_{n+1}, we get in the case of equally spaced points that

$$\prod_{i=1}^{k-1} (x_{n+1} - x_{n+1-i}) = \prod_{i=1}^{k-1} ih = h^{k-1}(k-1)!$$

In that case, the family of methods becomes:

$$\sum_{k=1}^{p} y[x_{n+1}, x_n, \ldots, x_{n+1-k}] h^{k-1}(k-1)! = f_{n+1}.$$

$p = 1$: $y[x_{n+1}, x_n] \approx f_{n+1}$. i.e., $y_{n+1} = y_n + hf_{n+1}$ (backward Euler method).

$p = 2$: $y[x_{n+1}, x_n] + hy[x_{n+1}, x_n, x_{n-1}] \approx f_{n+1}$. After some algebra, we obtain the method: $y_{n+1} = (4/3)y_n - (1/3)y_{n-1} + (2/3)hf_{n+1}$.

To determine the local truncation error for these methods, (i.e, how well the true solution satisfies the discrete equations), we write

$$y(x) = \sum_{k=0}^{p} y[x_{n+1}, x_n, \ldots, x_{n+1-k}] \prod_{i=0}^{k-1}(x - x_{n+1-i}) + y[x_{n+1}, x_n, \ldots, x_{n+1-p}, x] \prod_{i=0}^{p}(x - x_{n+1-i}).$$

Then

$$y'(x_{n+1}) = \sum_{k=1}^{p} y[x_{n+1}, x_n, \ldots, x_{n+1-k}]h^{k-1}(k-1)! + y[x_{n+1}, x_n, \ldots, x_{n+1-p}, x_{n+1}]h^p p!.$$

Since $y'(x_{n+1}) = f(x_{n+1}, y(x_{n+1}))$, the local truncation error, defined by

$$h \sum_{k=1}^{p} y[x_{n+1}, x_n, \ldots, x_{n+1-k}]h^{k-1}(k-1)! - hf(x_{n+1}, y(x_{n+1}))$$

is given by

$$= -h^{p+1} p! y[x_{n+1}, x_n, \ldots, x_{n+1-p}, x_{n+1}] = -h^{p+1} y^{(p+1)}(\xi)/(p+1).$$

To be A-stable, $A(\alpha)$-stable, or stiffly-stable, a linear multistep method must be implicit. Hence, it is used in predictor-corrector format, i.e., one solves the nonlinear equation given by the corrector with initial guess given by the predictor. Previously, we considered the simple iteration

$$y_{n+1}^{j+1} = hb_{-1} f(x_{n+1}, y_{n+1}^{j}) + g,$$

where g is already known from past values. For convergence, we required $h|Lb_{-1}| < 1$, where L is the Lipschitz constant. For stiff systems, L is extremely large and hence this condition poses a severe restriction on the stepsize h, about the same severity as that imposed by the stability requirements of a method. Hence, we solve this nonlinear equation using Newton's method or some variation. Recall, Newton's method for the nonlinear system $F(Y) = 0$ is given by:

$$Y^{j+1} = Y^j - J^{-1}(Y^j)F(Y^j),$$

where $J(Y)$ is the Jacobian matrix $\partial F_i / \partial Y_j$. In this case, $F(y) = y - hb_{-1} f(x, y) - g$.

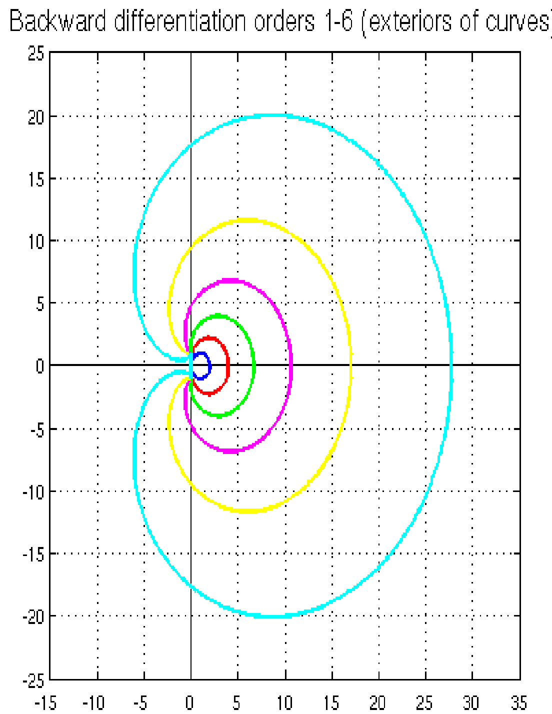

FIGURE 7. Stability regions for Backward Differentiation Formulas (1-6)