2.3. **Interpolation of moments.** In some applications, it is useful to interpolate quantities other than function or derivative values. One quantity that arises frequently is the moments of a function f over an interval $[a, b]$, i.e., quantities of the form $\int_a^b x^r f(x)\,dx$.

Example: Interpolation of f by a quadratic function $P_2(x)$ on the interval $[a, b]$. So far, we have defined $P_2(x)$ by requiring that it interpolate f at the points $x_0 = a$, $x_1 = b$, and a third point, say z in the interior of $[a, b]$. Another possibility is to replace interpolation of one or more of the point values of f by interpolation of moments. In particular, we could replace the requirement that $P_2(z) = f(z)$ by the condition $\int_a^b P_2(x)\,dx = \int_a^b f(x)\,dx$. To find such a function, we can save some work by writing it on the interval $[x_0, x_1]$ in the form

$$P_2(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) + A(x - x_0)(x - x_1).$$

Note that for any value of the constant A, $P_2(x_0) = f(x_0)$ and $P_2(x_1) = f(x_1)$. We now determine A by the integral condition, i.e.,

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} P_2(x)\,dx = \frac{x_1 - x_0}{2}[f(x_0) + f(x_1)] - A\frac{[x_1 - x_0]^3}{6}.$$

$$\text{Hence} \qquad A = \frac{3}{[x_1 - x_0]^2}\left\{[f(x_0) + f(x_1)] - \frac{2\int_{x_0}^{x_1} f(x)\,dx}{x_1 - x_0}\right\}.$$

This representation is useful for many purposes. However, we can also rewrite it in the form:

$$\begin{aligned} P_2(x) = &\left\{\frac{x - x_1}{x_0 - x_1} + \frac{3}{[x_1 - x_0]^2}(x - x_0)(x - x_1)\right\} f(x_0) \\ &+ \left\{\frac{x - x_0}{x_1 - x_0} + \frac{3}{[x_1 - x_0]^2}(x - x_0)(x - x_1)\right\} f(x_1) \\ &- \left\{\frac{6}{[x_1 - x_0]^3}(x - x_0)(x - x_1)\right\} \int_{x_0}^{x_1} f(x)\,dx. \end{aligned}$$

This representation shows we can write any quadratic polynomial in the form

$$\begin{aligned} P_2(x) = &\left\{\frac{x - x_1}{x_0 - x_1} + \frac{3}{[x_1 - x_0]^2}(x - x_0)(x - x_1)\right\} P_2(x_0) \\ &+ \left\{\frac{x - x_0}{x_1 - x_0} + \frac{3}{[x_1 - x_0]^2}(x - x_0)(x - x_1)\right\} P_2(x_1) \\ &- \left\{\frac{6}{[x_1 - x_0]^3}(x - x_0)(x - x_1)\right\} \int_{x_0}^{x_1} P_2(x)\,dx, \end{aligned}$$

i.e., in terms of the degrees of freedom $P_2(x_0)$, $P_2(x_1)$, and $\int_{x_0}^{x_1} P_2(x)\,dx$.

Abstractly, we are writing

$$P_2(x) = B_1(x)\phi_1(P_2) + B_2(x)\phi_2(P_2) + B_3(x)\phi_3(P_2)$$

where the B_i depend only on x, the ϕ_i are degrees of freedom of P_2 and we have the key property: $\phi_i(B_j) = 1$ if $i = j$ and 0 if $i \neq j$. This is exactly what we did when we constructed the Lagrange form of the interpolating polynomial. The basis B_i is the one that most simplifies this interpolation problem. Of course, one has to find it.

More generally, given a space of polynomial functions V, a set of degrees of freedom for the space V are a set of linear functionals, (i.e., linear mappings $\{\phi_i\}$ such that $\phi_i(p)$ is a real number for each $p \in V$), which can be assigned values arbitrarily to determine a unique polynomial $p \in V$. We refer to such a set as a *unisolvent* set of degrees of freedom. Checking that a proposed set of degrees of freedom is unisolvent can be done in the following way. First note that determining a unique polynomial from its proposed set of degrees of freedom is equivalent to solving a square linear system of equations for the unknown coefficients of the polynomial in an expansion of the polynomial in a fixed basis. (Of course we assume the number of linear functionals is equal to the dimension of the space V.) From the theory of solutions of linear equations, we know that a square linear system will have a unique solution for every right hand side if and only if the only solution of the homogeneous system (i.e., when the right hand side is zero) is the zero solution. So this is what we check. For example, to check that the degrees of freedom $P(x_0)$, $P(x_1)$, $\int_{x_0}^{x_1} P(x)\,dx$ are a unisolvent set of degrees of freedom for the space of polynomials of degree ≤ 2, we set these equal to zero and show the resulting quadratic polynomial must be zero. But if $P(x_0) = P(x_1) = 0$, then $P(x) = A(x - x_0)(x - x_1)$, for some constant A. Since $(x - x_0)(x - x_1) \leq 0$, $x \in [x_0, x_1]$, its integral over this interval $\neq 0$, so then $A = 0$ and hence $P(x) = 0$.

The basis we constructed at the beginning of this example is an illustration of a hierarchical basis, i.e., first we constructed a basis for the linear functions interpolating at 2 points. Then we added a quadratic that vanishes at those points. In a more general setting, we would choose basis functions in such a way that the linear system we have to solve to satisfy the conditions of the interpolation problem is triangular and therefore easy. If we don't choose special basis functions, then in general, the solution of the interpolation problem will require the solution of a linear system of equations involving a full matrix.

2.4. **Runge example.** One might infer from the error formula for polynomial interpolation that as one adds more and more interpolation points, one gets a better and better approximation. This fact is not true in general and depends on how the points are added.

Example: Runge $f(x) = 1/(1 + x^2)$, $x \in [-5, 5]$. Set $x_j = -5 + j\Delta x$, $j = 0, 1, \ldots, n$, $\Delta x = 10/n$. For each n, there is a unique polynomial $P_n(x)$ of degree $\leq n$ satisfying $P_n(x_j) = f(x_j)$. However, $|f(x) - P_n(x)|$ will become arbitrarily large at points in $[-5, 5]$ as n becomes large. One can show that for $n = 2r$,

$$f(x) - P_n(x) = \prod_{j=0}^{n} x(x - x_j) \frac{f(x)(-1)^{r+1}}{\prod_{j=0}^{r}(1 + x_j^2)}.$$

For $n = 2$,

$$|f(x) - P_2(x)| \leq \frac{|x^2(x + 5)(x - 5)}{26(1 + x^2)} \leq 1,$$

by looking at the graphs. For $n = 10$, $x = -4.5$, $|f(x) - P_n(x)| = 1.53166$, so the maximum error is not getting smaller as n increases.

3. Piecewise Polynomial Approximation

Consider a partition $\mathcal{P}$ of $[a, b]$ by points $x_0, \ldots, x_n$, i.e., $a = x_0 < x_1 < \ldots < x_n = b$.

Definition: We say $Q(x)$ is a C^r piecewise polynomial of degree $\leq k$ with respect to the partition $\mathcal{P}$ if $Q \in C^r[a, b]$ and Q has the form $Q(x) = q_j(x)$ for $x \in (x_{j-1}, x_j)$, $j = 1, \ldots n$, where $q_j(x)$ is a polynomial of degree $\leq k$ for each value of j.

Note that since $Q(x) \in C^r$ and the q_j are polynomials, its first r derivatives are continuous and its $r + 1$st derivative is defined everywhere except possibly at the points x_j.

Examples of piecewise polynomials:

$k = 0$: Piecewise constants (dimension $= n$). Degrees of freedom are: values of Q on each subinterval.

$k = 1$: Discontinuous piecewise linears (dimension $= 2n$). Degrees of freedom are: 2 values of Q on each subinterval. Continuous piecewise linears (dimension $= n + 1$). Degrees of freedom are: values of Q at points x_j. Note, in this case, we choose the mesh points x_j to insure continuity of Q, i.e., to have $q_j(x_j) = q_{j+1}(x_j)$.

$k = 2$: Discontinuous piecewise quadratics (dimension $= 3n$). Degrees of freedom are: 3 values of Q on each subinterval. Continuous piecewise quadratics (dimension $= 2n + 1$). Degrees of freedom are: values of Q at x_j and at one interior point in each subinterval. C^1 piecewise quadratics (dimension $= 2n + 1 - (n - 1) = n + 2$. Degrees of freedom are: values of Q at x_j and $Q'(x_0)$.

$k = 3$: C^{-1}, C^0, C^1, C^2 piecewise cubics. C^{-1}: (dimension $= 4n$). Degrees of freedom are: 4 values of Q on each subinterval. C^0: (dimension $= 3n + 1$). Degrees of freedom are: values of Q at x_j and at 2 interior points in each subinterval. C^1: (dimension $= 2n + 2$). Degrees of freedom are: values of Q and Q' at x_j. Note that this choice will guarantee that Q and Q' will be continuous across mesh points. C^2: (dimension $= 2n + 2 - (n - 1) = n + 3$). Degrees of freedom are: values of Q at x_j plus 2 additional conditions.

Note: A C^r piecewise polynomial of degree $\leq r$ is a global polynomial of degree $\leq r$. If $r = k - 1$, Q is called a spline function. To compute the dimension of the space of C^r piecewise polynomials of degree $\leq k$ starting from the dimension of the space of C^{r-1} piecewise polynomials of degree $\leq k$, we subtract the number of additional constraints imposed, i.e., $n - 1$, one at each interior mesh point.

Number of Degrees of Freedom

P_n	C^{-1}	C^0	C^1	C^2
0	n	1		
1	$2n$	$n+1$	2	
2	$3n$	$2n+1$	$n+2$	3
3	$4n$	$3n+1$	$2n+2$	$n+3$