13.12. **Discontinuous Galerkin methods for ordinary differential equations.** Reference: Delfour, Hager and Trochu, Math. Comp. (36) 1981, pp. 455-472.

Consider the problem $y' = f(t, y), \qquad y(0) = \alpha$.
Let $0 = t_0 < t_1 < \ldots < t_N = T$, $I_j = (t_{j-1}, t_j)$, and

$$V_h = \{v : v|_{I_j} \in P_k(I_j), j = 1, \cdots, N\}.$$

The approximation scheme is:
Find $y_h \in V_h$ and $y_h(0^-)$ or $y_h(T^+)$ such that for $1 \le j \le N$,

$$\int_{t_{j-1}}^{t_j} [dy_h/dt - f(t, y_h)]v\,dt + v(t_{j-1}^+)\alpha_{j-1}[y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] + v(t_j^-)(1-\alpha_j)[y_h(t_j^+) - y_h(t_j^-)] = 0, \quad \forall v \in P_k(I_j),$$

where $\alpha_0, \ldots, \alpha_N$ are scalars with $0 \le \alpha_j \le 1$ and we evaluate the integrals using a $k+1$ point quadrature formula exact for polynomials of degree $\le 2k+1$.

In this method, we produce a discontinuous piecewise polynomial approximation to the solution, i.e., the value $y_h(t_j^-)$ gives an approximation to $y(t_j)$ inside the interval (t_{j-1}, t_j) and $y_h(t_j^+)$ gives an approximation to $y(t_j)$ inside the interval (t_j, t_{j+1}). Note that the true solution of the initial value problem satisfies the equation given by the approximation scheme, since it satisfies the differential equation and for all j, the jump terms $y_h(t_j^+) - y_h(t_j^-) = 0$.

We will assume that either (i) $\alpha_0 = 0$ and $\alpha_j \ne 1$, $j = 1, \ldots N$, or (ii) $\alpha_N = 1$ and $\alpha_j \ne 0$, $j = 0, \ldots N-1$. In case (i), the unknown $y(0^-)$ disappears and in case (ii), the unknown $y(T^+)$ disappears. Hence we have $N(k+1)+1$ unknowns and $N(k+1)+1$ equations, where one of the unknowns and equations is determined by the initial condition and the rest from the above equation.

Consider some special cases: $k = 0$. Then y_h is a constant on each subinterval so $y_h(t_j^-) = y_h(t_{j-1}^+)$. Taking $v = 1$, and applying the midpoint quadrature rule (a one-point formula exact for linear polynomials), we get:

$$y_h(t_j^-) - y_h(t_{j-1}^+) - hf(t_{j-1/2}, y_h(t_{j-1/2})) + \alpha_{j-1}[y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] + (1-\alpha_{j-1})[y_h(t_j^+) - y_h(t_j^-)] = 0.$$

Now suppose $\alpha_j = 0$ for all j. Then

$$y_h(t_j^+) - y_h(t_{j-1}^+) = hf(t_{j-1/2}, y_h(t_{j-1}^+)), \qquad j = 1, \ldots, N.$$

In this case, $y_h(t_j^+)$ denotes the approximation to the true solution $y(t_j)$, so the method is similar to explicit Euler, except that t is evaluated at $t_{j-1/2}$ instead of t_{j-1}. In this case the starting value is $y_h(t_0^+)$ and $y_h(t_0^-)$ is not part of the method.

Next suppose $\alpha_j = 1$ for all j ($k = 0$). Then

$$y_h(t_j^-) - y_h(t_{j-1}^-) = hf(t_{j-1/2}, y_h(t_j^-)), \qquad j = 1, \ldots, N.$$

In this case, $y_h(t_j^-)$ denotes the approximation to the true solution $y(t_j)$, so the method is similar to implicit Euler, except that t evaluated at $t_{j-1/2}$ instead of t_j. In this case the starting value is $y_h(t_0^-)$ and $y_h(T^+)$ is not part of the method.

The methods with the choice $\alpha_j = 1$ are equivalent to implicit Runge-Kutta methods.

When $k = 1$, y_h will be a linear polynomial on the subinterval (t_{j-1}, t_j) which we write as:

$$y_h(t) = [1 - (t - t_{j-1})/h]y_h(t_{j-1}^+) + [(t - t_{j-1})/h]y_h(t_j^-).$$

If $\alpha_j = 1$, the equations are

$$\int_{t_{j-1}}^{t_j} [dy_h/dt - f(t, y_h)]v\,dt + v(t_{j-1}^+)[y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] = 0,$$

where we evaluate the integral using a 2-point Gauss formula. We now have two unknowns $y_h(t_{j-1}^+)$ and $y_h(t_j^-)$, and get two equations by choosing $v = 1$ and $v = t$. We thus need to solve a 2×2 system of non-linear equations.